Rigidity of time-flat surfaces in the Minkowski spacetime

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A time-flat condition on spacelike 2-surfaces in spacetime is considered here. This condition is analogous to the constant torsion condition for curves in a three-dimensional space and has been studied in [2, 5, 6, 13, 14]. In particular, any 2-surface in a static slice of a static spacetime is time-flat. In this paper, we address the question in the title and prove several local and global rigidity theorems for such surfaces in the Minkowski and Schwarzschild spacetimes. Higher-dimensional generalizations are also considered.

1. Introduction

The geometry of spacelike 2-surfaces in spacetime plays a crucial role in general relativity. Penrose’s singularity theorem predicts future singularity formation from the existence of a trapped 2-surface. A black hole is quasi-locally described by a marginally outer trapped 2-surface. These conditions can be expressed in terms of the mean curvature vector field $H$ of the 2-surface. $H$ is the unique normal vector field determined by the variation of the area functional and is ultimately connected to the warping of spacetime in the vicinity of the 2-surface. It is thus not surprising that several definitions of quasi-local mass in general relativity are closely related to the mean curvature vector field. In particular, both the Hawking mass [8] and the Brown–York–Liu–Yau mass [4, 10] involve the norm of the mean curvature vector field $|H|$. In the new definition of quasi-local mass in [13, 14], in addition to $|H|$, the direction of the mean curvature vector field is also utilized. When the mean curvature vector field is spacelike everywhere on $\Sigma$ (thus $|H| > 0$), the direction of $H$ defines a connection one-form $\alpha_H$ of the normal bundle (see Definition 2 for the precise definition of $\alpha_H$). The quasi-local mass in [13, 14] is defined in terms of the induced metric $\sigma$ on
the surface, $|H|$, and $\alpha_H$. In particular, the condition

\begin{equation}
\text{div}_\sigma(\alpha_H) = 0
\end{equation}

implies that the isometric embedding of $\Sigma$ into $\mathbb{R}^3 \subset \mathbb{R}^{3,1}$ is an optimal isometric embedding in the sense of [13, 14]. Recently, Bray and Jauregui [2] discovered a very interesting monotonicity property of the Hawking mass along surfaces that satisfy the condition (1.1). Such surfaces are said to be “time-flat” in [2] and include all 2-surfaces in a time-symmetric initial data set.

For curves in $\mathbb{R}^3$, the direction of the mean curvature vector corresponds to the normal of the curve [7, pages 16–17]. The connection 1-form $\alpha_H$ is nothing but the torsion of the curve (see Definition 2) and condition (1.1) simply says the torsion is constant. Weiner [16, 17] constructed simple closed curves with constant torsion in $\mathbb{R}^3$ that do not lie in a totally geodesic $\mathbb{R}^2$. On the other hand, in [3], Bray and Jauregui proved that a curve in $\mathbb{R}^3$ with constant torsion must lie in a totally geodesic $\mathbb{R}^2$ if it can be written as a graph over a simple closed curve in $\mathbb{R}^2$.

A natural rigidity question raised by Bray [1] is “Must a time-flat surface in the Minkowski spacetime lie in a totally geodesic $\mathbb{R}^3$?” From the above analogy between curves and surfaces, one expects some global condition is needed in order for the rigidity property of time-flat surfaces to hold. In this paper, we prove several global and local rigidity theorems for time-flat surfaces in the Minkowski and Schwarzschild spacetimes under various conditions.

We first prove a local rigidity theorem that holds in the Minkowski spacetime in all dimensions:

**Theorem 4.** Let $n \geq 3$. Suppose $\Sigma$ is a mean convex hypersurface which lies in a totally geodesic $\mathbb{R}^n$ in the $n + 1$ dimensional Minkowski spacetime $\mathbb{R}^{n,1}$, then $\Sigma$ is locally rigid as a time-flat $n - 1$ dimensional submanifold in $\mathbb{R}^{n,1}$. In other words, any infinitesimal deformation of $\Sigma$ that preserves the time-flat condition must be a deformation in the $\mathbb{R}^n$ direction, a deformation that is induced by a Lorentz transformation of $\mathbb{R}^{n,1}$, or a combination of these two types of deformations.

We also prove three global rigidity theorems:

**Theorem 6.** Suppose $\Sigma$ is a time-flat 2-surface in $\mathbb{R}^{3,1}$ such that $\alpha_H = 0$ and $\Sigma$ is a topological sphere, then $\Sigma$ lies in a totally geodesic hyperplane.
Theorem 7. Suppose $\Sigma$ is a time-flat 2-surface in $\mathbb{R}^{3,1}$ and $\Sigma$ is a topological sphere. If $\Sigma$ is invariant under a rotational Killing vector field, then $\Sigma$ lies in a totally geodesic hyperplane in $\mathbb{R}^{3,1}$.

The last global rigidity theorem holds for the $(n+1)$-dimensional Schwarzschild spacetime with metric

$$-\left(1 - \frac{2m}{r^{n-2}}\right) dt^2 + \frac{1}{1 - \frac{2m}{r^{n-2}}} dr^2 + r^2 g_{S^{n-1}},$$

where $m \geq 0$ is the mass and $g_{S^{n-1}}$ is the standard metric on a unit sphere $S^{n-1}$.

Theorem 10. Let $n \geq 3$ and $\Sigma^{n-1}$ be a connected spacelike codimension-2 submanifold in the $(n+1)$-dimensional Schwarzschild spacetime with mass $m$. Suppose $\alpha_H = 0$ and $\Sigma$ is star-shaped with respect to $Q$ (see Definition 9) where $Q = r dr \wedge dt$. Then, when $m > 0$, $\Sigma$ lies in a time-slice; when $m = 0$, $\Sigma$ lies in a totally geodesic hyperplane.

Note that $Q$ is a conformal Killing–Yano 2-form on the Schwarzschild spacetime, see [9]. Theorem 4 follows from applying the Reilly formula to the linearized equation (3.7). Theorem 6 is proved using the Codazzi equation and a theorem of Yau from [18]. The proof of Theorem 7 is reduced to Theorem 6 using the structure of axially symmetric surfaces in $\mathbb{R}^{3,1}$. Theorem 10 is proved using a Minkowski-type integral formula involving the conformal Killing–Yano 2-form on the Schwarzschild spacetime.

We review the geometry of spacetime surfaces in Section 2 and define the time-flat condition and the higher dimensional generalization. Theorem 4 is proved in Section 3, Theorem 6 is proved in Section 4, Theorem 7 is proved in Section 5, and Theorem 10 is proved in Section 6.

2. Geometry of spacelike 2-surface in spacetime

Let $N$ be a time-oriented spacetime. Denote the Lorentzian metric on $N$ by $\langle \cdot, \cdot \rangle$ and covariant derivative by $\nabla^N$. Let $\Sigma$ be a closed space-like two-surface embedded in $N$. Denote the induced Riemannian metric on $\Sigma$ by $\sigma$ and the gradient and Laplace operator of $\sigma$ by $\nabla$ and $\Delta$, respectively.

Given any two tangent vector $X$ and $Y$ of $\Sigma$, the second fundamental form of $\Sigma$ in $N$ is given by $\Pi(X, Y) = (\nabla^N_X Y) \perp$ where $(\cdot) \perp$ denotes the projection onto the normal bundle of $\Sigma$. The mean curvature vector is the
trace of the second fundamental form, or $H = \text{tr}_\Sigma \Pi = \sum_{a=1}^2 \Pi(e_a, e_a)$, where \{$e_1, e_2$\} is an orthonormal basis of the tangent bundle of $\Sigma$.

The normal bundle is of rank two with structure group $SO(1, 1)$ and the induced metric on the normal bundle is of signature $(-, +)$. Since the Lie algebra of $SO(1, 1)$ is isomorphic to $\mathbb{R}$, the connection form of the normal bundle is a genuine 1-form that depends on the choice of the normal frames. The curvature of the normal bundle is then given by an exact 2-form which reflects the fact that any $SO(1, 1)$ bundle is topologically trivial. Connections of different choices of normal frames differ by an exact form. We define (see [14]):

**Definition 1.** Let $e_3$ be a space-like unit normal along $\Sigma$, the connection one-form determined by $e_3$ is defined to be

\[
\alpha_{e_3} = \langle \nabla^N (\cdot) e_3, e_4 \rangle,
\]

where $e_4$ is the future-directed time-like unit normal that is orthogonal to $e_3$.

**Definition 2.** Suppose the mean curvature vector field $H$ of $\Sigma$ in $N$ is a spacelike vector field. The connection one-form in mean curvature gauge is

\[
\alpha_H = \langle \nabla^N (\cdot) e_3, e_4 \rangle,
\]

where $e_3 = -\frac{H}{|H|}$ and $e_4$ is the future-directed timelike unit normal that is orthogonal to $e_3$.

**Definition 3.** We say $\Sigma$ is time-flat if $\text{div}_\sigma(\alpha_H) = 0$.

**Remark 1.** For spacelike codimension-2 submanifolds in a time-oriented $(n + 1)$-dimensional spacetime $N$, the second fundamental form $\Pi$ and the mean curvature vector $H$ can be defined in the same manner. Let $e_n = -\frac{H}{|H|}$ and $e_{n+1}$ be the future timelike normal orthogonal to $e_n$. The connection one-form with respect to mean curvature gauge is defined to be

\[
\alpha_H = \langle \nabla^N (\cdot) e_n, e_{n+1} \rangle.
\]

3. **Local rigidity of mean convex hypersurfaces in $\mathbb{R}^n \subset \mathbb{R}^{n,1}$**

The local rigidity problem can be formulated as follows. Suppose $\Sigma$ is time-flat and is given by an embedding $X$. Suppose $V$ is a smooth vector field along $\Sigma$ such that the image of $X(s) = X + sV$ is infinitesimally time-flat
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in the sense the derivatives of $\text{div}_\sigma(\alpha_H)$ along the image with respect to $s$ is zero when $s = 0$. Do all such $V$ correspond to trivial deformations? It is easy to see that submanifolds lying in a totally geodesic slice is time-flat. We assume $\partial\Omega = \Sigma \subset \{t = 0\} = \mathbb{R}^n$. It is clear that any deformation in the $\mathbb{R}^n$ direction preserves the time-flat condition. On the other hand, a Lorentz transformation preserves the geometry of $\Sigma$ and thus preserves the time-flat condition.

Let $\nabla, \Delta$ denote the covariant derivative and Laplacian of the induced metric $\sigma$. Let $h_{ab}, h$ be the second fundamental form and mean curvature of $\Sigma \subset \mathbb{R}^n$ with respect to the outward unit normal $\nu$.

**Theorem 4.** Let $n \geq 3$. Suppose $\Sigma$ is a mean convex hypersurface which lies in a totally geodesic $\mathbb{R}^n$ in the $n + 1$ dimensional Minkowski spacetime $\mathbb{R}^{n,1}$, then $\Sigma$ is locally rigid as a time-flat $n - 1$ dimensional submanifold in $\mathbb{R}^{n,1}$. In other words, any infinitesimal deformation of $\Sigma$ that preserves the time-flat condition must be a deformation in the $\mathbb{R}^n$ direction, a deformation that is induced by a Lorentz transformation of $\mathbb{R}^{n,1}$, or a combination of these two types of deformations.

**Proof.** In this proof, we denote $\alpha_H$ by $\alpha$. Since $\delta(\text{div}_\sigma \alpha)$ depends linearly on infinitesimal deformation and any deformation in $\mathbb{R}^n$ corresponds to trivial deformations, it suffices to consider deformations in the time direction. Let $V = f \frac{\partial}{\partial t}$ for a smooth function $f$ defined on $\Sigma$ be such an infinitesimal deformation and $X(s) = (\tau(s), X^1(s), \ldots, X^n(s))$ be the corresponding deformation. Since we only vary the surface in the time direction, $X^i(s) = X^i(0)$ for $i = 1, \ldots, n$, and $\delta \tau = f$. Here $\delta$ stands for $\frac{\partial}{\partial s}\big|_{s=0}$. We start by computing the variation of $\text{div}_\sigma \alpha$. The induced metrics satisfy

\[(3.1) \quad \sigma(s)_{ab} = \sigma_{ab} - \frac{\partial \tau(s)}{\partial u^a} \frac{\partial \tau(s)}{\partial u^b}.\]

Since $\tau(0) = 0$, $\delta \sigma = 0$. Let $\Delta_s$ be the Laplacian of the induced metric on $X(s)$. We have

\[(3.2) \quad H = (\Delta_s \tau(s), \Delta_s X^1, \ldots, \Delta_s X^n).\]
δσ = 0 implies the infinitesimal variation of Laplacian is zero. Therefore, we have

\[(3.3)\]
\[\delta H = (\Delta f) \frac{\partial}{\partial t},\]

\[(3.4)\]
\[\delta |H|^2 = 2\langle \delta H, -he_n \rangle = 0,\]

\[(3.5)\]
\[\delta e_n = -\frac{\delta H}{h} + \frac{\delta |H|}{h^2} H = -\frac{\Delta f}{h} \frac{\partial}{\partial t}.\]

Since \(0 = \delta \langle e_{n+1}, \frac{\partial}{\partial u^a} \rangle = \delta \langle e_{n+1}, e_n \rangle\), we have

\[0 = \langle \delta e_{n+1}, \frac{\partial}{\partial u^a} \rangle + \langle e_{n+1}, \frac{\partial f}{\partial u^a} \frac{\partial}{\partial t} \rangle\]

and

\[0 = \langle \delta e_{n+1}, e_n \rangle + \langle e_{n+1}, -\frac{\Delta f}{h} \frac{\partial}{\partial t} \rangle.\]

Hence

\[(3.6)\]
\[\delta e_{n+1} = \nabla f - \frac{\Delta f}{h} e_n.\]

We are ready to compute the variation of \(\alpha\).

\[\langle \delta \alpha \rangle_a = \delta \langle D_a e_n, e_{n+1} \rangle\]
\[= \langle (\delta D)_a e_n, e_{n+1} \rangle + \langle D_{a(X)}^e e_n, e_{n+1} \rangle\]
\[+ \langle D_a(\delta e_n), e_{n+1} \rangle + \langle D_a e_n, \delta e_{n+1} \rangle.\]

Since \(\delta \sigma = 0, \delta D = 0\). By (3.5) and (3.6), we get

\[\langle \delta \alpha \rangle_a = \langle D_{\frac{\partial f}{\partial u^a}} e_n, e_{n+1} \rangle\]
\[= \langle D_a \left( -\frac{\Delta f}{h} \right) \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \rangle + \langle D_a e_n, \nabla f - \frac{\Delta f}{h} e_n \rangle\]
\[= \nabla_a \left( \frac{\Delta f}{h} \right) + h_{ab} \nabla^b f\]

and

\[(3.7)\]
\[\delta (\text{div}_\sigma \alpha) = \Delta \left( \frac{\Delta f}{h} \right) + \nabla^a \left( h_{ab} \nabla^b f \right).\]

We remark that the linearization of this operator was also derived in [5, 11].
To prove the theorem, it suffices to show that \( f \) is the restriction of linear coordinate functions on \( \Sigma \). Let \( u \) solve the Dirichlet problem

\[
\begin{align*}
\Delta u &= 0 \quad \text{in } \Omega, \\
u &= f \quad \text{on } \Sigma.
\end{align*}
\]

On \( \mathbb{R}^n \), the Reilly formula [12] reads

\[
\int_{\Omega} |D^2 u|^2 = - \int_{\Sigma} \left( h^{ab} \nabla_a f \nabla_b f + 2(\Delta f) e_n(u) + h(e_n(u))^2 \right).
\]

(3.8)

This was used in [11] to derive minimizing property of the Wang–Yau quasi-local energy. On the other hand, multiplying \( \delta(\text{div}_\sigma \alpha) = 0 \) by \( f \) and integrating over \( \Sigma \) yield

\[
\int_{\Sigma} \left( \frac{(\Delta f)^2}{h} - h^{ab} \nabla_a f \nabla_b f \right) = 0.
\]

(3.9)

Adding (3.8) and (3.9) together and completing square, we obtain

\[
\int_{\Omega} |D^2 u|^2 + \int_{\Sigma} \left( \frac{\Delta f}{\sqrt{h}} + \sqrt{h} e_n(u) \right)^2 = 0.
\]

Hence, \( u \) is a linear function up to a constant. \( \square \)

4. Global rigidity for surfaces with \( \alpha_H = 0 \)

We first recall a general sufficient condition of Yau that implies a submanifold lies in another submanifold which is totally geodesic.

**Theorem 5 ([18, Theorem 1]).** Let \( M \) be a submanifold in a \( p \)-dimensional pseudo-Riemannian manifold \( P \) with constant curvature. Let \( N_1 \) be a subbundle of the normal bundle of \( M \) with fiber dimension \( k \). Suppose the second fundamental form of \( M \) with respect to any direction in \( N_1 \) vanishes and \( N_1 \) is parallel in the normal bundle. Then \( M \) lies in a \( (p - k) \)-dimensional totally geodesic submanifold.

Although in [18], the theorem is proved only for Riemannian manifold, the same argument works in the pseudo-Riemannian case. We apply Theorem 5 to prove the following global rigidity theorem.

**Theorem 6.** Suppose \( \Sigma \) is a time-flat 2-surface in \( \mathbb{R}^{3,1} \) such that \( \alpha_H = 0 \) and \( \Sigma \) is a topological sphere, then \( \Sigma \) lies in a totally geodesic \( \mathbb{R}^3 \).
Proof. Denote by $e_3 = \frac{-H}{|H|}$ and $e_4$ to be the unit future timelike normal that is orthogonal to $e_3$. The second fundamental form of $\Sigma$ can be written as $h^{31}_{ab}e_3 - h^{41}_{ab}e_4$ and $h^{41}_{ab}$ is trace-free. The Codazzi equation for $h^{41}_{ab}$ reads

$$\nabla^a h^{41}_{ab} - \nabla_b \text{tr}_a h^4 + (\alpha_H)^a h^{31}_{ab} - \text{tr}_a h^3 (\alpha_H)_b = 0.$$ 

Since $\alpha_H = 0$ and $h^{41}_{ab}$ is trace-free, this reduces to

$$\nabla^a h^{41}_{ab} = 0.$$ 

A divergence-free symmetric trace-free 2-tensor corresponds to a holomorphic quadratic differential, which must vanish since $\Sigma$ is a topological sphere. Let $N_1$ be the subbundle of the normal bundle spanned by $e_4$. Since $\alpha_H = 0$, $N_1$ is parallel in the normal bundle. Hence by Theorem 5 above, $\Sigma$ lies in a totally geodesic hyperplane in $\mathbb{R}^{3,1}$. □

5. Global rigidity for axially symmetric and time-flat surfaces in $\mathbb{R}^{3,1}$

In this section, we prove the following global rigidity theorem for time-flat axially symmetric surfaces in $\mathbb{R}^{3,1}$.

**Theorem 7.** Suppose $\Sigma$ is a time-flat 2-surface in $\mathbb{R}^{3,1}$ and $\Sigma$ is a topological sphere. If $\Sigma$ is invariant under a rotational Killing vector field, then $\Sigma$ lies in a totally geodesic hyperplane in $\mathbb{R}^{3,1}$.

**Proof.** Without loss of generality, we assume that the rotational Killing vector field is $\frac{\partial}{\partial \phi}$ where $(t, r, \theta, \phi)$ is the standard spherical coordinate in $\mathbb{R}^{n,1}$, and in terms of this coordinate system $\Sigma$ is locally given by the embedding

$$F : (\theta, \phi) \rightarrow (t(\theta), r(\theta), \theta, \phi).$$

A basis of the tangent space of $\Sigma$ consists of

$$t' \frac{\partial}{\partial t} + r' \frac{\partial}{\partial r} + \frac{\partial}{\partial \theta} \quad \text{and} \quad \frac{\partial}{\partial \phi},$$

where $(\cdot)'$ stands for differentiation with respect to $\theta$. On the other hand, a basis of the normal bundle is given by

$$Y_3 = r^2 \frac{\partial}{\partial r} - r' \frac{\partial}{\partial \theta} \quad \text{and} \quad Y_4 = r' \frac{\partial}{\partial t} + t' \frac{\partial}{\partial r}.$$
By axisymmetry, both normal vectors $e_3 = -\frac{H}{|H|}$ and its orthogonal complement $e_4$ can be written as linear combinations of $Y_3$ and $Y_4$ with coefficients depending only on $\theta$. We compute

$$\alpha_H \left( \frac{\partial}{\partial \phi} \right) = \langle D_{\frac{\partial}{\partial \phi}} e_3, e_4 \rangle = \left\langle a(\theta) \Gamma_{\phi r}^{\phi} \frac{\partial}{\partial \phi} + b(\theta) \Gamma_{\phi \theta}^{\phi} \frac{\partial}{\partial \phi}, e_4 \right\rangle = 0.$$ 

Hence $\alpha_H = \varphi(\theta) d\theta$ and $d\alpha_H = 0$. Since $\Sigma$ is a topological 2-sphere, $d\alpha_H = \text{div} \alpha_H = 0$ imply $\alpha_H = 0$. By Theorem 6, $\Sigma$ lies in a totally geodesic hyperplane.

6. Global rigidity of codimension-2 submanifolds with $\alpha_H = 0$ in the Schwarzschild spacetime

We generalize Theorem 6 to “star-shaped” (see Definition 9) codimension-2 submanifolds in the $(n + 1)$ dimensional Schwarzschild spacetime with mass $m \geq 0$. In Schwarzschild coordinates, the metric $\bar{g}$ takes the form

$$\bar{g} = -(1 - \frac{2m}{r^{n-2}})dt^2 + \frac{1}{1 - \frac{2m}{r^{n-2}}} dr^2 + r^2 g_{S^{n-1}}.$$ 

We include the case $m = 0$ which corresponds to the Minkowski spacetime.

Let $Q = rdr \wedge dt$ be the conformal Killing–Yano 2-form and $Q^2$ be the symmetric 2-tensor given by

$$(Q^2)_{\alpha \beta} = Q_{\alpha}^\gamma Q_{\beta \gamma}.$$ 

We need the following lemma from [15, Lemma B.1] relating the curvature tensor and $Q$.

**Lemma 8.** The curvature tensor $\bar{R}_{\alpha \beta \gamma \delta}$ of the Schwarzschild metric $\bar{g}$ can be expressed as

$$\bar{R}_{\alpha \beta \gamma \delta} = \frac{2m}{r^n} (g_{\alpha \gamma} g_{\beta \delta} - g_{\alpha \delta} g_{\beta \gamma}) - \frac{n(n-1)m}{r^{n+2}} \times \left( \frac{2}{3} Q_{\alpha \beta} Q_{\gamma \delta} - \frac{1}{3} Q_{\alpha \gamma} Q_{\delta \beta} - \frac{1}{3} Q_{\alpha \delta} Q_{\beta \gamma} \right)$$

$$(6.1)$$

$$- \frac{n m}{r^{n+2}} \left( \bar{g} \circ Q^2 \right)_{\alpha \beta \gamma \delta}$$

where $(\bar{g} \circ Q^2)_{\alpha \beta \gamma \delta} = \bar{g}_{\alpha \gamma} (Q^2)_{\beta \delta} - \bar{g}_{\alpha \delta} (Q^2)_{\beta \gamma} + \bar{g}_{\beta \delta} (Q^2)_{\alpha \gamma} - \bar{g}_{\beta \gamma} (Q^2)_{\alpha \delta}.$
Let $\Sigma$ be a spacelike codimension-2 submanifold in the Schwarzschild spacetime of $(n+1)$-dimension. Let $e_n$, $e_{n+1}$ and $\alpha_H$ be as in Remark 1. We consider a natural condition that generalizes the star-shaped condition for hypersurfaces in the Euclidean space.

**Definition 9.** $\Sigma$ is said to be star-shaped with respect to $Q$ if $Q(e_n, e_{n+1}) > 0$.

We are now ready to prove the main theorem in this section:

**Theorem 10.** Let $n \geq 3$ and $\Sigma^{n-1}$ be a connected spacelike codimension-2 submanifold in the $(n+1)$-dimensional Schwarzschild spacetime with mass $m \geq 0$. Suppose $\alpha_H = 0$ and $\Sigma$ is star-shaped with respect to $Q$ where $Q = rdr \wedge dt$. Then, when $m > 0$, $\Sigma$ lies in a time-slice; when $m = 0$, $\Sigma$ lies in a totally geodesic hyperplane.

**Proof.** Let $\sigma$ denote the induced metric on $\Sigma$. Let $h_n$ and $h_{n+1}$ be the second fundamental forms with respect to $e_n$ and $e_{n+1}$, respectively. We pick a tangential basis $\partial_a, a = 1, \ldots, n-1$ and express the components of a tensor in terms of this basis together with $e_n$ and $e_{n+1}$. For example, $Q_{nb} = Q(e_n, \partial_b)$. In the following computations, we lower and raise indices with respect to the induced metric $\sigma_{ab} = \sigma(\partial_a, \partial_b)$ and its inverse $\sigma^{ab}$. Consider the divergence quantity on $\Sigma$:

$$\nabla_a \left[ \left( \text{tr}_\sigma(h_{n+1})\sigma^{ab} - h_{n+1}^{ab} \right) Q_{nb} \right].$$

By the assumption $\alpha_H = 0$ and the Codazzi equation (see, for example, [15, Theorem 2.1]), we obtain

$$\nabla_a \left( \text{tr}_\sigma(h_{n+1})\sigma^{ab} - h_{n+1}^{ab} \right) = -\bar{R}^{ab}_{a,n+1}.$$

On the other hand, $Q$ satisfies the conformal Killing–Yano equation [9, Definition 1]

$$D_\alpha Q_{\beta\gamma} + D_\beta Q_{\alpha\gamma} = \frac{2}{n} \left( \tilde{g}_{\alpha\beta}\xi_\gamma - \frac{1}{2} \tilde{g}_{\alpha\gamma}\xi_\beta - \frac{1}{2} \tilde{g}_{\beta\gamma}\xi_\alpha \right)$$

where $\xi^\beta = D_\alpha Q^{\alpha\beta}$. In our case $Q = rdr \wedge dt$ and $\xi = -n \frac{\partial}{\partial t}$. Therefore,

$$\frac{1}{2} \left( \nabla_a(Q(e_n, \partial_b)) + \nabla_b(Q(e_n, \partial_a)) \right) = \sigma_{ab} \left( \frac{\partial}{\partial t}, e_n \right) - h_{n+1,ab}Q_{n+1,n}$$

$$- \frac{1}{2}(Q_{bc}h^c_{na} + Q_{ac}h^c_{nb}).$$
The assumption $\alpha_H = 0$ and Ricci equation (see, for example, [15, Theorem 2.1]) together imply

$$Q_{bc} h_{na} c_{n+1}^{ab} = \frac{1}{2} \bar{R}_{a,n+1,n}^{ab} Q_{ba}.$$

Therefore, (6.2) becomes

$$-|h_{n+1}|^2 Q_{n,n+1} - \bar{R}_{a,n+1}^{ab} Q_{nb} + \frac{1}{2} \bar{R}_{n+1,n}^{ab} Q_{ba}.$$

Here we use the fact $\text{tr}_\sigma(h_{n+1}) = 0$. In the following, we apply the curvature formula (6.1) and an algebraic relation of components of $Q$ to simplify the last two terms.

By (6.1),

$$\bar{R}_{a,n+1}^{ab} = -\frac{n(n-1)m}{p^{n+2}} Q_{a,n+1}^{ab} Q_{a,n+1} + \frac{n(n-2)m}{p^{n+2}} Q_{n,n+1}^{ab} \left(-Q_{a,n+1}^{ab} + Q_{b}^{b} Q_{n,n+1}^{ab}\right),$$

$$\bar{R}_{n+1,n}^{ab} = -\frac{n(n-1)m}{p^{n+2}} \left(\frac{2}{3} Q_{a,n+1}^{ab} Q_{n,n+1} - \frac{1}{3} Q_{n+1,n}^{ab} Q_{n} - \frac{1}{3} Q_{n}^{a} Q_{b}^{b} Q_{n,n+1}^{ab}\right),$$

and thus

$$-\bar{R}_{a,n+1}^{ab} Q_{nb} + \frac{1}{2} \bar{R}_{n+1,n}^{ab} Q_{ba}$$

$$= \frac{nm}{p^{n+2}} \left[(n-1) Q_{a,n+1}^{ab} Q_{n,n+1} - (n-2) Q_{a,n+1}^{ab} Q_{n+1,n} + (n-2) Q_{b}^{b} Q_{n,n+1}^{ab} Q_{n,n+1} + (n-2) Q_{b}^{b} Q_{n,n+1}^{ab} Q_{n,n+1}\right].$$

From [15, Lemma B.3], we have

$$Q_{a,n+1}^{ab} Q_{n+1,n} = -\frac{1}{2} Q_{a,n+1}^{ab} Q_{n,n+1}.$$

Therefore,

$$\bar{R}_{a,n+1}^{ab} Q_{nb} - \frac{1}{2} \bar{R}_{n+1,n}^{ab} Q_{ba}$$

$$= \frac{n(n-2)m}{p^{n+2}} \left(\frac{1}{2} Q_{a,n+1}^{ab} Q_{n,n+1} + Q_{n}^{b} Q_{bn} Q_{n,n+1}\right),$$
and we obtain

\[ 0 = \int_\Sigma \nabla_a \left[ \left( \text{tr}_a (h_{n+1}) \sigma^{ab} - h_{n+1}^{ab} \right) Q_{nb} \right] d\mu \]

\[ = - \int_\Sigma \left( |h_{n+1}|^2 + \frac{n(n - 2)m}{r^{n+2}} \left( \frac{1}{2} Q_{ab} Q_{ab} + Q_{bn} Q_{bn} \right) \right) Q_{n,n+1} d\mu. \]

If \( m > 0 \), we obtain \( Q_{ab} = Q_{bn} = 0 \). As \( Q = rdr \wedge dt \), it is not hard to see that \( \frac{\partial}{\partial t} \) is orthogonal to \( \Sigma \). Hence \( \Sigma \) lies on a time-slice. If \( m = 0 \), we can only deduce \( h_{n+1} = 0 \). However, this is the case of the Minkowski spacetime on which Theorem 5 is applicable. We conclude that \( \Sigma \) lies in a totally geodesic hyperplane of the Minkowski spacetime. \( \square \)

**Remark 2.** Theorem 10 holds on a class of spherically symmetric space-times that satisfy null convergence condition. See [15, Theorem 5.11].

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