Semisimple Complex Lie Algebras

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ABSTRACT
This paper examines Lie algebras with particular focus on finite-dimensional Lie algebras over $\mathbb{C}$, building up to the classification of semisimple complex Lie algebras. We draw heavily from Erdmann and Wildon’s book *Introduction to Lie Algebras* [3]. After some brief historical background, we introduce properties like solvability and semisimplicity, define the classical matrix Lie algebras, then give a whirlwind tour of the classification by root systems and Dynkin diagrams. The appendix includes solutions to many exercises from [3].

1 Historical notes

In the 1862-1863, Sophus Lie (1842-1899) attended a series of lectures by Sylow on Galois theory while studying at the University of Oslo [6]. This may have been the kernel of Lie’s great inspiration: to look at differential equations in the way that Galois looked at polynomial equations, that is, by considering symmetry groups. First known as “continuous groups” or “infinitesimal groups,” these symmetry groups are now known as Lie groups [2]. In his 1874 paper *Zur Theorie des Integrabilitetsfaktors*, Lie demonstrated the validity of his intuition with a theorem relating the stability group of a differential equation to its solvability via integration.

Wilhelm Killing (1847-1923) began studying the closely related structures now known as Lie algebras from a different starting point than Lie. While Lie began with differential equations, Killing had a geometrical interest in “the problem of classifying infinitesimal motions of a rigid body in any type of space” [2]. In the second of a series of four papers published in 1888-1890 called *Die Zusammensetzung der stetigen, endlichen Transformationsgruppen*, Killing gave a full classification of simple complex Lie algebras. In this paper, he introduced what are now known as Cartan subalgebras, Cartan matrices, and root systems [2].


Regardless, it is certain that Killing’s paper lacked some amount of details and rigor, and Cartan’s thesis gave a fuller presentation of the classification. In his thesis, Cartan introduced what is now called the Killing form, as well as his “criterion for solvability” and “criterion for semisimplicity.” Cartan went on to do more work in the classification of Lie algebras, including his classification of simple real Lie algebras.
2 Properties of Lie Algebras

We assume that the reader is familiar with the linear algebra concepts of vector spaces, linear maps, and representing linear maps by matrices.

Definition 2.1. A Lie algebra is a vector space $L$ over a field $F$ with a bilinear bracket $[\cdot, \cdot] : L \times L \to L$ that satisfies $[x, x] = 0$ and $[x, [y, z]] + [y, [z, x]] + [z, [y, x]] = 0$ for all $x, y, z \in L$.

Specifically, the bilinear property of the bracket is equivalent to the following:

$$\sum_i a^i x_i, \sum_j b^j y_j = \sum_{i,j} a^i b^j [x_i, y_j]$$

Proposition 2.2. Let $L$ be a Lie algebra. Then $[x, y] = -[y, x]$ for $x, y \in L$.


This paper is mostly concerned with finite-dimensional Lie algebras over the fields $\mathbb{R}$ and $\mathbb{C}$, so frequently finite-dimensionality is taken as an unstated assumption. Working in finite dimensions simplifies things because it allows us to always represent a linear map by a matrix. The field can be left unspecified for many results, but many of the later theorems restrict to the case where the field is $\mathbb{C}$.

Lie algebras have many analogous concepts to groups and rings, including subalgebras (corresponding to subgroups), ideals, being abelian, center (denoted $Z(L)$), homomorphisms, isomorphisms (denoted $\cong$), and quotient algebras. For definitions of these, see chapters 1 and 2 of [3]. As just one example of the analogy between Lie algebras and rings, take the following proposition.

Proposition 2.3 (Exercise 1.6\textsuperscript{1}). Let $L_1, L_2$ be Lie algebras and let $\phi : L_1 \to L_2$ be a Lie algebra homomorphism. Then $\ker \phi$ is an ideal of $L_1$ and $\im \phi$ is a subalgebra of $L_2$.

Proof. First we show that $\ker \phi$ is an ideal of $L_1$. Let $x \in L_1, y \in \ker \phi$. Then

$$\phi([x, y]) = [\phi(x), \phi(y)] = [\phi(x), 0] = 0 \implies [x, y] \in \ker \phi$$

Now we show that $\im \phi$ is a subalgebra of $L_2$. Let $x, y \in \im \phi$. Then there exist $x', y' \in L_1$ such that $\phi(x') = x$ and $\phi(y') = y$. Then

$$[x', y'] \in L_1 \implies [x, y] = [\phi(x'), \phi(y')] = \phi([x', y']) \in \im \phi$$

The above proposition is exactly parallel to one about rings: the kernel of a ring homomorphism is an ideal of the domain, and the image is a subring of the codomain.

\textsuperscript{1} All exercise numbers refer to Erdmann and Wildon, Introduction to Lie Algebras.
2.1 Solvable and Nilpotent

Because of Proposition 2.3 above, we know that Lie algebra isomorphisms must preserve properties related to ideals, so constructions involving ideals are central to classifying Lie algebras.

**Definition 2.4.** Let $L$ be a Lie algebra with ideals $I, J$. We define the bracket of $I, J$ by

$$[I, J] = \text{span}\{[x, y] : x \in I, y \in J\}$$

**Proposition 2.5.** Let $L$ be a Lie algebra with ideals $I, J$. Then $[I, J]$ is an ideal of $L$.


In particular, we use the symbol $L'$ or $L^{(1)}$ to refer to $[L, L]$, which is called the derived algebra of $L$. We can also consider the derived algebra of $L'$, which is an ideal of $L'$, and so on. We refer to the $k$th derived algebra by $L^{(k)}$, so we have the derived series of ideals of $L$:

$$L \supset L^{(1)} \supset L^{(2)} \supset L^{(3)} \supset \ldots$$

**Definition 2.6.** A Lie algebra $L$ is solvable if $L^{(k)} = 0$ for some $k \geq 1$.

Note that the symbol “0” here refers not to the additive identity of the vector space $L$, but to the singleton set containing that element. This common abuse of notation is unfortunate, since “0” is also used for the zero vector in $L$ and the zero element of the underlying field $F$.

The adjective “solvable” is applied to both Lie algebras and to groups, and the parallel usage is not coincidental. The next two lemmas indicate how a requirement analogous to the definition of solvable group makes $L$ a solvable Lie algebra.

**Lemma 2.7 (Lemma 4.1 of [3]).** Suppose that $L$ is an ideal of $L$. Then $L/I$ is abelian if and only if $I$ contains the derived algebra $L'$.

**Proof.** (Proof quoted from [3] page 28.) The algebra $L/I$ is abelian if and only if for all $x, y \in L$ we have

$$[x + I, y + I] = [x, y] + I = I$$

or, equivalently, for all $x, y \in L$ we have $[x, y] \in I$. Since $I$ is a subspace of $L$, this holds if and only if the space spanned by the brackets $[x, y]$ is contained in $I$; that is, $L' \subseteq I$.

**Lemma 2.8 (Lemma 4.3 of [3]).** If $L$ is a Lie algebra with ideals

$$L = I_0 \supseteq I_1 \supseteq \ldots \supseteq I_{m-1} \supseteq I_m = 0$$

such that $I_{k-1}/I_k$ is abelian for $1 \leq i \leq m$, then $L$ is solvable.

**Proof.** (Proof quoted from [3] pages 29-29.) We shall show that $L^{(k)}$ is contained in $I_k$ for $k$ between 1 and $m$. Putting $k = m$ will then give $L^{(m)} = 0$.

Since $L/I_1$ is abelian, we have from Lemma 4.1 that $L' \subseteq I_1$. For the inductive step, we suppose that $L^{(k-1)} \subseteq I_{k-1}$, where $k \geq 2$. The Lie algebra $I_{k-1}/I_k$ is abelian. Therefore by Lemma 4.1, this time applied to the Lie algebra $I_{k-1}$, we have $[I_{k-1}, I_{k-1}] \subseteq I_k$. But $L^{(k-1)}$ is contained in $I_{k-1}$ by our inductive hypothesis, so we deduce that

$$L^{(k)} = [L^{(k-1)}, L^{(k-1)}] \subseteq [I_{k-1}, I_{k-1}]$$

and hence $L^{(k)} \subseteq I_k$. 


Now we introduce a important particular ideal of every Lie algebra, the radical.

**Definition 2.9.** The **radical** of a Lie algebra $L$, denoted $\text{rad } L$, is the unique maximal solvable ideal of $L$, that is, if $I \subset L$ is a solvable ideal, then $I \subset \text{rad } L$.

One must actually prove that $\text{rad } L$ is well-defined; this is shown in Corollary 4.5 of [3].

In addition to the derived series of ideals $L^{(k)}$, there is another important series of ideals of $L$, called the **central lower series**, denoted by $L^k$. In the case of the derived series, we had

$$L^{(k)} = [L^{(k-1)}, L^{(k-1)}]$$

The central lower series is defined by the similar recursive formula

$$L^k = [L, L^{k-1}]$$

and one gets a similar sequence of containments:

$$L \supset L^1 \supset L^2 \supset L^3 \supset \ldots$$

**Definition 2.10.** A Lie algebra $L$ is **nilpotent** if $L^k = 0$ for some $k \geq 1$.

As one would expect, a subalgebra of a solvable or nilpotent Lie algebra inherits being solvable or nilpotent, respectively.

**Proposition 2.11** (Lemma 4.4(a) of [3]). If $L$ is a solvable Lie algebra, then every subalgebra of $L$ is solvable.

**Proof.** Let $L$ be solvable with subalgebra $A$. Then $L^{(m)} = 0$ for some $m$. Notice that $A^{(k)} \subseteq L^{(k)}$ for all $k$, so $A^{(m)} \subseteq L^{(m)} = 0$, thus $A^{(m)} = 0$. □

**Proposition 2.12** (Lemma 4.9(a) of [3]). If $L$ is a nilpotent Lie algebra, then every subalgebra of $L$ is nilpotent.

**Proof.** Let $L$ be nilpotent with subalgebra $A$. Then $L^m = 0$ for some $m$ and $A^k \subseteq L^k$ for all $k$, so $A^k \subseteq L^k = 0$, hence $A^k = 0$. □

### 2.2 Isomorphism Theorems and Direct Sums

We assume the reader is familiar with definitions of subspaces, cosets, and quotient spaces of vector spaces. Once again, Lie algebras have analogous structures - one can consider cosets of an ideal and impose a bracket structure on them to make the space of cosets a Lie algebra (see [3] section 2.2). Lie algebras have analogous isomorphism theorems to vector spaces and groups.

**Theorem 2.13** (Isomorphism theorems).

1. Let $\phi : L_1 \rightarrow L_2$ be a Lie algebra homomorphism. Then $L_1/\ker \phi \cong \text{im } \phi$.
2. Let $I, J$ be ideals of a Lie algebra. Then $(I + J)/(J \cong I/(I \cap J)$
3. Let $I, J$ be ideals of a Lie algebra with $I \subset J$. Then $J/I$ is an ideal of $L/I$ and $(L/I)/(J/I) \cong L/J$. 


Another construction inherited from vector spaces is that of direct sums.

**Definition 2.14.** Let \( L_1, L_2 \) be Lie algebras. We define \( L_1 \oplus L_2 \) to be the vector space direct sum of \( L_1, L_2 \) as vector spaces, and give it the bracket

\[
[(x_1, x_2), (y_1, y_2)] = ([x_1, y_1], [x_2, y_2])
\]

For a proof that this bracket satisfies the necessary properties, see the solution to Exercise 2.6 in the appendix. As one would expect, the projections

\[
L_1 \oplus L_2 \to L_1 \quad \text{and} \quad L_1 \oplus L_2 \to L_2
\]

\[
(x, 0) \mapsto x \quad \text{and} \quad (0, x) \mapsto x
\]

are surjective Lie algebra homomorphisms, so \( L_1 \oplus L_2 \) contains subalgebras isomorphic to \( L_1 \) and \( L_2 \). (For more on this, see solution to Exercise 2.7 in the appendix.)

### 2.3 Simple and Semisimple

**Definition 2.15.** A Lie algebra is **simple** if it has no nonzero proper ideals and it is not abelian.

Note that if \( \dim L > 2 \) and \( L \) has no nonzero proper ideals, then it is simple.

**Definition 2.16.** A Lie algebra is **semisimple** if has no nonzero solvable ideals.

There are many other equivalent characterizations of semisimple Lie algebras, as illustrated by the following proposition.

**Proposition 2.17.** Let \( L \) be a Lie algebra. The following are equivalent:

1. \( L \) is semisimple.
2. \( \text{rad } L = 0 \).
3. \( L \) has no nonzero abelian ideals.
4. \( L \) can be written as a direct sum of simple Lie algebras.

**Proof.** The equivalence of (1) and (2) is immediate from the definitions. If \( L \) is semisimple, then the only solvable ideal is the zero ideal, so \( \text{rad } L = 0 \). If \( \text{rad } L = 0 \), then there are no bigger solvable ideals, so \( L \) is semisimple.

The equivalence of (2) and (3) is Exercise 4.6. We begin by showing that (2) implies (3). Let \( L \) be a semisimple Lie algebra. Let \( I \) be a nonzero ideal of \( L \). Suppose \( I \) is abelian. Then \([I, I] = 0\), so \( I \) is solvable. However, this contradicts the fact that \( L \) has no nonzero solvable ideals, so \( I \) must not be abelian. Thus \( L \) has no nonzero abelian ideals.

Now we show that (3) implies (2). Let \( L \) be a Lie algebra with no nonzero abelian ideals. Suppose \( L \) has a nonzero solvable ideal \( I \). Then \( I^{(k)} = 0 \) for some \( k \). Let \( m \) be the minimum over such \( k \), so that \( I^{(m)} = 0 \) but \( I^{(m-1)} \neq 0 \). Then \([I^{(m-1)}, I^{(m-1)}] = I^{(m)} = 0 \) so \( I^{(m-1)} \) is a nonzero abelian ideal of \( L \), which is a contradiction. Thus we conclude that \( L \) has no solvable ideals. Thus we have shown that (1) is equivalent to (2).

The equivalence of (1) and (4) is proven in Theorem 9.11 of [3]. We defer the proof until later (Theorem 5.5), since it requires machinery that has not yet been defined. \( \Box \)
3 Matrix Lie Algebras

Most important examples of Lie algebras are matrix algebras, and as one eventually discovers in the classification, every finite-dimensional semisimple complex Lie algebra is isomorphic to a matrix algebra, except for five exceptions. For all matrix Lie algebras, the bracket is given by the matrix commutator, \([x, y] = xy - yx\). One can quickly confirm that this bracket always satisfies \([x, x] = 0\) and the Jacobi identity. Before getting into matrix algebras, we need the notation \(e_{ij}\).

**Definition 3.1.** The matrix \(e_{ij}\) is the matrix with a one in the \(ij\)th place and zero elsewhere.

**Lemma 3.2.** \([e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{il}e_{kj}\)

*Proof.* See appendix, Proposition 6.10. \(\square\)

**Definition 3.3.** \(\text{gl}(n, F)\) is the set of \(n \times n\) matrices with entries in \(F\).

It has dimension \(n^2\), since \(\{e_{ij} : 1 \leq i, j \leq n\}\) is a basis. \(\text{gl}(n, F)\) is closely related to the following, as indicated by the parallel notation.

**Definition 3.4.** Let \(V\) be a vector space over a field \(F\). Then \(\text{gl}(V)\) is the set of linear maps from \(V\) to \(V\).

For any vector space \(V\), \(\text{gl}(V)\) is a Lie algebra under the bracket \([x, y] = x \circ y - y \circ x\). (For proof that this bracket satisfies the Jacobi identity, see solution to Exercise 1.3 in the appendix.)

The parallel “\(\text{gl}\)” notation is justified by the fact that if \(V\) is finite-dimensional, then \(\text{gl}(V)\) is isomorphic to \(\text{gl}(n, F)\). If one fixes a basis of \(V\), then one can represent every map \(x \in \text{gl}(V)\) uniquely by a matrix in \(\text{gl}(n, F)\), and every matrix in \(\text{gl}(n, F)\) gives a linear map in \(\text{gl}(V)\). This bijection preserves the bracket since composition of linear maps corresponds to multiplication of matrix representations. Note that in this correspondence, to compute the “action” of a matrix as a linear map on a vector, one multiplies the matrix by the vector, with the matrix as the left multiplicand.

Because of this isomorphism, people often refer to the elements of \(\text{gl}(V)\) as matrices, and the same terminology is used to refer to matrices and linear maps. Here’s a table summarizing the equivalence. Let \(V\) be any finite-dimensional vector space over \(F\) (isomorphic to \(F^n\)), \(v \in V\), and \(\lambda \in F\).

<table>
<thead>
<tr>
<th>ambient space</th>
<th>(\text{Linear maps})</th>
<th>(\text{Matrices})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\text{gl}(V))</td>
<td>(\text{gl}(n, F))</td>
<td></td>
</tr>
<tr>
<td>binary operation</td>
<td>composition</td>
<td>multiplication</td>
</tr>
<tr>
<td>zero element</td>
<td>zero map, (v \mapsto 0)</td>
<td>matrix of all zeros</td>
</tr>
<tr>
<td>identity element</td>
<td>identity map, (v \mapsto v)</td>
<td>identity matrix, (\text{diag}(1,1,\ldots))</td>
</tr>
<tr>
<td>nilpotent</td>
<td>(\exists n) such that (x^n(v) = 0, \forall v)</td>
<td>(\exists n) such that (x^n = 0)</td>
</tr>
<tr>
<td>eigenvectors</td>
<td>(v) such that (x(v) = \lambda v)</td>
<td>(v) such that (xv = \lambda v)</td>
</tr>
<tr>
<td>eigenvalues</td>
<td>(\lambda) such that (x(v) = \lambda v)</td>
<td>(\lambda) such that (xv = \lambda v)</td>
</tr>
<tr>
<td>diagonalizable</td>
<td>(\exists) basis of (V) of eigenvectors</td>
<td>(\exists) basis of (V) of eigenvectors</td>
</tr>
</tbody>
</table>
3.1 The Adjoint Representation

Definition 3.5. Let $L$ be a Lie algebra over a field $F$. A representation of $L$ is a Lie algebra homomorphism $\phi : L \to \mathfrak{gl}(V)$ where $V$ is a vector space over $F$.

While technically the word “representation” refers to the map $\phi$, often this map is described without being named, and instead the vector space $V$ is called a representation of $L$. Assuming that $V$ is finite-dimensional, $\mathfrak{gl}(V)$ is isomorphic to $\mathfrak{gl}(n, F)$, so a representation allows one to work with the elements of $L$ as if they were matrices, by considering their images under $\phi$. This explains the use of the term “representation;” a representation allows one to represent the elements of $L$ as matrices.

The most important example of a representation is the adjoint representation.

Definition 3.6. Let $L$ be a Lie algebra. The adjoint representation of $L$ is the map $\text{ad} : L \to \mathfrak{gl}(L)$ where $\text{ad} x$ is the map $y \mapsto [x, y]$. That is, $\text{ad} x(y) = [x, y]$.

One can check that $\text{ad}$ is a Lie algebra homomorphism (see [3] pages 4-5). If $L$ is finite-dimensional, the adjoint representation always allows one to represent $L$ as a Lie subalgebra of $\mathfrak{gl}(n, F)$. You may lose some information about the structure of $L$, though, since $\text{ad}$ is not necessarily one-to-one. (The word faithful is often used to refer to a one-to-one representation.) However, even when $\text{ad}$ is not faithful, it is always possible to find a faithful representation as a matrix algebra, due to Ado’s Theorem.

Theorem 3.7 (Ado). Let $L$ be a finite-dimensional Lie algebra. There exists a faithful representation of $L$, $\phi : L \to \mathfrak{gl}(V)$ where $V$ is a finite-dimensional vector space.

Proof. See [9]. For more on why the result holds over arbitrary fields, see [8].

This theorem justifies focusing study of the subalgebras of $\mathfrak{gl}(n, F)$, since such an approach is in fact the study of all finite-dimensional Lie algebras.

3.2 Matrices with Trace Zero

Definition 3.8. $\mathfrak{sl}(n, F)$ is the set of $n \times n$ matrices with entries in $F$ and trace zero.

It has dimension $n^2 - 1$, since $\{e_{ij} : i \neq j\} \cup \{e_{ii} - e_{i+1,i+1} : 1 \leq i \leq n - 1\}$ is a basis. To confirm that $\mathfrak{sl}(n, F)$ is a Lie algebra, one needs to check that the bracket of two matrices with trace zero also has trace zero. In fact, the bracket of any two matrices has trace zero, as shown in the next proposition.

Proposition 3.9. Let $x, y$ be matrices. Then $\text{tr}[x, y] = 0$.

Proof. Let $x_{ij}$ and $y_{ij}$ be the $ij$th entries of $x, y$ respectively. Then

$$\text{tr}(xy) = \sum_i (xy)_{ii} = \sum_i \sum_j x_{ij}y_{ji} = \sum_j \sum_i y_{ji}x_{ij} = \sum_j (yx)_{jj} = \text{tr}(yx)$$

Hence

$$\text{tr}[x, y] = \text{tr}(xy - yx) = \text{tr}(xy) - \text{tr}(yx) = 0$$
This establishes that $\mathfrak{sl}(n,F)$ is closed under the bracket. It also establishes that the derived algebra $\mathfrak{gl}(n,F)'$ is contained in $\mathfrak{sl}(n,F)$. In fact, the derived algebra is not merely a subset of $\mathfrak{sl}(n,F)$, but equal to it.

**Proposition 3.10** (Exercise 2.10). The derived algebra of $\mathfrak{gl}(n,F)$ is $\mathfrak{sl}(n,F)$.

**Proof.** Using the above lemma, we compute:

\[
[e_{i1}, e_{1j}] = \delta_{11} e_{ij} = e_{ij} \quad \text{for } i \neq j
\]

\[
[e_{i,i+1}, e_{i+1,i}] = \delta_{i+1,i+1} e_{ii} - \delta_{ii} e_{i+1,i+1} = e_{ii} - e_{i+1,i+1} \quad \text{for } 1 \leq i < n
\]

Thus $e_{ij}, e_{ii} - e_{i+1,i+1} \in \mathfrak{gl}(n,F)'$, so $\mathfrak{gl}(n,F)'$ contains the basis described above for $\mathfrak{sl}(n,F)$. We already know that $\mathfrak{gl}(n,F)' \subset \mathfrak{sl}(n,F)$, so now that we know $\mathfrak{gl}(n,F)'$ is a subspace of equal or greater dimension, $\mathfrak{gl}(n,F)'$ must be equal to $\mathfrak{sl}(n,F)$.

One particularly important instance of $\mathfrak{sl}(n,F)$ is $\mathfrak{sl}(n,\mathbb{C})$, since it appears in the final classification of semisimple complex Lie algebras. It also has the significant property of being simple.

**Proposition 3.11** (Exercise 4.7). $\mathfrak{sl}(n,\mathbb{C})$ is a simple Lie algebra for $n \geq 2$.

**Proof.** See Exercise 4.7 in appendix.

### 3.3 Upper Triangular Matrices

We introduce two more matrix Lie algebras, and then examine several of their properties.

**Definition 3.12.** $\mathfrak{b}(n,F)$ is the set of $n \times n$ upper triangular matrices with entries in $F$.

To check that $\mathfrak{b}(n,F)$ is closed under the bracket, note that the product of upper triangular matrices is upper triangular.

**Definition 3.13.** $\mathfrak{n}(n,F)$ is the set of $n \times n$ strictly upper triangular matrices with entries in $F$.

$\mathfrak{n}(n,F)$ is also closed under the bracket, so $\mathfrak{b}(n,F)$ and $\mathfrak{n}(n,F)$ are subalgebras of $\mathfrak{gl}(n,F)$. Note that every strictly upper triangular matrix is nilpotent.

**Proposition 3.14.** $\mathfrak{b}(n,F)' = \mathfrak{n}(n,F)$.

**Proposition 3.15.** $\mathfrak{n}(n,F)$ is nilpotent.

**Proposition 3.16.** $\mathfrak{b}(n,F)$ is solvable. Additionally, if $n \geq 2$, then $\mathfrak{b}(n,F)$ is not nilpotent.

For proofs of the above propositions, see Exercises 4.4 and 4.5 in the appendix. At first glance, $\mathfrak{b}(n,F)$ being solvable and $\mathfrak{n}(n,F)$ being nilpotent seem like very narrow results, but actually there is a sense in which $\mathfrak{b}(n,F)$ is a model for many solvable Lie algebras, and $\mathfrak{n}(n,F)$ is a model for many nilpotent Lie algebras.
Proposition 3.17 (Exercise 5.4i). Let $L$ be a Lie subalgebra of $\mathfrak{gl}(V)$. Suppose there is a basis of $V$ such that every $x \in L$ is represented by a strictly upper triangular matrix. Then $L$ is isomorphic to a subalgebra of $\mathfrak{n}(n, F)$.

Proposition 3.18 (Exercise 5.4ii). Let $L$ be a Lie subalgebra of $\mathfrak{gl}(V)$. Suppose there is a basis of $V$ such that every $x \in L$ is represented by an upper triangular matrix. Then $L$ is isomorphic to a subalgebra of $\mathfrak{b}(n, F)$.

See appendix for proofs. Note that one can make the above into “if and only if” statements, since the other direction is straightforward. (If $L \cong \mathfrak{b}(n, F)$, then that isomorphism gives a representation of $L$ in which every $x$ is upper triangular, and likewise for $\mathfrak{n}(n, F)$.)

These propositions say that $\mathfrak{b}(n, F)$ and $\mathfrak{n}(n, F)$ are not merely basic instances of solvable and nilpotent Lie algebras; in a loose sense, they encapsulate much of the general structure of solvable and nilpotent Lie algebras. The next two major results extend the above pair of propositions and make this “loose sense” rigorous.

Theorem 3.19 (Engel’s Theorem). Let $V$ be a vector space, and let $L$ be a Lie subalgebra of $\mathfrak{gl}(V)$ such that every element of $L$ is a nilpotent linear transformation of $V$. Then there is a basis of $V$ in which every element of $L$ is represented by a strictly upper triangular matrix.


We already knew that if $L$ can be represented as all strictly upper triangular matrices, then $L$ is nilpotent, because of Propositions 3.15 and 3.17. Engel’s Theorem says that if there is a faithful representation of $L$ in which the representation of each $x \in L$ is nilpotent, then there is a faithful matrix representation of $L$ in all $x \in L$ are strictly upper triangular. Engel’s Theorem is frequently written in a slightly different form:

Theorem 3.20 (Engel’s Theorem). A Lie algebra $L$ is nilpotent if and only if for all $x \in L$ the map $\text{ad} \ x$ is nilpotent.


Lie’s Theorem gives a similar characterization for upper triangular matrices, but requires that the underlying field be $\mathbb{C}$.

Theorem 3.21 (Lie’s Theorem). Let $V$ be a finite-dimensional vector space over $\mathbb{C}$, and let $L$ be a solvable Lie subalgebra of $\mathfrak{gl}(V)$. Then there is a basis of $V$ in which every element of $L$ is represented by an upper triangular matrix.


3.4 Classical Lie Algebras

We have already defined $\mathfrak{sl}(n, F)$; $\mathfrak{sl}(n, \mathbb{C})$ is the same, just with the field being $\mathbb{C}$. There are two other families of classical Lie algebras that appear in the classification of semisimple complex Lie algebras.
Definition 3.22. so(n, \mathbb{C}) is the set of n × n orthogonal matrices with determinant one. That is,
\[
so(n, \mathbb{C}) = \{ x \in gl(n, \mathbb{C}) : x^t = x^{-1}, \det x = 1 \}
\]
In order to define sp(2n, \mathbb{C}), let \( \Omega \) be the following 2n × 2n block matrix.
\[
\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}
\]
Definition 3.23. sp(2n, \mathbb{C}) is the set of 2n × 2n matrices with entries in \( \mathbb{C} \) that satisfy \( x^t \Omega x = \Omega \). That is,
\[
sp(2n, \mathbb{C}) = \{ x \in gl(2n, \mathbb{C}) : x^t \Omega x = \Omega \}
\]
Note that so(n, \mathbb{C}) is a Lie subalgebra of sl(n, \mathbb{C}) and sp(2n, \mathbb{C}) is a subalgebra of sl(2n, \mathbb{C}) (see [3] page 33).

3.5 Exceptional Lie Algebras
The exceptional Lie algebras are known as \( e_6, e_7, e_8, f_4, \) and \( g_2 \). There’s not much else to say about them here, except that they’re finite-dimensional semisimple complex Lie algebras. They’ll show up in the big classification of Dynkin diagrams later.

4 Modules
Even though we already have the language of representations to think of any finite-dimensional Lie algebra as a matrix algebra, we introduce another set of notation for thinking of a Lie algebra as a set of linear maps acting on some vector space \( V \), the language of modules.

Definition 4.1. Let \( L \) be a Lie algebra over \( F \). An L-module is a vector space \( V \) over \( F \) with a map
\[
L \times V \to V
\]
\[
(x, v) \mapsto x \cdot v
\]
with the properties
\[
(\lambda x + \mu y) \cdot v = \lambda (x \cdot v) + \mu (y \cdot v) \quad (M1)
\]
\[
x \cdot (\lambda v + \mu w) = \lambda (x \cdot v) + \mu (x \cdot w) \quad (M2)
\]
\[
[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v) \quad (M3)
\]
There is an exact correspondence between \( L \)-modules and representations of \( L \). As mentioned before, one often doesn’t care about the name of the map in a representation of \( L \), and viewing it as a module allows one to ignore the map \( \phi \). Here is a short table summarizing the correspondence between \( L \)-modules and representations of \( L \).
<table>
<thead>
<tr>
<th>Representations</th>
<th>Modules</th>
</tr>
</thead>
<tbody>
<tr>
<td>main mapping</td>
<td>$\phi : L \to \text{gl}(V)$</td>
</tr>
<tr>
<td>image of one element</td>
<td>$\phi(x)(v)$</td>
</tr>
<tr>
<td>linearity in first entry</td>
<td>$\phi$ is linear</td>
</tr>
<tr>
<td>linearity in second entry</td>
<td>$\phi(x)$ is linear $\forall x$</td>
</tr>
<tr>
<td>preserves bracket</td>
<td>$\phi$ is homomorphism</td>
</tr>
<tr>
<td>submodule/subrepresentation</td>
<td>$x \in L, w \in W \to \phi(x)(w) \in W$</td>
</tr>
<tr>
<td>homomorphism</td>
<td>$\theta(\phi(x)v) = \psi(x)\theta(v)$</td>
</tr>
</tbody>
</table>

As shown on pages 55-56 of [3], one can make a representation into an $L$-module and one can make an $L$-module into a representation.

**Definition 4.2.** Let $V$ be an $L$-module. A **submodule** of $V$ is a subspace $W$ such that for $x \in L, w \in W$, we have $x \cdot w \in W$.

When first encountering submodules, one often sees the confusing interchangeability of “ submodule of $V$” and “submodule of $L$.” Even though the phrase “submodule of $L$” implies a subspace of $L$, it refers to the same thing, that is, subspaces of $V$.

**Definition 4.3.** Let $V$ be an $L$-module. $V$ is **irreducible** if it is nonzero and has no nonzero proper submodules.

There are, of course, many modules that are not irreducible. One would hope that we could always decompose a module as a direct sum of irreducible submodules. While this is not true for all Lie algebras, it is true for semisimple complex Lie algebras; this result is Weyl’s Theorem.

**Theorem 4.4** (Weyl’s Theorem). Let $L$ be a complex semisimple Lie algebra. Every finite dimensional $L$-module can be written as a direct sum of irreducible submodules.

**Proof.** See appendix B of [3].

As with any algebraic structure, modules/representations have their own notion of homomorphism.

**Definition 4.5.** Let $L$ be a Lie algebra and let $V, W$ be $L$-modules. An $L$-**module homomorphism** is a linear map $\theta : V \to W$ with

$$\theta(x \cdot v) = x \cdot \theta(v)$$

for all $v \in V, x \in L$.

**Definition 4.6.** Let $L$ be a Lie algebra and let $\phi : L \to \text{gl}(V), \psi : L \to \text{gl}(W)$ be representations. A representation homomorphism is a linear map $\theta : V \to W$ with

$$\theta(\phi(x)v) = \psi(x)\theta(v)$$

for all $v \in V, x \in L$.
Of course, the definitions for homomorphism are equivalent, but we have separated them into two different definitions since the notation looks quite different. As usual, an isomorphism (denoted \( \cong \)) is a bijective homomorphism.

**Lemma 4.7** (Schur’s Lemma). Let \( L \) be a complex Lie algebra and let \( S \) be a finite-dimensional irreducible \( L \)-module. A map \( \theta : S \to S \) is an \( L \)-module homomorphism if and only if \( \theta \) is a scalar multiple of the identity transformation, that is, if \( \theta = \lambda 1_S \) for some \( \lambda \in \mathbb{C} \).


### 4.1 Classification of \( \mathfrak{sl}(2, \mathbb{C}) \) modules

Somewhat surprisingly, the irreducible \( \mathfrak{sl}(2, \mathbb{C}) \) modules are completely determined by dimension. First we need to define the common basis \( e, f, h \) for \( \mathfrak{sl}(2, \mathbb{C}) \).

\[
e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

**Definition 4.8.** \( V_d \) is the vector space of homogenous polynomials of degree \( d \) in two variables with complex coefficients.

The natural basis for \( V_d \) is \( \{x^d, x^{d-1} y, \ldots, y^{d-1} x, y^d\} \). It has dimension \( d+1 \). \( V_d \) can be viewed as an \( \mathfrak{sl}(2, \mathbb{C}) \) module by defining

\[
e \cdot x = x \frac{\partial}{\partial y} \quad f \cdot x = y \frac{\partial}{\partial x} \quad h \cdot x = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}
\]

For details on how this is in fact a module, and that it is irreducible, see pages 67-70 of [3].

**Theorem 4.9** (Theorem 8.5 of [3]). Let \( V \) be a finite-dimensional irreducible \( \mathfrak{sl}(2, \mathbb{C}) \) module. Then \( V \) is isomorphic to \( V_d \) for some \( d \).


### 5 Classification of Semisimple Complex Lie Algebras

We now look at the structures needed for the classification theorem of Killing and Cartan.

#### 5.1 The Killing Form

As mentioned in the historical notes, the Killing form is somewhat improperly named, as it was actually first introduced by Cartan. But someone (not Cartan) decided to call this map the Killing form, and the name stuck.

**Definition 5.1.** Let \( L \) be a complex Lie algebra. The **Killing form** on \( L \) is the symmetric bilinear form \( \kappa : L \times L \to \mathbb{C} \) defined by

\[
\kappa(x, y) = \text{tr}(\text{ad} x \circ \text{ad} y)
\]
Note that \( \kappa \) is symmetric because \( \text{tr}(ab) = \text{tr}(ba) \) for matrices \( a, b \). The values of the Killing form allow one to deduce a lot about the structure of \( L \), as seen in the following two propositions.

**Theorem 5.2** (Cartan’s First Criterion). A complex Lie algebra \( L \) is solvable if and only if \( \kappa(x, y) = 0 \) for all \( x \in L, y \in L' \).


**Theorem 5.3** (Cartan’s Second Criterion). A complex Lie algebra \( L \) is semisimple if and only if the Killing form on \( L \) is non-degenerate.

*Proof.* See pages 82-83 of [3]

We now have the required machinery to prove the equivalence of (1) and (4) asserted in Proposition 2.17, that a Lie algebra is semisimple if and only if it can be written as a direct sum of simple ideals. First we prove one more lemma.

**Lemma 5.4** (Lemma 9.10 of [3]). If \( I \) is a non-trivial proper ideal in a complex semisimple Lie algebra \( L \), then \( L = I \oplus I^\perp \). The ideal \( I \) is a semisimple Lie algebra in its own right.

*Proof.* (This proof quoted from [3], page 83.) As usual, let \( \kappa \) denote the Killing form on \( I \). The restriction of \( \kappa \) to \( I \cap I^\perp \) is identically 0, so by Cartan’s First Criterion, \( I \cap I^\perp = 0 \). It now follows by dimension counting that \( L = I \oplus I^\perp \).

We shall show that \( I \) is semisimple using Cartan’s Second Criterion. Suppose that \( I \) has a non-zero solvable ideal. By the “only if” direction of Cartan’s Second Criterion, the Killing form on \( I \) is degenerate. We have seen that the Killing form on \( I \) is given by restricting the Killing form on \( L \), so there exists \( a \in I \) such that \( \kappa(a, x) = 0 \) for all \( x \in I \). But as \( a \in I, \kappa(a, y) = 0 \) for all \( y \in I^\perp \) as well. Since \( L = I \oplus I^\perp \), this shows that \( \kappa \) is degenerate, a contradiction.

**Theorem 5.5** (Theorem 9.11 of [3]). Let \( L \) be a complex Lie algebra. Then \( L \) is semisimple if and only if there are simple ideals \( L_1, \ldots, L_r \) of \( L \) such that \( L = L_1 \oplus L_2 \oplus \ldots \oplus L_r \).

*Proof.* (This proof quoted for [3], pages 83-84.) We begin with the “only if” direction, working by induction on \( \dim L \). Let \( I \) be an ideal in \( L \) of the smallest possible non-zero dimension. If \( I = L \), we are done. Otherwise \( I \) is a proper simple ideal of \( L \). (It cannot be abelian as by hypothesis \( L \) has non non-zero abelian ideals.) By the preceding lemma, \( L = I \oplus I^\perp \), where, as an ideal of \( L \), \( I^\perp \) is a semisimple Lie algebra of smaller dimension than \( L \). So, by induction, \( I^\perp \) is a direct sum of simple ideals, \( I^\perp = L_2 \oplus \ldots \oplus L_r \).

Each \( L_i \) is also an ideal of \( L \), as \([I, L_i] \subseteq I \cap I^\perp = 0\), so putting \( L_1 = I \) we get the required decomposition.

Now for the “if” direction. Suppose that \( L = L_1 \oplus \ldots \oplus L_r \), where the \( L_i \) are simple ideals. Let \( I = \text{rad} L \); our aim is to show that \( I = 0 \). For each ideal \( L_i \), \([I, L_i] \subseteq I \cap L_i \) is a solvable ideal of \( L_i \). But the \( L_i \) are simple, so

\([I, L] \subseteq [I, L_1] \oplus \ldots \oplus [I, L_r] = 0\)
This shows that \( I \) is contained in \( Z(L) \). But by Exercise 2.6(ii) (see appendix)
\[
Z(L) = Z(L_1) \oplus \ldots \oplus Z(L_r).
\]
We know that \( Z(L_1) = \ldots = Z(L_r) = 0 \) as the \( L_i \) are simple ideals, so \( Z(L) = 0 \) and \( I = 0 \).

\[\square\]

### 5.2 Root space decomposition

In order to classify the semisimple complex Lie algebras, we will see that they can all be decomposed as a direct sum of a Cartan subalgebra with a bunch of root spaces. The big picture is that the structure of the root spaces determines the Lie algebra up to isomorphism.

**Definition 5.6.** Let \( L \) be a Lie algebra. A **Cartan subalgebra** is a maximal Lie subalgebra with two properties: \( H \) is abelian, and for \( h \in H \), \( \text{ad} h \) is diagonalizable.

On page 95, [3] shows that every semisimple complex Lie algebra has a Cartan subalgebra. However, it is not unique.

**Definition 5.7.** Let \( L \) be a semisimple complex Lie algebra with Cartan subalgebra \( H \). A **root space** corresponding to the root \( \alpha \) is the space
\[
L_\alpha = \{ x \in L : [h, x] = \alpha(h)x, \forall h \in H \}
\]
In general, \( \alpha \in H^* \) is only called a root when the corresponding root space \( L_\alpha \) is nonzero.

It does not seem obvious at all from the definition, but all of the root spaces \( L_\alpha \) are always one-dimensional ([3] Proposition 10.9).

The Killing form \( \kappa \) is a symmetric bilinear form on \( L \), but eventually we plan to view the roots \( \alpha \in H^* \) as living in an inner product space, so that we can think of them as root systems (to be defined shortly). We use \( \kappa \) to define this inner product, though the definition is rather involved.

For \( h \in H \), define \( \theta_h : H \to \mathbb{C} \) by \( \theta_h(k) = \kappa(h, k) \). The map \( H \to H^* \) given by \( h \to \theta_h \) is an isomorphism. (For more details than you probably wanted, see page 99 of [3]). Now we can define a bilinear form on \( H^* \).
\[
(\cdot, \cdot) : H^* \times H^* \to \mathbb{C} \quad (\theta_h, \theta_k) = \kappa(h, k)
\]

**Proposition 5.8** (Proposition 10.15 of [3]). Let \( \beta \) be a basis for \( H^* \) consisting of roots of \( L \). The above form is a real-valued inner product on the real span of \( \beta \).

As a useful convention, we define another binary operator on the same space, in terms of this inner product.
\[
\langle \alpha, \beta \rangle = \frac{2(\alpha, \beta)}{(\beta, \beta)}
\]

**Definition 5.9.** Let \( L \) be a semisimple complex Lie algebra with Cartan subalgebra \( H \). The **root space decomposition** of \( L \) is the direct sum expression
\[
L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha
\]
where \( \Phi \) is the set of roots, that is,
\[
\Phi = \{ \alpha \in H^* : \alpha \neq 0, L_\alpha \neq 0 \}
\]
5.3 Root systems

Before we can define a root system, we need to define the reflection \( s_\alpha \). For \( \alpha \neq 0 \) in an inner product space \( E \), \( s_\alpha \) is the reflection through the hyperplane perpendicular to \( \alpha \). The following formula is useful for computations.

\[
s_\alpha(\beta) = \beta - \frac{2(\alpha, \beta)}{\langle \beta, \beta \rangle} \beta = \beta - (\alpha, \beta) \beta
\]

**Definition 5.10.** A **root system** is a subset \( R \) of a real inner-product space \( E \) satisfying

1. \( R \) is finite, \( R \) spans \( E \), and \( 0 \not\in R \).
2. For \( \alpha \in R \), the only multiples of \( \alpha \) in \( R \) are \( \pm \alpha \).
3. For \( \alpha \in R \), the reflection \( s_\alpha \) permutes \( R \).
4. For \( \alpha, \beta \in R \), \( \langle \beta, \alpha \rangle \in \mathbb{Z} \).

For our purposes, the most salient fact is that the roots of a semisimple complex Lie algebra are a root system, given in the following proposition.

**Proposition 5.11.** Let \( L \) be a semisimple complex Lie algebra with Cartan subalgebra \( H \) and roots \( \Phi \). Let \( E \) be the real span of \( \Phi \). Then \( \Phi \) is a root system in \( E \).

**Proof.** See Example 11.2 on page 110 of [3].

We noted previously that a given Lie algebra may have various Cartan subalgebras, the root space decomposition is not unique. Since we plan to show that a root system of a Lie algebra determines the Lie algebra up to isomorphism, we need to establish that different root space decompositions give rise to the same root systems. In order to talk about the “sameness” of root systems, we need a notion of isomorphism.

**Definition 5.12.** Let \( R, R' \) be root systems in the real inner-product spaces \( E, E' \) respectively. A **root system isomorphism** is a vector space isomorphism \( \phi: E \to E' \) such that \( \phi(R) = R' \) and \( \langle \alpha, \beta \rangle = \langle \phi(\alpha), \phi(\beta) \rangle \) for \( \alpha, \beta \in R \). If there is an isomorphism between \( \Phi_1 \) and \( \Phi_2 \), we write \( \Phi_1 \cong \Phi_2 \).

This allows us to state the following proposition, which resolves the “sameness” question raised above.

**Theorem 5.13** (Theorem 12.6 of [3]). Let \( L \) be a complex semisimple Lie algebra. Let \( L \) be a semisimple complex Lie algebra with Cartan subalgebras \( H_1, H_2 \) and corresponding root systems \( \Phi_1, \Phi_2 \). Then \( \Phi_1 \cong \Phi_2 \).

**Proof.** See appendix C of [3].

As with modules, root systems are unwieldy in general, so we try to decompose them into more basic versions that have restrictive properties.

**Definition 5.14.** A root system \( R \) is **irreducible** if \( R \) cannot be expressed as a disjoint union of two nonempty subset \( R_1, R_2 \) such that \( \langle \alpha, \beta \rangle = 0 \) for all \( \alpha \in R_1, \beta \in R_2 \).
This property is important because of the following lemma.

**Lemma 5.15** (Lemma 11.8 of [3]). *Every root system may be written as a disjoint union of irreducible root systems.*

The following proposition makes an important link between irreducible root systems and simple complex Lie algebras.

**Proposition 5.16.** *Let $L$ be a complex semisimple Lie algebra with Cartan subalgebra $H$ and root system $\Phi$. $\Phi$ is irreducible if and only if $L$ is simple.*

*Proof.* The “if” direction is Proposition 12.4 of [3] (found on pages 128-129), and the “only if” direction is Proposition 14.2 (found on page 154).

Since every root system can be written as a disjoint union of irreducible root systems, and the irreducible root systems correspond to simple Lie algebras, and semisimple Lie algebras can be written as direct sums of simple Lie algebras, we see why perhaps semisimple Lie algebras are determined by their root systems. To make that determination rigorous, we need Dynkin diagrams.

### 5.4 Cartan Matrix and Dynkin Diagrams

We now consider an important substructure of a root system.

**Definition 5.17.** A *base* for a root system $R$ is a subset $B \subset R$ such that $B$ is a basis for $E$ and every $\beta \in R$ can be written in the form

$$\beta = \sum_{\alpha \in B} k_\alpha \alpha$$

where $k_\alpha \in \mathbb{Z}$ and all the nonzero $k_\alpha$ have the same sign.

This allows us to sort the elements of $R$ into two buckets: ones where the nonzero $k_\alpha$ are positive, and ones where the $k_\alpha$ are negative. These two subsets are called $R^+$ and $R^-$ respectively.

**Definition 5.18.** Let $B = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be an ordered base for the root system $R$. The *Cartan matrix* is the $n \times n$ matrix where the $ij$th entry is $\langle \alpha_i, \alpha_j \rangle$.

This definition clearly depends upon the choice of base and ordering. Surprisingly, the Cartan matrix of a root system does not depend on the choice of base (see Theorem 11.16 and appendix D of [3]).

**Definition 5.19.** Let $B$ be an ordered base of a root system $R$. The *Dynkin diagram* of $R$ is the graph with one vertex for each root $\alpha \in B$, and between each pair of vertices $\alpha, \beta$, there are $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle$ lines. Additionally, whenever $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle > 1$, there is an arrow pointing towards whichever of $\alpha, \beta$ is longer. Alternately, one can replace arrows and instead color longer vertices a different color.
Note that $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle$ is always an integer between 0 and 4 inclusive (see Lemma 11.4 of [3]), so it makes sense to talk about that as a number of lines. Note that a Dynkin diagram contains exactly the same information as a Cartan matrix.

**Proposition 5.20.** Let $\Phi$ be a root system. The $\Phi$ is irreducible if and only if its Dynkin diagram is connected.

**Proof.** We will prove the contrapositive of both directions. First suppose that a Dynkin diagram corresponding to $R$ is disconnected. Then there are two disjoint sets of vertices $R_1, R_2$ which have no lines between them. Thus for $\alpha \in R_1, \beta \in R_2$,
\[
0 = \langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = \frac{2(\alpha, \beta)}{(\beta, \beta)} \frac{2(\alpha, \beta)}{(\alpha, \alpha)} = \frac{4(\alpha, \beta)^2}{(\alpha, \alpha)(\beta, \beta)}
\]
hence $(\alpha, \beta) = 0$. Thus $R$ is reducible.

Now suppose that $R$ is reducible. Then there exist disjoint subsets $R_1, R_2$ with $(\alpha, \beta) = 0$ for $\alpha \in R_1, \beta \in R_2$. Then $R_1, R_2$ correspond to disjoint subgraphs of the Dynkin diagram. Hence the Dynkin diagram is disconnected. \hfill \Box

Next, we connect Dynkin diagrams to root systems.

**Proposition 5.21** (Proposition 11.21 of [3]). Let $R, R'$ be root systems in the real vector spaces $E, E'$ respectively. If the Dynkin diagrams of $R$ and $R'$ are the same, then the root systems are isomorphic.

**Proof.** See pages 122-123 of [3]. \hfill \Box

Finally, we get the theorem that relates the Cartan matrix to the isomorphism class of a semisimple complex Lie algebra, though the statement sort of goes the opposite direction. Don’t worry, though, the useful part (for classifying) comes as a neatly stated corollary. First, we need to define generators of a Lie algebra.

**Definition 5.22.** Let $L$ be a Lie algebra. A set $\{x_1, \ldots x_n\}$ generates $L$ if every $x \in L$ can be written as a linear combination of $x_1, \ldots x_n$ and brackets of $x_1, \ldots x_n$.

**Theorem 5.23** (Serre’s Theorem). Let $C$ be the Cartan matrix of a root system. Let $L$ be the complex Lie algebra which is generated by elements $e_i, f_i, h_i$ for $1 \leq i \leq l$ satisfying
\[
[h_i, h_j] = 0 \quad [h_i, e_j] = c_{ij}e_j \quad [h_i, f_j] = -c_{ij}f_j \quad [e_i, f_i] = h_i
\]
for all $i, j$, and
\[
[e_i, f_j] = 0 \quad (\text{ad } e_i)^{1-c_{ij}}(e_j) = 0 \quad (\text{ad } f_i)^{1-c_{ij}}(f_j) = 0
\]
for $i \neq j$. Then $L$ is finite-dimensional and semisimple with Cartan subalgebra $H$ spanned by $\{h_1, \ldots h_l\}$, and its root system has Cartan matrix $C$.

**Theorem 5.24** (Corollary of Serre’s Theorem). Let $L_1, L_2$ be complex semisimple Lie algebras with Cartan matrix $C$. Then $L_1$ is isomorphic to $L_2$.

So finally we have demonstrated what has been hinted at, that the Cartan matrix of a complex semisimple Lie algebra allows one to categorize the Lie algebra into an isomorphism class. Since we’ve also related the Cartan matrix to a root system and a Dynkin diagram, all that remains is to classify the Dynkin diagrams.
5.5 Final Classification

Theorem 5.25 (Theorem 13.1 of [3]). Given an irreducible root system R, the unlabelled Dynkin diagram associated to R is either a member of the four families

\[
A_n \quad \bullet \cdots \bullet \quad B_n \quad \bullet \cdots \bullet \quad C_n \quad \bullet \cdots \bullet \quad D_n
\]

where each of the diagrams above has n vertices, or one of the five exceptional diagrams

\[
G_2 \quad \bullet \cdots \bullet \\
F_4 \\
E_6 \\
E_7 \\
E_8
\]

Proof. See chapter 13 of [3].

This gives rise to the classification theorem for simple complex Lie algebras.

Theorem 5.26. Every finite-dimensional simple complex Lie algebra is isomorphic to one of

\[
\text{sl}(n, \mathbb{C}) \quad \text{so}(n, \mathbb{C}) \quad \text{sp}(2n, \mathbb{C})
\]

for some n, or to one of the five exceptional Lie algebras, e_6, e_7, e_8, f_4, g_2.

Proof. See chapter 12 of [3].

As implied by the labelling scheme, the simple Lie algebras correspond to the capitalized Dynkin diagram names. For the others, this table gives the correspondence.
Keep in mind that the above classification of Dynkin diagrams is a classification of connected Dynkin diagrams, but of course any Dynkin diagram is a union of connected components. Each connected component corresponds to an irreducible root system, which corresponds to a simple complex Lie algebra. A disconnected Dynkin diagram $D$ corresponds to a semisimple complex Lie algebra, where each direct summand is a simple Lie algebra corresponding to the connected components of $D$.

<table>
<thead>
<tr>
<th>Graph</th>
<th>Lie algebra</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_n$</td>
<td>$\text{sl}(n+1, \mathbb{C})$</td>
</tr>
<tr>
<td>$B_n$</td>
<td>$\text{so}(2n+1, \mathbb{C})$</td>
</tr>
<tr>
<td>$D_n$</td>
<td>$\text{so}(2n, \mathbb{C})$</td>
</tr>
<tr>
<td>$C_n$</td>
<td>$\text{sp}(2n, \mathbb{C})$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Lie algebra</th>
<th>Root system</th>
<th>Dynkin diagram</th>
</tr>
</thead>
<tbody>
<tr>
<td>direct sum</td>
<td>disjoint union</td>
<td>union of disjoint, connected subgraphs</td>
</tr>
<tr>
<td>simple</td>
<td>irreducible</td>
<td>connected</td>
</tr>
</tbody>
</table>
Appendix: Solutions to Exercises

6 Chapter 1 Exercises

Proposition 6.1 (Exercise 1.1i). Let $L$ be a Lie algebra, and let $v \in L$. Then $[v,0] = [0,v] = 0$.

Proof. By bilinearity of the bracket,

\[ [v,0] = [v,v-v] = [v,v] - [v,v] = 0 \]
\[ [0,v] = [v-v,v] = [v,v] - [v,v] = 0 \]

Note that in the following proposition the symbol “0” is used for both the additive identity in the field $F$ and the additive identity vector in $L$. When added to a vector, “0” refers to a vector; when multiplied by a bracket or vector “0” refers to the identity for $F$.

Lemma 6.2 (Lemma for Exercise 1.1ii). Let $L$ be a Lie algebra over $F$. Let $x,y \in L$ and $a \in F$. Then $a[x,y] = [ax,y] = [x,ay]$.

Proof.

\[ a[x,y] = a[x,y] + 0[0,y] = [ax + 0,y] = [ax,y] \]
\[ a[x,y] = 0[x,0] + a[x,y] = [x,ay + 0] = [ax,y] \]

Proposition 6.3 (Exercise 1.1ii). Let $L$ be a Lie algebra with $x,y \in L$ such that $[x,y] \neq 0$. Then $x$ and $y$ are linearly independent over $F$.

Proof. Let $x,y \in L$ such that $[x,y] \neq 0$. Suppose $x,y$ are linearly dependent over $F$. Then there exist $a,b \in F$ with $a \neq 0, b \neq 0$ such that $ax + by = 0$, or equivalently $ax = -by$.

Let $v = [x,y]$. Then $av = a[x,y] = [ax,y]$ and $-bv = -b[x,y] = [ax,-by]$. Since $a \neq 0, b \neq 0$ and $v \neq 0$, thus $-abv \neq 0$ so $[ax,-by] \neq 0$. However, we showed that $ax = -by$, so by the property of the Lie bracket, $[ax,-by] = 0$. Thus we reject our hypothesis that $x,y$ are linearly dependent and conclude that they are linearly independent.

Lemma 6.4 (Lemma for Exercise 1.2). For any vectors $u,v,w \in \mathbb{R}^3$, \[ u \times (v \times w) = (u \cdot w)v - (u \cdot v)w. \]
Proposition 6.5 (Exercise 1.2). The Jacobi identity holds for the cross product of vectors in \( \mathbb{R}^3 \).

Proof. Using the above proposition,

\[
[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = (x \cdot z)y - (x \cdot y)z \\
+ (y \cdot x)z - (y \cdot z)x \\
+ (z \cdot y)x - (z \cdot x)y \\
= (x \cdot z)y - (z \cdot x)y \\
+ (y \cdot x)z - (x \cdot y)z \\
+ (z \cdot y)x - (y \cdot z)x \\
= 0 + 0 + 0 \\
= 0
\]

\[\square\]

Proposition 6.6 (Exercise 1.3). Let \( V \) be a finite-dimensional vector space over \( F \) and let \( \text{gl}(V) \) be the set of all linear maps from \( V \) to \( V \). We define a Lie bracket on this space by

\[
[x, y] := x \circ y - y \circ x
\]

where \( \circ \) denotes map composition. We claim that the Jacobi identity holds for this bracket operator.
Proof.

\[
[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = (x \circ y \circ z - z \circ y \circ x) \\
+ (y \circ z \circ x - x \circ z \circ y) \\
+ (z \circ x \circ y - y \circ x \circ z) \\
= (x \circ y \circ z - x \circ z \circ y) \\
+ (y \circ z \circ x - y \circ x \circ z) \\
+ (z \circ x \circ y - z \circ y \circ x) \\
= x \circ [y, z] + y \circ [z, x] + z \circ [x, y]
\]

\[
[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = (x \circ y \circ z - y \circ x \circ z) \\
+ (z \circ x \circ y - x \circ z \circ y) \\
+ (y \circ z \circ x - z \circ y \circ x) \\
= [x, y] \circ z + [z, x] \circ y + [y, z] \circ x
\]

Thus we reach

\[
x \circ [y, z] + y \circ [z, x] + z \circ [x, y] = [x, y] \circ z + [z, x] \circ y + [y, z] \circ x
\]

Now we can subtract to have one side equal zero,

\[
0 = x \circ [y, z] - [y, z] \circ x + y \circ [z, x] - [z, x] \circ y + z \circ [x, y] - [x, y] \circ z \\
= [x, [y, z]] + [y, [z, x]] + [z, [x, y]]
\]

which is precisely the Jacobi identity.

Note that for \(n \times n\) matrices \(A, B\) with entries \(A_{ij}, B_{ij}\), the entries of the matrix product \(AB\) are

\[
(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj} \quad (6.1)
\]

**Definition 6.7.** The **Kronecker delta function** is the function

\[
\delta_{ij} = \begin{cases} 
1 & i = j \\
0 & i \neq j 
\end{cases} \quad (6.2)
\]

Sometimes the alternate notation \(\delta_{i}^{j}\) is used (equivalent to \(\delta_{ij}\)).

**Definition 6.8.** In \(\text{gl}(n, F)\), \(e_{ij}\) is the matrix with a 1 in position \(ij\) and zero everywhere else. Using the delta function, we can say that the \(ab\)-th entry of \(e_{ij}\) is \(\delta_{ai}\delta_{bj}\).

**Proposition 6.9 (Lemma for Page 3).** In \(\text{gl}(n, F)\), the product of the matrices \(e_{ij}\) and \(e_{kl}\) is given by the formula:

\[
(e_{ij}e_{kl})_{ab} = \delta_{ai}\delta_{bl}\delta_{jk} \quad (6.3)
\]
Proof. By the general rule for square matrix products,

\[(e_{ij} e_{kl})_{ab} = \sum_{x=1}^{n} (e_{ij})_{ax} (e_{kl})_{xb}\]

Note that the term to be summed over is equal to 1 when \(a = i, j = x, k = x, \) and \(l = b.\) Since \(x\) ranges over \(1, 2, \ldots n\) and \(1 \leq j, k \leq n,\) this summand is 1 when \(a = i, b = l,\) and \(j = k = x.\) In each summand, \(x\) takes a different value, so no two terms can both have \(x = j = k,\) but if \(j = k,\) then in one term we will have \(x = j = k.\) Thus

\[(e_{ij} e_{kl})_{ab} = \sum_{x=1}^{n} (e_{ij})_{ax} (e_{kl})_{xb} = \begin{cases} 1 & a = i, b = l, \text{and } j = k \\ 0 & \text{otherwise} \end{cases} = \delta_{ai} \delta_{bl} \delta_{jk}\]

Proposition 6.10 (Page 3). In \(\text{gl}(n, F),\) the bracket of two basis matrices is given by

\[[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{kj}\]  \hspace{1cm} (6.4)

Proof.

\[[e_{ij}, e_{kl}]_{ab} = (e_{ij} e_{kl} - e_{kl} e_{ij})_{ab} = \delta_{ai} \delta_{bl} \delta_{jk} - \delta_{ak} \delta_{bj} \delta_{li}\]

\[(\delta_{jk} e_{il} - \delta_{il} e_{kj})_{ab} = \delta_{jk} (e_{il})_{ab} - \delta_{il} (e_{kj})_{ab} = \delta_{jk} \delta_{ai} \delta_{lb} - \delta_{il} \delta_{ka} \delta_{bj}\]

Since \(\delta_{ij} = \delta_{ji}\) for any \(i, j,\) by commuting these expressions we see that they are equal. Since the two matrices have equal entries for all \(1 \leq a, b \leq n,\) they are equal matrices.

Proposition 6.11 (Exercise 1.4). Let \(b(n, F)\) be the subset of upper triangular matrices in \(\text{gl}(n, F).\) Then \(b(n, F)\) is a Lie algebra with the same bracket as in \(\text{gl}(n, F).\)

Proof. We need to show that \(b(n, F)\) is closed under the bracket. Let \(x, y \in b(n, F).\) Since the product or sum of two upper triangular matrices is itself upper triangular, \([x, y] = xy - yx\) is upper triangular, so it is also an element of \(b(n, F).\)

Proposition 6.12 (Exercise 1.4). Let \(n(n, F)\) be the strictly upper triangular matrices in \(\text{gl}(n, F).\) Then \(n(n, F)\) is a Lie algebra with the same bracket.

Proof. Same argument as previous proposition.
Proposition 6.13 (Exercise 1.5). Let $L = \text{sl}(2, F)$. If char($F$) $\neq 2$ then
\[ Z(\text{sl}(2, F)) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \]
and if char($F$) = 2 then
\[ Z(\text{sl}(2, F)) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \]

Proof. Let $L = \text{sl}(2, F)$. The following matrices form a basis for $L$.
\[ e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]
And the bracket products of these are
\[ [e, f] = g, \quad [f, g] = 2f, \quad [g, e] = 2e \]
by straightforward computation. Let $y \in L$ and let $x \in Z(L)$. Then $xy = yx$. We write $x$ and $y$ as linear combinations of the basis matrices $e, f, g$
\[ x = x_{11}g + x_{12}e + x_{21}f = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & -x_{11} \end{pmatrix} \]
\[ y = y_{11}g + y_{12}e + y_{21}f = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & -y_{11} \end{pmatrix} \]
Then we use the linearity of the bracket to compute
\[ [x, y] = \begin{vmatrix} x_{12} & x_{21} \\ y_{12} & y_{21} \end{vmatrix} g + 2 \begin{vmatrix} x_{11} & x_{12} \\ y_{1} & y_{12} \end{vmatrix} e - 2 \begin{vmatrix} x_{11} & x_{21} \\ y_{11} & y_{21} \end{vmatrix} f \]
Since $g, e, f$ are linearly independent over $F$, from this we know that
\[ \begin{vmatrix} x_{12} & x_{21} \\ y_{12} & y_{21} \end{vmatrix} = x_{12}y_{21} - x_{21}y_{12} = 0 \]
for all $y_1, y_2 \in F$. This is true only if $x_{12} = x_{21} = 0$, regardless of the characteristic of $F$. Similarly, the coefficients of $e$ and $f$ must be zero, which is true only when $2 = 0$ or when $x_{11} = 0$. If char($F$) = 2, then we see that $g$ is actually the identity matrix, which clearly commutes with everything, but in this case there are no other possible values for $x_{11}$, so the center of $L$ is just the zero matrix and the identity matrix.

However, if char($F$) $\neq 2$, then we must have $x_{11} = 0$, so the only matrix in the center of $L$ is the zero matrix. \[ \square \]

Proposition 6.14 (Exercise 1.6). Let $L_1, L_2$ be Lie algebras and let $\phi : L_1 \to L_2$ be a homomorphism. Then $\ker \phi$ is an ideal of $L_1$.

Proof. We need to show that for $x \in L_1, y \in \ker \phi = \{ v \in L_1 : \phi(v) = 0 \}$, we have $[x, y] \in \ker \phi$. Let $x \in L_1, y \in \ker \phi$. Then $\phi([x, y]) = [\phi(x), \phi(y)] = [\phi(x), 0] = 0$. \[ \square \]
**Proposition 6.15** (Exercise 1.6). Let \( \phi : L_1 \to L_2 \) be a Lie algebra homomorphism. Then \( \text{im} \phi \) is a subalgebra of \( L_2 \).

**Proof.** We need to show that for \( x, y \in \text{im} \phi \), we have \([x, y] \in \text{im} \phi \). Let \( x, y \in \text{im} \phi \). Then there exist \( x', y' \in L_1 \) such that \( \phi(x') = x \), \( \phi(y') = y \). Then \([x', y'] \in L_1 \), so \( \phi([x', y']) \in \text{im} \phi \). Since \( \phi([x', y']) = [\phi(x'), \phi(y')] = [x, y] \), we see that \([x, y] \in \text{im} \phi \). \( \square \)

**Proposition 6.16** (Exercise 1.7). Let \( L \) be a Lie algebra such that for all \( a, b \in L \), we get \([a, b] \in Z(L)\). Then the Lie bracket is associative.

**Proof.** Let \( x, y, z \in L \). Then \([x, y] \in Z(L), \) so \([x, y], z] = 0\). We also know that \([y, z] \in Z(L),\) so we get that \([x, [y, z]] = [-[y, z], x] = 0\) so \([x, [y, z]] = [[x, y], z] = 0\). \( \square \)

**Proposition 6.17** (Exercise 1.7). Let \( L \) be a Lie algebra such that the bracket is associative. Then for \( x, y \in L \), \([x, y] \in Z(L)\).

**Proof.** Let \( x, y, z \in L \). We need to show that \([x, y], z] = 0\). Using anti-communitivity, linearity, and associativity we get

\[
\begin{align*}
[z, [x, y]] &= -[[x, y], z] = -[[y, x], z] = [[y, x], z] = [y, [x, z]] = [y, -[z, x]] = -[y, [z, x]]
\end{align*}
\]

Then using the Jacobi identity and substituting \(-[y, [z, x]]\) for \([z, [x, y]]\)

\[
\begin{align*}
[x, [y, z]] + [y, [z, x]] + [z, [x, y]] &= 0 \\
[x, [y, z]] + [y, [z, x]] - [y, [z, x]] &= 0 \\
[x, [y, z]] &= 0 \\
[[x, y], z] &= 0
\end{align*}
\]

\( \square \)

**Proposition 6.18** (Exercise 1.8i). Let \( D, E \) be derivations of an algebra \( A \). Then \([D, E] = D \circ E - E \circ D\) is a derivation (of \( A \)).

**Proof.** We need to show that \([D, E](xy) = x[D, E](y) + [D, E](x)y\). First we compute \( D \circ E(xy) \) and \( E \circ D(xy) \).

\[
\begin{align*}
D \circ E(xy) &= D(xE(y) + E(x)y) \\
&= D(xE(y)) + D(E(x)y) \\
&= xD \circ E(y) + D(x)E(y) + D \circ E(xy) + E(x)D(y)
\end{align*}
\]

\[
\begin{align*}
E \circ D(xy) &= xE \circ D(y) + E(x)D(y) + D(x)E(y) + E \circ D(x)y
\end{align*}
\]

Now that we’ve done that we can easily compute \([D, E](xy)\).

\[
\begin{align*}
[D, E](xy) &= (D \circ E - E \circ D)(xy) \\
&= D \circ E(xy) - E \circ D(xy) \\
&= xD \circ E(y) + D \circ E(x)y - xE \circ D(y) - E \circ D(x)y \\
&= x(D \circ E(y) - E \circ D(y)) + (D \circ E(x) - E \circ D(x))y \\
&= x[D, E](y) - [D, E](x)y
\end{align*}
\]

\( \square \)
Proposition 6.19 (Exercise 1.9). Let \( L_1, L_2 \) be Lie algebras such that there exist bases \( \beta_1 \) for \( L_1 \) and \( \beta_2 \) for \( L_2 \) such that the structure constants of \( L_1 \) with respect to \( \beta_1 \) are equal to the structure constants of \( L_2 \) with respect to \( \beta_2 \). Then \( L_1 \cong L_2 \).

**Proof.** Let \( \beta_1 = \{x_1, x_2 \ldots x_n\} \) and \( \beta_2 = \{y_1, y_2 \ldots y_n\} \) be bases for \( L_1, L_2 \) as described. Then since the structure constants are the same,

\[
[x_i, x_j] = \sum_{k=1}^{n} a_{ij}^k x_k
\]

\[
[y_i, y_j] = \sum_{k=1}^{n} a_{ij}^k y_k
\]

We define a linear map \( \phi : L_1 \rightarrow L_2 \) by \( \phi(x_i) = y_i \). Since \( \phi \) maps \( \beta_1 \) to \( \beta_2 \), \( \phi \) is a bijection. Also,

\[
\phi([x_i, x_j]) = \phi \left( \sum_{k=1}^{n} a_{ij}^k x_k \right) = \sum_{k=1}^{n} a_{ij}^k \phi(x_k) = \sum_{k=1}^{n} a_{ij}^k y_k = [y_i, y_j] = [\phi(x_i), \phi(x_j)]
\]

so \( \phi \) is an isomorphism. \( \square \)

Proposition 6.20 (Exercise 1.9). Let \( L_1, L_2 \) be isomorphic Lie algebras. Then there exist bases \( \beta_1 \) for \( L_1 \) and \( \beta_2 \) for \( L_2 \) such that the structure constants for \( L_1 \) with respect to \( \beta_1 \) are equal to the structure constants for \( L_2 \) with respect to \( \beta_2 \).

**Proof.** Let \( \beta_1 = \{x_1, x_2 \ldots x_n\} \) be any basis for \( L_1 \) and let \( \phi : L_1 \rightarrow L_2 \) be an isomorphism. Let \( \beta_2 = \{\phi(x_1), \phi(x_2), \ldots \phi(x_n)\} \). Since \( \phi \) is a linear bijection, \( \beta_2 \) is a basis for \( L_2 \). Let \( a_{ij}^k \) be the structure constants of \( L_1 \) with respect to \( \beta_1 \). Then

\[
[\phi(x_i), \phi(x_j)] = \phi([x_i, x_j]) = \phi \left( \sum_{k=1}^{n} a_{ij}^k x_k \right) = \sum_{k=1}^{n} a_{ij}^k \phi(x_k)
\]

so by definition \( a_{ij}^k \) are the structure constants of \( L_2 \) with respect to \( \beta_2 \). \( \square \)

Proposition 6.21 (Exercise 1.10). Let \( L \) be a Lie algebra with basis \( \{x_1, x_2 \ldots x_n\} \) and structure constants with respect to this basis \( a_{ij}^k \). Then

\[
a_{jk}^m a_{im}^t + a_{ki}^m a_{jm}^t + a_{ij}^m a_{km}^t = 0
\]

for \( 1 \leq i, j, k, m, t \leq n \).
Proof. From the Jacobi identity

$$0 = [x_1, [x_j, x_k]] + [x_j, [x_k, x_i]] + [x_i, [x_j, x_k]]$$

$$= [x_i, \sum_{m=1}^{n} a_{jm}^m x_m] + [x_j, \sum_{m=1}^{n} a_{ki}^m x_m] + [x_k, \sum_{m=1}^{n} a_{ij}^m x_m]$$

$$= \sum_{m=1}^{n} a_{jk}^m [x_i, x_m] + \sum_{m=1}^{n} a_{ki}^m [x_j, x_m] + \sum_{m=1}^{n} a_{ij}^m [x_k, x_m]$$

$$= \sum_{m=1}^{n} a_{jk}^m \left( \sum_{t=1}^{n} a_{im}^t x_t \right) + \sum_{m=1}^{n} a_{ki}^m \left( \sum_{t=1}^{n} a_{jm}^t x_t \right) + \sum_{m=1}^{n} a_{ij}^m \left( \sum_{t=1}^{n} a_{km}^t x_t \right)$$

$$= \sum_{m=1}^{n} \sum_{t=1}^{n} \ (a_{jk}^m a_{im}^t + a_{ki}^m a_{jm}^t + a_{ij}^m a_{km}^t) x_t$$

since $x_t$ for $t = 1, 2, \ldots n$ are linearly independent, this implies that the coefficient $a_{jk}^m a_{im}^t + a_{ki}^m a_{jm}^t + a_{ij}^m a_{km}^t$ is equal to zero for all values of $i, j, k, m, t$. \hfill \Box

Lemma 6.22 (Lemma for Exercise 1.11). Let $V, W$ be $n$-dimensional vector spaces over a field $F$. Then $V \cong W$. (vector space isomorphism)

Proof. Let $\{v_i\}_{i=1}^{n}, \{w_i\}_{i=1}^{n}$ be bases for $V$ and $W$ respectively. Let $\phi : V \to W$ be a linear map defined on $v_i$ by $\phi(v_i) = (w_i)$ for $i = 1, 2, \ldots n$. Then for a general vector in $V$ given by $a_1 v_1 + a_2 v_2 + \ldots + a_n v_n$, we compute

$$\phi(a_1 v_1 + a_2 v_2 + \ldots + a_n v_n) = a_1 \phi(v_1) + a_2 \phi(v_2) + \ldots + a_n \phi(v_n)$$

$$= a_1 w_1 + a_2 w_2 + \ldots + a_n w_n$$

Since every element of $W$ can be written uniquely as a linear combination of $w_1, \ldots w_n$, from this we get that $\phi$ is one-to-one and onto. Thus $\phi$ is an isomorphism. \hfill \Box

Proposition 6.23 (Exercise 1.11). Let $L_1, L_2$ be $n$-dimensional abelian Lie algebras over $F$. Then $L_1 \cong L_2$. (Lie algebra isomorphism)

Proof. As shown above, $L_1$ and $L_2$ are isomorphic as vector space via the map $\phi$. We can see that $\phi$ is also a Lie algebra isomorphism for abelian Lie algebras since

$$\phi([x, y]) = \phi(0) = 0 = [\phi(x), \phi(y)]$$

\hfill \Box

Proposition 6.24 (Exercise 1.8ii). Let $A = C^\infty \mathbb{R}$ be the vector space of infinitely differentiable functions $\mathbb{R} \to \mathbb{R}$. Let $D : A \to A$ be the usual derivative $D(f) = f'$. Then $D \circ D$ is not a derivation for the algebra $A$.
Proof.

\[ D \circ D(fg) = D(f'g + fg') = D(f'g) + D(fg') = f''g + f'g' + fg'' + fg'g = D \circ D(f)g + fD \circ D(g) + 2f'g' \]

If \( D \circ D \) were a derivation, it would satisfy the above equation only when \( 2f'g' = 0 \) for all \( f, g \in A \). But this is not true, since \( f(x) = x, g(x) = x \) gives \( 2f'g' = 2 \).

\[ \frac{\partial}{\partial t} \exp(-tH) \exp(-tH) \exp(tH) \exp(tH) = \exp(-2tH) \]

Lemma 6.25 (Lemma for Exercise 1.11). Let \( V, W \) be finite-dimensional isomorphic \( F \)-vector spaces. Then \( \dim V = \dim W \).

Proof. Let \( \{ v_i \}_{i=1}^n \) be a basis for \( V \), and let \( \phi : V \to W \) be an isomorphism. We claim that \( \{ \phi(v_i) \}_{i=1}^n \) is a basis for \( W \). To do this, we just need to show that \( \{ \phi(v_i) \}_{i=1}^n \) is linearly independent.

Let \( a_1, a_2, \ldots, a_n \in F \) such that

\[ a_1\phi(v_1) + a_2\phi(v_2) + \ldots + a_n\phi(v_n) = 0 \]

Then by linearity of \( \phi \),

\[ \phi(a_1v_1 + a_2v_2 + \ldots + a_nv_n) = 0 \]

Since \( \phi \) is one-to-one, \( \ker \phi = \{0\} \), so the above equation implies that

\[ a_1v_1 + a_2v_2 + \ldots + a_nv_n = 0 \]

Since \( \{ v_i \} \) is a basis, it is linearly independent, so the above implies that \( a_i = 0 \) for \( i = 1, 2, \ldots, n \). Thus \( \{ \phi(v_i) \} \) is linearly independent and thus a basis of size \( n \) for \( W \), so \( \dim W = n = \dim V \).

Proposition 6.26 (Exercise 1.11). Let \( L_1, L_2 \) be finite-dimensional, isomorphic abelian Lie algebras. Then \( \dim L_1 = \dim L_2 \).

Proof. By the above lemma, \( L_1, L_2 \) have equal dimension as vector spaces.

Solution 0.1 (Exercise 1.12). The structure constants of \( sl(2, F) \) with respect to the basis

\[ x_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad x_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad x_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

are

\[ a_{12}^1 = 0 \quad a_{13}^1 = -2 \quad a_{23}^1 = 0 \]
\[ a_{12}^2 = 0 \quad a_{13}^2 = 0 \quad a_{23}^2 = 2 \]
\[ a_{12}^3 = 0 \quad a_{13}^3 = 0 \quad a_{23}^3 = 0 \]

Proposition 6.27 (Exercise 1.13). \( sl(2, \mathbb{C}) \) has no non-trivial proper ideals.

28
Proof. We will show that any non-zero ideal of \( \text{sl}(2, \mathbb{C}) \) is the whole space. Suppose \( I \subseteq \text{sl}(2, \mathbb{C}) \) with \( I \neq \{0\} \). Then there exists \( x \in I \) with \( x \neq 0 \). Let \( x = ae + bf + ch \), where \( e, f, h \) are the matrices \( x_1, x_2, x_3 \) from Exercise 1.12. Since \( x \neq 0 \), at least one of \( a, b, c \neq 0 \).

Since \( I \) is an ideal, \([h, x] \in I \) and we compute
\[
[h, x] = a[h, e] + b[h, f] = 2ae - 2bf
\]
Furthermore, since \( I \) is an ideal, \([e, [h, x]] \in I \) and \([f, [h, x]] \in I \), and we compute
\[
[e, [h, x]] = -2b[e, f] = -2bh
\]
\[
[f, [h, x]] = -2ah
\]
If \( a \neq 0 \) or \( b \neq 0 \), then \( h \in I \). If \( h \in I \), then \( e, f \in I \) because \([f, h] = 2f \) and \([h, e] = 2e \).

Suppose \( a = 0 \) and \( b = 0 \). Then \( x = ch \) so \( h \in I \), so then \( e, f \in I \). Thus for any values of \( a, b, c, e, f, h \in I \), so \( I \) contains a basis for \( \text{sl}(2, \mathbb{C}) \), so \( I = \text{sl}(2, \mathbb{C}) \).

Proposition 6.28 (Exercise 1.14i). Let \( L \) be the 3-dimensional Lie algebra over \( \mathbb{C} \) with basis \( \{x, y, z\} \) where the bracket is defined by
\[
[x, y] = z, \quad [y, z] = x, \quad [z, x] = y
\]
Then \( L \) is isomorphic to the Lie subalgebra of \( \text{gl}(3, \mathbb{C}) \) consisting of antisymmetric matrices.

Proof. Let \( A \) be the subalgebra of \( \text{gl}(3, \mathbb{C}) \) of antisymmetric matrices. Take the basis \( a, b, c \) for \( A \) where
\[
a = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix} \quad b = \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \quad c = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & 0
\end{pmatrix}
\]
One can crunch numbers and compute that
\[
[a, b] = c, \quad [b, c] = a, \quad [c, a] = b
\]
We define a map \( \phi : L \to A \) by \( \phi(x) = a, \phi(y) = b, \phi(z) = c \). Since \( a, b, c \) and \( x, y, z \) are bases, \( \phi \) is a bijection, and by the above in conjunction with bilinearity, \( \phi \) preserves bracket products. Thus \( \phi \) is an isomorphism.

Proposition 6.29 (Exercise 1.14ii). Let \( L \) be the 3-dimensional Lie algebra over \( \mathbb{C} \) with basis \( \{x, y, z\} \) where the bracket is defined by
\[
[x, y] = z, \quad [y, z] = x, \quad [z, x] = y
\]
Then \( L \cong \text{sl}(2, \mathbb{C}) \). (Lie algebra isomorphism)

Proof. Let \( (\hat{x}, \hat{y}, \hat{z}) \) be the matrices
\[
\hat{x} = \frac{1}{2} \begin{pmatrix}
  i & 0 \\
  0 & -i
\end{pmatrix} \quad \hat{y} = \frac{1}{2} \begin{pmatrix}
  0 & 1 \\
  -1 & 0
\end{pmatrix} \quad \hat{z} = \frac{1}{2} \begin{pmatrix}
  0 & i \\
  i & 0
\end{pmatrix}
\]

We claim that these matrices are linearly independent over \( \mathbb{C} \). Let \( u, v, w \in \mathbb{C} \) such that 
\[
 u \hat{x} + v \hat{y} + w \hat{z} = 0.
\]
Then clearly \( u = 0 \), since the upper left entry of \( \hat{y} \) and \( \hat{z} \) are zero. From the other matrix positions we get the equations
\[
\frac{v}{2} + \frac{w}{2} = 0 \\
-\frac{v}{2} + \frac{w}{2} = 0
\]
Adding these equations gives \( w = 0 \), and then plugging in \( w = 0 \) into one of them gives \( v = 0 \). Thus, \( (\hat{x}, \hat{y}, \hat{z}) \) are linearly independent over \( \mathbb{C} \), and since \( \text{sl}(2, \mathbb{C}) \) is a three dimensional vector space, they must also span it, and thus be a basis for \( \text{sl}(2, \mathbb{C}) \).

Straightforward computation of the bracket products in \( \text{sl}(2, \mathbb{C}) \) of these matrices gives
\[
[\hat{x}, \hat{y}] = \frac{1}{4} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \frac{1}{4} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix} = \hat{z}
\]
\[
[\hat{y}, \hat{z}] = \frac{1}{4} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \frac{1}{4} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} = \hat{x}
\]
\[
[\hat{z}, \hat{x}] = \frac{1}{4} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \frac{1}{4} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = \hat{y}
\]
Finally, we define a map \( \phi : \text{sl}(2, \mathbb{C}) \to L \) by \( \phi(\hat{x}) = x, \phi(\hat{y}) = y, \phi(\hat{z}) = z \). Since both \( (x, y, z) \) and \( (\hat{x}, \hat{y}, \hat{z}) \) are bases, \( \phi \) is a bijection, and by our computations we’ve seen that \( \phi \) preserves the bracket operator. Thus, \( \phi \) is an isomorphism of Lie algebras.

**Proposition 6.30** (Exercise 1.15i). Let \( S \in \text{gl}(n, F) \). Define
\[
\text{gl}_S(n, F) := \{ x \in \text{gl}(n, F) : x^T S = -S x \}
\]
where \( x^T \) denotes the transpose of \( x \). Then \( \text{gl}_S(n, F) \) is a subalgebra of \( \text{gl}(n, F) \).

**Proof.** We need to show that \( \text{gl}_S(n, F) \) is closed under vector addition, scalar multiplication, and the bracket product. First we show closure under addition. Let \( x, y \in \text{gl}_S(n, F) \). Then
\[
(x + y)^T S = (x^T + y^T) S = x^T S + y^T S = -S x - S y = -S(x + y)
\]
so \( x + y \in \text{gl}_S(n, F) \). Now let \( x \in \text{gl}_S(n, F) \) and \( a \in F \). Then
\[
(ax)^T S = a(x^T) S = a(x^T S) = a(-S x) = -S(ax)
\]
so \( ax \in \text{gl}_S(n, F) \). Finally, for \( x, y \in \text{gl}_S(n, F) \),
\[
[x, y]^T S = (xy - yx)^T S = (xy)^T S - (yx)^T S = y^T x^T S - x^T y^T S
\]
\[
= -y^T S x + x^T S y = Sy x - S x y = S(y x - x y) = S[y, x]
\]
Thus \( [x, y] \in \text{gl}_S(n, F) \). \( \square \)
Proposition 6.31 (Exercise 1.15ii). Let $S$ be
\[
S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]
Then
\[
\text{gl}_S(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{R} \right\}
\]
Proof. Let $x = \text{gl}_S(2, \mathbb{R})$ where
\[
x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]
Then
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} -c & d \\ 0 & 0 \end{pmatrix}
\]
Thus $a = d, c = 0, d = 0$. Thus
\[
\text{gl}_S(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{R} \right\}
\]

Proposition 6.32 (Exercise 1.15iii). There is no matrix $S \in \text{gl}(2, \mathbb{R})$ such that
\[
\text{gl}_S(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{R} \right\}
\]
Proof. Let $D$ be
\[
D = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{R} \right\}
\]
and suppose there exists $S \in \text{gl}(2, \mathbb{R})$ such that $\text{gl}_S(2, \mathbb{R}) = D$. let $x \in D$, where
\[
x = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}
\]
for some fixed $x_1, x_2 \in \mathbb{R}$. Since $x \in \text{gl}_S(2, \mathbb{R})$, $x^T S = -S x$ so $x S = -S x$.
Proof. this we also will get that ad
Then \( x \)
Let 
Proof. Let 
Proposition 6.35 (Exercise 1.17) 
\[ \text{Proposition 6.34} \]
Thus no such \( S \) exists. 

Proposition 6.33 (Exercise 1.15iv). \( \text{gl}_I(3, \mathbb{R}) \cong \mathbb{R}^3 \) (Lie algebra isomorphism) 
Proof. Let \( i = (1, 0, 0), j = (0, 1, 0), k = (0, 0, 1) \) be the standard basis for \( \mathbb{R}^3 \), and note that \( i \times j = k, j \times k = i, \) and \( k \times i = j \). It turns out that \( \text{gl}_I(3, \mathbb{R}) \) is the subalgebra of anti-symmetric matrices, which as shown previously has basis
\[
\begin{align*}
a &= \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad b &= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad c &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},
\end{align*}
\]
where \([a, b] = c, [b, c] = a, \) and \([c, a] = b\). We define \( \phi : \mathbb{R}^3 \to \text{gl}_I(3, \mathbb{R}) \) by \( \phi(i) = a, \phi(b) = j, \phi(k) = c \). Since \( i, j, k \) and \( a, b, c \) are bases and \( \phi \) preserves brackets, \( \phi \) is an isomorphism. 

Proposition 6.34 (Exercise 1.16). If \( F \) is a field of characteristic 2, then there exist algebras over \( F \) which satisfy anticommutivity \([x, y] = -[y, z]\) and the Jacobi identity but not \([x, x] = 0\). 
Proof. Let \( A \) be the algebra on \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) with basis \( x_1 = (1, 0), x_2 = (0, 1) \). We define a bilinear map \([\cdot, \cdot] : \mathbb{Z}_2 \oplus \mathbb{Z}_2 \times \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) by the table

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_1 + x_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
<td>( x_1 )</td>
<td>0</td>
<td>( x_1 + x_2 )</td>
<td>( x_1 + x_2 )</td>
<td>0</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>0</td>
<td>( x_1 + x_2 )</td>
<td>( x_1 + x_2 )</td>
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<tr>
<td>( x_1 + x_2 )</td>
<td>0</td>
<td>0</td>
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<td>0</td>
</tr>
</tbody>
</table>

This bracket is symmetric as visible from the table, and since everything has order 2, it is thus antisymmetric. The Jacobi identity is seen to be true because every bracket product is either 0 or \( x_1 + x_2 \), and every bracket involving 0 or \( x_1 + x_2 \) is zero. Thus \([x, [y, z]] = 0\) for \( x, y, z \in \mathbb{Z}_2 \oplus \mathbb{Z}_2 \). However, \([x_1, x_1] \neq 0\), so this is not a Lie bracket. 

Proposition 6.35 (Exercise 1.17). Let \( V \) be a vector space over \( \mathbb{C} \) with basis \( \beta = \{v_i\}_{i=1}^n \). Let \( x : V \to V \) be a diagonalisable linear map with eigenvalues \( \lambda_i \), that is, \( x(v_i) = \lambda_i v_i \). Then \( \text{ad} x : \text{gl}(V) \to \text{gl}(V) \) is diagonalisable with eigenvalues \( \lambda_i - \lambda_j \) for \( 1 \leq i, j \leq n \).
Proof. Let \( y_{ij} : V \to V \) be the map with matrix \( c_{ij} \in \text{gl}(n, \mathbb{C}) \). We will show that \( \text{ad} x(y_{ij}) = (\lambda_i - \lambda_j)y_{ij} \), so \( y_{ij} \) for \( 1 \leq i, j \leq n \) are eigenvectors of \( \text{ad} x \) with eigenvalues \( \lambda_i - \lambda_j \). From this we also will get that \( \{y_{ij}\}_{1 \leq i, j \leq n} \) is a basis for \( \text{gl}(V) \). 

32
First we will show that $e_{ij}[x] = \lambda_j e_{ij}$ and $[x] e_{ij} = \lambda_i e_{ij}$. The matrix of $x$ is

$$
[x] = \begin{pmatrix}
\lambda_1 & 0 & 0 & 0 \\
0 & \lambda_2 & 0 & 0 \\
0 & 0 & \lambda_3 & 0 \\
0 & \vdots & \vdots & 0 \\
0 & 0 & 0 & \ldots & \lambda_n
\end{pmatrix}
$$

so the $ab$th entry of $[x]$ is $[x]_{ab} = \delta_{ab} \lambda_a = \delta_{ab} \lambda_b$. The $ab$th entry of $e_{ij}$ is $(e_{ij})_{ab} = \delta_{ai} \delta_{jb}$. Using the formula $(AB)_{ab} = \sum_{k=1}^{n} A_{ak} B_{kb}$ for the product of two matrices,

$$(e_{ij} [x])_{ab} = \sum_{k=1}^{n} (e_{ij})_{ak} [x]_{kb} = \sum_{k=1}^{n} \delta_{ai} \delta_{jk} \delta_{kb} \lambda_b = \delta_{ai} \lambda_b \sum_{k=1}^{n} \delta_{jk} \delta_{kb}$$

Since $1 \leq j, b \leq n$, in the sum $\sum_{k=1}^{n} \delta_{jk} \delta_{kb}$, all terms will be zero except when $j = k = b$, and if $j = b$ then there will be a nonzero term since $k$ ranges from 1 to $n$. Thus this sum term is equal to $\delta_{jb}$. Note also that because of the factor $\delta_{jb}$, the only time this product is nonzero is when $j = b$, so we can replace $\lambda_b$ with $\lambda_j$. Thus

$$(e_{ij} [x])_{ab} = \delta_{ai} \lambda_b \delta_{jb} = \lambda_j (e_{ij})_{ab}$$

Thus

$$e_{ij} [x] = \lambda_j e_{ij}$$

Using the product formula for matrices again,

$$( [x] e_{ij} )_{ab} = \sum_{k=1}^{n} [x]_{ak} (e_{ij})_{kb} = \sum_{k=1}^{n} \delta_{ak} \lambda_a \delta_{jk} \delta_{jb} = \lambda_a \delta_{jb} \sum_{k=1}^{n} \delta_{ak} \delta_{ik} = \lambda_a \delta_{ai} \delta_{jb} = \lambda_i (e_{ij})_{ab}$$

so

$$[x] e_{ij} = \lambda_i e_{ij}$$

Now we are able to compute ad $x(y_{ij})$.

$$\text{ad } x(y_{ij}) = [x, y_{ij}] = x \circ y_{ij} - y_{ij} \circ x$$

We know that $x \circ y_{ij}$ has matrix $[x] e_{ij} = \lambda_i e_{ij}$ and that $y_{ij} \circ x$ has matrix $e_{ij}[x] = \lambda_j e_{ij}$. Then by the previous lemmas about the linearity of the matrix of a transformation, $x \circ y_{ij} - y_{ij} \circ x$ has matrix $\lambda_i e_{ij} - \lambda_j e_{ij} = (\lambda_i - \lambda_j)e_{ij}$. Then the map corresponding to this matrix must be $(\lambda_i - \lambda_j)y_{ij}$. Thus $\text{ad } x(y_{ij}) = (\lambda_i - \lambda_j)y_{ij}$ as desired. \( \square \)

**Proposition 6.36** (Exercise 1.18). Let $L$ be a Lie algebra over $F$. Let

$$\text{IDer}(L) = \{ \text{ad } x : x \in L \}$$

Then $\text{IDer}(L)$ is an ideal of $\text{Der}(L)$. 33
Proof. First we show that $\text{IDer}(L)$ is closed under addition. Let $x, y \in L$, $\text{ad } x, \text{ad } y \in \text{IDer}(L)$.

$$(\text{ad } x + \text{ad } y)(z) = \text{ad } x(z) + \text{ad } y(z) = [x, z] + [y, z] = [x + y, z] = \text{ad}(x + y)(z)$$

Thus $\text{ad } x + \text{ad } y$ is in $\text{IDer}(L)$. Now we show that $\text{IDer}(L)$ is closed under scalar multiplication. Let $a \in F$.

$$a \text{ad } x(z) = a[x, z] = [ax, z] = \text{ad}(ax)(z)$$

Thus $a \text{ad } x$ is in $\text{IDer}(L)$. Now we show that $\text{IDer}$ satisfies the ideal property. Let $D \in \text{Der}(L), \text{ad } x \in \text{IDer}(L)$. Then

$$[D, \text{ad } x](z) = D \circ \text{ad } x(z) - \text{ad } x \circ D(z)$$

$$= D([x, z]) - [x, D(z)]$$

$$= [x, D(z)] + [D(x), z] - [x, D(z)]$$

$$= [D(x), z]$$

$$= \text{ad}(D(x))(z)$$

Thus $[D, \text{ad } x] = \text{ad } D(x)$, so $[D, \text{ad } x] \in \text{IDer}(L)$.

\[\square\]

**Proposition 6.37** (Exercise 1.19). Let $A$ be an algebra and let $\delta : A \to A$ be a derivation. Then

$$\delta^n(xy) = \sum_{r=0}^{n} \binom{n}{r} \delta^r(x)\delta^{n-r}(y)$$

for all $x, y \in A$.

Proof. Clearly this is true for $n = 1$ by the definition of a derivation. We proceed by induction on $n$. Suppose the statement is true for $n - 1$, that is,

$$\delta^{n-1}(xy) = \sum_{r=0}^{n-1} \binom{n-1}{r} \delta^r(x)\delta^{n-1-r}(y)$$

for all $x, y \in A$. Then

$$\delta^n(xy) = \delta(\delta^{n-1}(xy))$$

$$= \sum_{r=0}^{n-1} \binom{n-1}{r} \delta^{r+1}(x)\delta^{n-(r+1)}(y) + \delta^r(x)\delta^{n-r}(y)$$

$$= \sum_{r=0}^{n-1} \binom{n-1}{r} \delta^{r+1}(x)\delta^{n-(r+1)}(y) + \sum_{r=0}^{n-1} \binom{n-1}{r} \delta^r(x)\delta^{n-r}(y)$$

34
Now set $s = r + 1$. Then this is equal to

$$
\begin{align*}
&= \sum_{s=1}^{n} \binom{n-1}{s-1} \delta^s(x)\delta^{n-s}(y) + \sum_{r=0}^{n-1} \binom{n-1}{r} \delta^r(x)\delta^{n-r}(y) \\
&= \sum_{s=0}^{n} \binom{n-1}{s-1} \delta^s(x)\delta^{n-s}(y) + \sum_{r=0}^{n-1} \binom{n-1}{r} \delta^r(x)\delta^{n-r}(y) \\
&= \sum_{r=0}^{n} \left( \binom{n-1}{r} + \binom{n-1}{r-1} \right) \delta^r(x)\delta^{n-r}(y) \\
&= \sum_{r=0}^{n} \binom{n-1}{r} \delta^r(x)\delta^{n-r}(y)
\end{align*}
$$

It is a basic identity that

$$
\binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r}
$$

thus we finally have

$$
\delta^n(xy) = \sum_{r=0}^{n} \delta^r(x)\delta^{n-r}(y)
$$

Since the statement is true for $n = 1$ and if true for $n - 1$ it must be true for $n$, thus the statement is true for $n \in \mathbb{N}$ by the principle of induction.

\[\square\]
7 Chapter 2 Exercises

Proposition 7.1 (Exercise 2.1). Let $I, J$ be ideals of a Lie algebra $L$. Then

$$I + J := \{x + y : x \in I, y \in J\}$$

is an ideal of $L$.

Proof. We need to show that $I + J$ is a vector subspace of $L$ and that for $a \in L, b \in I + J$, we have $[a, b] \in I + J$.

Let $v, w \in I + J$. Then $v = v_i + v_j$ and $w = w_i + w_j$ where $v_i, w_i \in I$ and $v_j, w_j \in J$. Then $v + w = (v_i + v_j) + (w_i + w_j) = (v_i + w_i) + (v_j + w_j)$. Since $I, J$ are vector subspaces, $v_i + w_i \in I$ and $v_j + w_j \in J$. Thus $v + w \in I + J$.

Let $\lambda \in F$. Then $\lambda v = \lambda (v_i + v_j) = \lambda v_i + \lambda v_j$. Since $I, J$ are vector subspaces, $\lambda v_i \in I$ and $\lambda v_j \in J$. Thus $\lambda v \in I + J$.

Let $a \in L, b \in I + J$. Then $b = b_i + b_j$ so

$$[a, b] = [a, b_i + b_j] = [a, b_i] + [a, b_j]$$

Since $I, J$ are ideals of $L$, $[a, b_i] \in I$ and $[a, b_j] \in J$. Thus $[a, b] \in I + J$. \qed

Definition 7.2. Let $I, J$ be ideals of a Lie algebra $L$. Then we define

$$[I, J] := \text{Span}\{[x, y] : x \in I, y \in J\}$$

Proposition 7.3 (Exercise 2.2). $sl(2, \mathbb{C})' = sl(2, \mathbb{C})$

Proof. Take the basis for $sl(2, \mathbb{C})$ given by

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We compute the bracket products

$$[e, f] = g, \quad [f, g] = 2f, \quad [g, e] = 2e$$

Since $g, e, f$ are linearly independent over $\mathbb{C}$, so are $g, 2f, 2e$. Thus $sl(2, \mathbb{C})'$ has dimension at least 3. Since $sl(2, \mathbb{C})'$ is a span of vectors in $sl(2, \mathbb{C})$, it is a subspace. A subspace of equal dimension must be the whole space. Thus $sl(2, \mathbb{C})' = sl(2, \mathbb{C})$. \qed

Proposition 7.4 (Exercise 2.4). Let $L$ be a Lie algebra. Then $L/Z(L)$ is isomorphic to a subalgebra of $gl(L)$.

Proof. Consider the map $ad : L \to gl(L)$ where $ad(x) = ad_x$ is the map $ad_x : L \to L$ given by $ad_x(y) = [x, y]$. As shown on pages 4-5 of Erdmann and Wildon, $ad$ is linear and bracket-preserving, with $\ker(ad) = Z(L)$. As shown in Exercise 1.6, $\text{im}(ad)$ is a subalgebra of $gl(L)$. By the First Isomorphism Theorem,

$$L/\ker(ad) \cong \text{im}(ad)$$

Thus $L/Z(L)$ is isomorphic to a subalgebra of $gl(L)$. \qed

36
Proposition 7.5 (Exercise 2.3i). Let $L$ be a Lie algebra over $F$, and let $I$ be an ideal of $L$. We define a bracket on $L/I$ by

$$[w + I, z + I] = [w, z] + I$$

We claim that this bracket is bilinear.

Proof. Let $\lambda_1, \lambda_2 \in F$ and $v_1, v_2, w \in L$. Then

$$[\lambda_1(v_1 + I) + \lambda_2(v_2 + I), w + I] = [(\lambda_1 + \lambda_2v_2) + I, w + I]$$

$$= [(\lambda_1v_1 + \lambda_2v_2), w + I]$$

$$= [(\lambda_1v_1, w)] + [\lambda_2v_2, w] + I$$

$$= (\lambda_1v_1w) + ([\lambda_2v_2, w]) + I$$

$$= (\lambda_1[v_1w] + I) + ([\lambda_2v_2, w] + I)$$

$$= (\lambda_1[v_1w] + I) + (\lambda_2[v_2, w] + I)$$

thus the bracket is linear in the first component.

$$[w + I, \lambda_1(v_1 + I) + \lambda_2(v_2 + I)] = [w + I, (\lambda_1v_1 + \lambda_2v_2) + I]$$

$$= [w, \lambda_1v_1 + \lambda_2v_2] + I$$

$$= [w, \lambda_1v_1] + [w, \lambda_2v_2] + I$$

$$= (\lambda_1[v_1w] + I) + ([\lambda_2v_2, w] + I)$$

$$= (\lambda_1[v_1w] + I) + (\lambda_2[v_2, w] + I)$$

thus the bracket is linear in the second component. This shows that the bracket is bilinear.

Proposition 7.6 (Exercise 2.3i). Let $L$ be a Lie algebra over $F$ and let $I$ be an ideal of $L$. Then the bracket on $L/I$ satisfies

$$[x, x] = 0$$

for $x \in L/I$.

Proof. Let $v \in L$, so $v + I \in L/I$. Then

$$[v + I, v + I] = [v, v] + I = 0 + I$$

where $0 + I$ is the identity for $L/I$.

Proposition 7.7 (Exercises 2.3i). Let $L$ be a Lie algebra over $F$, and let $I$ be an ideal of $L$. The the bracket on $L/I$ satisfies the Jacobi identity.

Proof. Let $u + I, v + I, w + I \in L/I$. Then

$$[u + I, [v + I, w + I]] + [v + I, [w + I, u + I]] + [w + I, [u + I, v + I]]$$

$$= [u + I, [v, w] + I] + [v + I, [w, u] + I] + [w + I, [u, v] + I]$$

$$= ([u, v] + I) + ([v, w] + I) + ([w, u] + I)$$

$$= 0 + I$$

where $0 + I$ is the additive identity of $L/I$. 

37
Proposition 7.8 (Exercise 2.3ii). Let \( I \) be an ideal of Lie algebra \( L \) over \( F \). Define
\[
\pi : L \to L/I
\]
by \( \pi(z) = z + I \). Then \( \pi \) is a Lie algebra homomorphism.

**Proof.** First we show that \( \pi \) is a linear map. Let \( a \in F \) and \( u,v \in L \)
\[
\pi(au + v) = (au + v) + I = (au + I) + (v + I) = a(u + I) + (v + I) = a\pi(u) + \pi(v)
\]
so \( \pi \) is linear. Now we show that \( \pi \) preserves the bracket.
\[
\pi([u,v]) = [u,v] + I = [u + I, v + I] = [\pi(u), \pi(v)]
\]
Thus \( \pi \) is a Lie algebra homomorphism. \( \square \)

Proposition 7.9 (Exercise 2.5). Let \( L \) be a Lie algebra, and let \( v \in L' \). Then \( \text{tr} \, \text{ad} \, v = 0 \).

**Proof.** Let \( x,y,z \in L \). Then
\[
\text{ad}[x,y](z) = [[x,y],z] = -[z,[x,y]] \quad \text{by anticommutativity}
\]
\[
= [x,[y,z]] + [y,[z,x]] \quad \text{by Jacobi identity}
\]
\[
= [x,[y,z]] - [y,[x,z]]
\]
\[
= \text{ad} \, x \circ \text{ad} \, y(z) - \text{ad} \, y \circ \text{ad} \, x(z)
\]
Thus \( \text{tr} \, \text{ad}[x,y] = \text{tr}(\text{ad} \, x \circ \text{ad} \, y - \text{ad} \, y \circ \text{ad} \, x) = 0 \). Since \( v \) is a linear combination of \( [x_i,y_i] \) and \( \text{tr} \) is linear,
\[
\text{tr} \, \text{ad} \, v = \text{tr} \, \text{ad} \sum_i a_i [x_i,y_i] = \sum_i a_i \text{tr} \, \text{ad} [x_i,y_i] = \sum_i a_i 0 = 0
\]
\( \square \)

Proposition 7.10 (Exercise 2.6i). \( \mathfrak{gl}(2,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathbb{C} \)

**Proof.** Let
\[
\mathcal{C} = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{C} \right\}
\]
Then \( \mathbb{C} \cong \mathcal{C} \) by the isomorphism
\[
a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}
\]
We claim that \( \mathfrak{gl}(2,\mathbb{C}) = \mathfrak{sl}(2,\mathbb{C}) \oplus \mathcal{C} \). If we think of \( \mathfrak{sl}(2,\mathbb{C}) \oplus \mathcal{C} \) not as ordered tuples but as sums of elements from \( \mathfrak{sl}(2,\mathbb{C}) \) and \( \mathcal{C} \), then we see that \( \mathfrak{sl}(2,\mathbb{C}) \oplus \mathcal{C} \subseteq \mathfrak{gl}(2,\mathbb{C}) \). Then since
\[
\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{C})
\]
and
\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in \mathfrak{sl}(2, \mathbb{C}) \oplus \tilde{\mathbb{C}}
\]
\[
\begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathfrak{sl}(2, \mathbb{C}) \oplus \tilde{\mathbb{C}}
\]
Thus $\mathfrak{sl}(2, \mathbb{C})$ contains a basis for $\mathfrak{gl}(2, \mathbb{C})$ so it is the entire space.

\[\text{Proposition 7.11 (Exercise 2.6ii). Let } L_1, L_2 \text{ be Lie algebras. Then}\]
\[Z(L_1 \oplus Z_2) = Z(L_1) \oplus Z(L_2) \tag{7.1}\]

\[\text{Proof.}\]
\[Z(L_1 \oplus L_2) = \{(x_1, x_2) : [(x_1, x_2), (y_1, y_2)] = 0 \text{ for all } (y_1, y_2) \in L_1 \oplus L_2\} = \{(x_1, x_2) : ([x_1, y_1], [x_2, y_2]) = (0, 0) \text{ for all } y_1 \in L_1, y_2 \in L_2\} = \{(x_1, x_2) : x_1 \in Z(L_1), x_2 \in Z(L_2)\} = Z(L_1) \oplus Z(L_2)\]

\[\text{Proposition 7.12 (Exercise 2.6ii). Let } L_1, L_2 \text{ be Lie algebras. Then } L_1' \oplus L_2' = (L_1 \oplus L_2)'\]

\[\text{Proof. Let } L = L_1 \oplus L_2.
\]
\[L' = \text{span}\{(x, y) : x, y \in L\} = \text{span}\{[x_1, x_2], (y_1, y_2) : x_1, y_1 \in L_1, x_2, y_2 \in L_2\} = \text{span}\{[x_1, y_1], [x_2, y_2] : x_1, y_1 \in L_1, x_2, y_2 \in L_2\} = \text{span}\{[x_1, y_1] : x, y \in L\} = L_1' \oplus L_2'\]

\[\text{Lemma 7.13 (Lemma for Exercise 2.6ii). Let } L_1, L_2, L_3 \text{ be Lie algebras. Then}\]
\[(L_1 \oplus L_2) \oplus L_3 \cong L_1 \oplus (L_2 \oplus L_3)\]

\[\text{Proof. The map } ((x_1, x_2), x_3) \mapsto (x_1, (x_2, x_3)) \text{ is clearly an isomorphism.}\]

\[\text{Proposition 7.14 (Exercise 2.6ii). If } L = \bigoplus_{i=1}^{k} L_i, \text{ then } Z(L) = \bigoplus_{i=1}^{k} Z(L_i) \text{ for } k \in \mathbb{N}.\]

\[\text{Proof. We have already showed that this is true for } k = 2 \text{ and it is obviously true for } k = 1.\]
Suppose it is true for $k = n$. Then
\[Z\left(\bigoplus_{i=1}^{n+1} L_i\right) = Z\left(\bigoplus_{i=1}^{n} L_i \oplus L_{n+1}\right) = Z\left(\bigoplus_{i=1}^{n} L_i\right) \oplus Z(L_{n+1}) = \left(\bigoplus_{i=1}^{n} Z(L_i)\right) \oplus Z(L_{n+1}) = \bigoplus_{i=1}^{n+1} Z(L_i)\]
so then it is true for $k = n + 1$. Thus it is true for $k \in \mathbb{N}$.\]
Proposition 7.15 (Exercise 2.6ii). If $L = \oplus_{i=1}^{k} L_i$, then $L' = \oplus_{i=1}^{k} L'_i$ for $k \in \mathbb{N}$.

Proof. It is obvious for $k = 1$, and we have shown this is true for $k = 2$. Suppose it is true for $k = n$. Then

\[
\left( \bigoplus_{i=1}^{n+1} L_i \right)' = \left( \bigoplus_{i=1}^{n} L_i \oplus L_{n+1} \right)' = \left( \bigoplus_{i=1}^{n} L_i \right)' \oplus (L_{n+1})' = \left( \bigoplus_{i=1}^{n} (L_i)' \right) \oplus (L_{n+1})' = \bigoplus_{i=1}^{n+1} (L_i)'
\]

so then it is true for $k = n + 1$. Thus it is true for $k \in \mathbb{N}$. \qed

Proposition 7.16 (Exercise 2.7i). Let $L_1, L_2$ be Lie algebras over $F$. Then

\[
\begin{align*}
p_1 : L_1 \oplus L_2 & \to L_1 \quad p_1(x_1, x_2) = x_1 \\
p_2 : L_1 \oplus L_2 & \to L_2 \quad p_2(x_1, x_2) = x_2
\end{align*}
\]

are Lie algebra homomorphisms.

Proof. Let $a, b \in F, x_1, y_1 \in L_1, x_2, y_2 \in L_2$. We show $p_1$ is linear.

\[
p_1(a(x_1, x_2) + (y_1, y_2)) = p_1(ax_1 + y_1, ax_2 + y_2) = ax_1 + y_1 = ap_1(x_1, x_2) + p_1(y_1, y_2)
\]

We show $p_1$ is bracket-preserving.

\[
p_1([x_1, x_2], (y_1, y_2)) = p_1([x_1, y_1], [x_2, y_2]) = [x_1, y_1] = [p_1(x_1, x_2), p_1(y_1, y_2)]
\]

We show that $p_2$ is linear.

\[
p_2(a(x_1, x_2) + (y_1, y_2)) = p_2(ax_1 + y_1, ax_2 + y_2) = ax_2 + y_2 = ap_2(x_1, x_2) + p_2(y_1, y_2)
\]

We show that $p_2$ is bracket-preserving.

\[
p_2([x_1, x_2], (y_1, y_2)) = p_2([x_1, y_1], [x_2, y_2]) = [x_2, y_2] = [p_2(x_1, x_2), p_2(y_1, y_2)]
\]

Thus $p_1, p_2$ are Lie algebra homomorphisms. \qed

Proposition 7.17 (Exercise 2.7i). Let $L_1, L_2$ be Lie algebras. Then

\[
I_1 = \{ (x_1, 0) : x_1 \in L_1 \} \\
I_2 = \{ (0, x_2) : x_2 \in L_2 \}
\]

are ideals of $L_1 \oplus L_2$ with $I_1 \cong L_1$ and $I_2 \cong L_2$.

Proof. We showed that $p_1, p_2$ defined above were Lie algebra homomorphisms. Since $\ker p_1 = I_2$ and $\ker p_2 = I_1$, we know that $I_1, I_2$ are ideals of $L_1 \oplus L_2$. We define $\phi_1 : I_1 \to L_1$ by $\phi_1(x_1, 0) = x_1$ and $\phi_2 : I_2 \to L_2$ by $\phi_2(0, x_2) = x_2$. $\phi_1, \phi_2$ are easily seen to be isomorphisms. \qed
Proposition 7.18 (Exercise 2.7ii). Let $L_1, L_2$ be Lie algebras with no non-trivial proper ideals. Define $I_1, I_2$ by

$$I_1 = \{(x_1, 0) : x_1 \in L_1\}$$
$$I_2 = \{(0, x_2) : x_2 \in L_2\}$$

Let $J$ be a non-trivial proper ideal of $L_1 \oplus L_2$ such that $J \cap I_1 = 0$ and $J \cap I_2 = 0$. Then the projections

$$p_1 : J \rightarrow I_1 \quad p_1(x_1, x_2) = (x_1, 0)$$
$$p_2 : J \rightarrow I_2 \quad p_2(x_1, x_2) = (0, x_2)$$

are isomorphisms.

Proof. We have shown in part (i) that $p_1, p_2$ are homomorphisms, so we just need to show that they are bijections. Since $J \cap I_1 = 0$ and $J \cap I_2 = 0$, for all $(x_1, x_2) \in J$ either $x_1 = x_2 = 0$ or neither of $x_1, x_2$ are zero. Thus $\ker p_1 = \ker p_2 = 0$, so $p_1, p_2$ are one-to-one.

We claim that $p_1$ is onto. First we show that $\text{im} p_1$ is an ideal of $I_1$. Let $(x_1, 0) \in \text{im} p_1$ and $(y_1, 0) \in I_1$. Then there exists $(x_1, x_2) \in J$. Then

$$[(x_1, 0), (y_1, 0)] = ([x_1, y_1], [0, 0]) = ([x_1, y_1], 0)$$

Since $J$ is an ideal and $(x_1, x_2) \in J$ and $(y_1, 0) \in L$, $[(x_1, x_2), (y_1, 0)] \in J$. Thus $([x_1, y_1], [x_2, 0]) \in J$, so $p_1([x_1, y_1], [x_2, 0]) = ([x_1, y_1], 0) \in \text{im} p_1$. Thus $\text{im} p_1$ is an ideal of $I_1$. Since $I_1$ has no non-trivial proper ideals, $\text{im} p_1 = I_1$, so $p_1$ is onto.

The same argument works to show that $\text{im} p_2$ is an ideal of $I_2$ and so $p_2$ is onto. Thus $p_1, p_2$ are isomorphisms.

Lemma 7.19 (Lemma for Exercise 2.7iii). Let $I, J$ be ideals of $L$. Then $I \cap J$ is an ideal of $L$.

Proof. Let $x \in I \cap J, y \in L$. Then $x \in I$ and $x \in J$ so $[x, y] \in I$ and $[x, y] \in J$ so $[x, y] \in I \cap J$.

Lemma 7.20. Let $I, J$ be ideals of $L$. Then $I \cup J$ is an ideal of $L$.

Proof. Let $x \in I \cup J$ and $y \in L$. Then $x \in I$ or $x \in J$, so $[x, y] \in I$ or $[x, y] \in J$. Thus $[x, y] \in I \cup J$.

Proposition 7.21 (Exercise 2.7iii). Let $L_1, L_2$ be non-isomorphic Lie algebras each with no non-trivial proper ideals and let $L = L_1 \oplus L_2$. Then $L$ has exactly two non-trivial proper ideals, which are respectively isomorphic to $L_1$ and $L_2$.

Proof. We have shown in part (ii) that

$$I_1 = \{(x_1, 0) : x_1 \in L_1\}$$
$$I_2 = \{(0, x_2) : x_2 \in L_2\}$$

41
are ideals of \( L \) with \( I_1 \cong L_1 \) and \( I_2 \cong L_2 \). Thus we must show that there are no other non-trivial proper ideals of \( L \).

Let \( J \) be a non-trivial proper ideal of \( L \) with \( J \) not equal to \( I_1 \) or \( I_2 \). If \( J \cap I \neq 0 \) then \( J \cap I_1 \) is a non-trivial proper ideal of \( I_1 \), but \( I_1 \) has no non-trivial proper ideals. Thus \( J \cap I_1 = 0 \). For analogous reasons, \( J \cap I_2 = 0 \). Then by part (ii), \( J \) is isomorphic to \( I_1 \) and \( I_2 \), so \( J \cong L_1 \cong L_2 \). But by hypothesis, \( L_1 \ncong L_2 \). Thus no such \( J \) exists. Thus \( L \) has exactly two non-trivial proper ideals.

\[ Q \]

**Proposition 7.22** (Exercise 2.7iv). Let \( L_1 \) be a one-dimensional Lie algebra over an infinite field \( F \). Let \( L_2 \cong L_1 \). Then \( L = L_1 \oplus L_2 \) has infinitely many different ideals.

**Proof.** Let \( L_1 = \text{span}\{x\} \). Since \( L_1 \) is one-dimensional, all bracket products are zero, so any subalgebra is an ideal. Since \( F \) is infinite, it contains a subset isomorphic to \( \mathbb{Q} \) which will contain a subset isomorphic to \( \mathbb{Z} \). Then the principal ideals

\[
< x > = \{nx : n \in \mathbb{Z}\}
\]
\[
< 2x > = \{n(2x) : n \in \mathbb{Z}\}
\]
\[
\vdots < kx > = \{n(kx) : n \in \mathbb{Z}\}
\]

are infinitely many different ideals of \( L_1 \). Thus from any of these we can make an ideal of \( L_1 \oplus L_2 \) by adding a zero in the \( L_2 \) position.

\[ Q \]

**Proposition 7.23** (Exercise 2.8a). Let \( \phi : L_1 \to L_2 \) be an onto homomorphism of Lie algebras. Then \( \phi(L_1t) = L_2t \).

**Proof.** By definition, \( L_2t = \{[x_2, y_2] : x_2, y_2 \in L_2\} \). Since \( \phi \) is onto, for all \( x, y \in L_2 \) there exist \( x_1, y_1 \in L_1 \) such that \( \phi(x_1) = x \) and \( \phi(y_1) = y \). Thus

\[
L_1t = \{[\phi(x_1), \phi(y_1)] : x_1, y_1 \in L_1\} = \{[\phi([x_1, y_1])] : x_1, y_1 \in L_1\} = \phi(L_1t)
\]

\[ Q \]

**Proposition 7.24** (Exercise 2.8b). Let \( \phi : L_1 \to L_2 \) be an isomorphism. Then \( \phi(Z(L_1)) = Z(L_2) \).

**Proof.** Let \( x \in Z(L_1) \). We need to show that for all \( b \in L_2 \), \( [\phi(x), b] = 0 \). Let \( b \in L_2 \). Then there exists \( y \in L_1 \) such that \( \phi(y) = b \). Since \( x \in Z(L_1) \), we know that \( [x, y] = 0 \), so

\[
[\phi(x), b] = [\phi(x), \phi(y)] = \phi([x, y]) = \phi(0) = 0
\]

Thus \( \phi(Z(L_1)) \subseteq Z(L_2) \).

Now suppose \( b \in Z(L_2) \). We need to show that there exists \( x \in Z(L_1) \) such that \( \phi(x) = b \). Since \( \phi \) is onto, there exists \( x \in L_1 \) such that \( \phi(x) = b \). Then for \( y \in L_1 \),

\[
[\phi(x),\phi(y)] = 0 = \phi([x, y]).
\]

Since \( \phi \) is one-to-one, this implies that \( [x, y] = 0 \). Thus \( x \in Z(L_1) \), so \( Z(L_2) \subseteq \phi(Z(L_1)) \).

\[ Q \]

**Proposition 7.25** (Exercise 2.8b). Let \( \phi : L_1 \to L_2 \) be an onto Lie algebra homomorphism. Then \( Z(L_2) \) is not necessarily contained in \( \phi(Z(L_1)) \).

42
Proof. We provide a specific counterexample. Let $L_1$ be the Lie algebra

$$L_1 = \text{span}\{z_1, z_2, z_3, x, y\}$$

then $Z(L_1) = \text{span}\{z_1, z_2, z_3\}$ and $L_1' = \text{span}\{x\}$. Let $L_2$ be the three dimensional abelian Lie algebra spanned by $w_1, w_2, w_3$. Note that $Z(L_2) = L_2$.

Let $\phi : L_1 \to L_2$ be the linear map defined by

$$\begin{align*}
\phi(x) &= 0 \\
\phi(y) &= w_1 \\
\phi(z_1) &= \phi(z_2) = w_2 \\
\phi(z_3) &= w_3
\end{align*}$$

Then since $\phi(L_1)$ contains a basis for $L_2$, $\phi$ is onto. It is also a homomorphism, because

$$[\phi(a), \phi(b)] = 0 \phi([a, b]) = \phi(\lambda x) = \lambda \phi(x) = 0$$

for any $a, b \in L_1$, for some $\lambda \in F$. Thus $\phi$ is a homomorphism. However, $\phi(Z(L_1)) = \text{span}\{w_2, w_2\} \neq Z(L_2) = L_2$. 

\textbf{Proposition 7.26} (Exercise 2.8c). Let $L_1, L_2$ be Lie algebras over $F$. If $\phi : L_1 \to L_2$ is an isomorphism and $x \in L_1$ such that $\text{ad} x$ diagonalisable, then $\text{ad} \phi(x)$ is diagonalisable.

\textbf{Proposition 7.27} (Exercise 2.8c). Let $L_1, L_2$ be Lie algebras over $F$, and let $\phi : L_1 \to L_2$ be an onto homomorphism. Let $x \in L_1$ such that $\text{ad} x$ is diagonalisable. Then $\text{ad} \phi(x)$ is diagonalisable.

\textbf{Proposition 7.28} (Exercise 2.9). $\mathbb{R}_x^3 \cong L = \{ x \in \text{gl}(3, \mathbb{R}) : x^t = -x \}$. 
Proof. We know that \( i = (1,0,0), j = (0,1,0), k = (0,0,1) \) forms a basis for \( \mathbb{R}^3_\wedge \) with
\[
[i, j] = k \quad [j, k] = i \quad [k, i] = j
\]
We also know that
\[
e = \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\quad f = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix}
\quad g = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{pmatrix}
\]
forms a basis for \( L \) with
\[
[e, f] = g \quad [f, g] = e \quad [g, e] = f
\]
Thus \( \phi : \mathbb{R}^3_\wedge \to L \) defined by \( \phi(i) = e, \phi(j) = f, \phi(k) = h \) is an isomorphism. \( \square \)

**Proposition 7.29** (Exercise 2.9). Let
\[
L_1 = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{R} \right\}
\]
\[
L_2 = \left\{ \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & c \end{pmatrix} : a, b, c \in \mathbb{R} \right\}
\]
then \( L_1 \nless L_2 \) as Lie algebras.

*Proof.* First we claim that \( L'_2 \subseteq Z(L_2) \). Let \( A, B \in L_2 \) be
\[
A = \begin{pmatrix}
0 & a & b \\
0 & 0 & c \\
0 & 0 & 0
\end{pmatrix}
\quad B = \begin{pmatrix}
0 & e & f \\
0 & 0 & g \\
0 & 0 & 0
\end{pmatrix}
\]
We compute the bracket product of \( A \) and \( B \):
\[
[A, B] = AB - BA = \begin{pmatrix}
0 & 0 & ag - ce \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]
Thus
\[
L'_2 = \text{span} \left\{ C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}
\]
Then for \( A \in L_2, C \in L'_2 \), we compute \([A, C] = 0\). Thus \( C \in Z(L) \), so \( L'_2 \subseteq Z(L_2) \). Now we claim that \( L'_1 \not\subseteq Z(L_1) \). We know that
\[
L'_1 = \text{span} \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}
\]
44
but we compute
\[
\begin{pmatrix}
0 & 1 \\
0 & 1
\end{pmatrix} \cdot \begin{pmatrix}
0 & -1 \\
0 & 0
\end{pmatrix} \neq \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}
\]
so
\[
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix} \notin Z(L_1)
\]
Thus \(L'_1 \nsubseteq Z(L_1)\).

Suppose \(\phi : L_1 \to L_2\) were an isomorphism. Then \(\phi\) would preserve the property of the derived algebra being contained in the center, but since these algebras do not share this property, no such isomorphism exists. Thus \(L_1 \not\cong L_2\).

To summarize our results for Exercise 2.9, we showed that (i) and (iv) are isomorphic and that (ii) and (iii) are not isomorphic. It is easy to see that (i) is not isomorphic to (ii) or (iii), since the derived algebra for (i) has dimension 3 and the derived algebras for (ii) and (iii) have dimension 1.

**Proposition 7.30** (Exercise 2.10). Let \(F\) be a field. Then \(\text{gl}(n,F)' = \text{sl}(n,F)\).

**Proof.** Since \([x,y] = xy - yx\) has trace zero for any \(x,y \in \text{gl}(n,F)\), clearly \(\text{gl}(n,F)' \subseteq \text{sl}(n,F)\). To show equality, we will show that \(\text{gl}(n,F)'\) contains a basis for \(\text{sl}(n,F)\). This makes it a subspace of equal dimension.

A basis for \(\text{sl}(n,F)\) is given by \(e_{ij}\) for \(i \neq j\) and \(e_{ii} - e_{i+1,i+1}\) for \(1 \leq i < n\), as stated on page 3 of Erdmann and Wildon. Using the formula for bracket products of \(e_{ij}\) also on page 3, we compute
\[
[e_{ii}, e_{ij}] = \delta_{11} e_{ij} = e_{ij} \quad \text{for} \ i \neq j
\]
\[
[e_{i,i+1}, e_{i+1,i}] = \delta_{i+1,i+1} e_{ii} - \delta_{ii} e_{i+1,i+1} = e_{ii} - e_{i+1,i+1} \quad \text{for} \ 1 \leq i < n
\]
Thus \(\text{gl}(n,F)'\) contains this basis for \(\text{sl}(n,F)\), so it is a subspace of equal dimension, so \(\text{gl}(n,F)' = \text{sl}(n,F)\). \( \square \)

**Proposition 7.31** (Exercise 2.11). Let \(S \in \text{gl}(n,F)\) and let \(P\) be an invertible matrix in \(\text{gl}(n,F)\). Let \(A = P^T S P\). Then \(\text{gl}_A(n,F) \cong \text{gl}_S(n,F)\) (Lie algebra isomorphism).

**Proof.** Define \(\phi : \text{gl}_S(n,F) \to \text{gl}_A(n,F)\) by \(\phi(x) = P^{-1} x P\). First we show that \(\phi\) actually maps into \(\text{gl}_A(n,F)\), so we need to show that for \(x \in \text{gl}_S(n,F)\), \(\phi(x) \in \text{gl}_A(n,F)\). Let \(x \in \text{gl}_S(n,F)\). Then
\[
x^T S = -Sx
\]
\[
P^T x^T S = -P^T S x
\]
\[
P^T x^T S P = -P^T S x P
\]
\[
P^T x^T (P^T)^{-1} P^T S P = -P^T S P P^{-1} x P
\]
\[
\phi(x)^T P^T S P = -P^T S P \phi(x)
\]
\[
\phi(x)^T A = -A \phi(x)
\]
thus \( \phi(x) \in \mathfrak{gl}_A(n,F) \).

Now we show that \( \phi \) is one-to-one. Let \( x, y \in \mathfrak{gl}_S(n,F) \) with \( x = y \). Then
\[
\begin{align*}
P^{-1}x &= -1 y \\
P^{-1}xP &= P^{-1}yP \\
\phi(x) &= \phi(y)
\end{align*}
\]
thus \( \phi \) is one-to-one.

Now we show that \( \phi \) is onto. Let \( z \in \mathfrak{gl}_A(n,F) \). We claim that \( PzP^{-1} \in \mathfrak{gl}_S(n,F) \) and that \( \phi(PzP^{-1}) = z \). Since \( z \in \mathfrak{gl}_A(n,F) \),
\[
\begin{align*}
z^T P^T SP &= -P^T SPz \\
(P^{-1})^T z^T p^T SP &= -SPz \\
(PzP^{-1})^T SP &= -SPz \\
(PzP^{-1})^T S &= -S(PzP^{-1})
\end{align*}
\]
thus \( PzP^{-1} \in \mathfrak{gl}_S(n,F) \). Then \( \phi(PzP^{-1}) = P^{-1}PzP^{-1}P = z \). Thus \( \phi \) is onto.

Finally, we show that \( \phi \) is a homomorphism. Let \( x, y \in \mathfrak{gl}_S(n,F) \). Then
\[
\begin{align*}
\phi([x, y]) &= P^{-1}(xy - yx)P \\
&= P^{-1}xyP - P^{-1}yxp \\
&= P^{-1}xyPP^{-1}yP - P^{-1}yPP^{-1}xP \\
&= \phi(x)\phi(y) - \phi(y)\phi(x) \\
&= [\phi(x), \phi(y)]
\end{align*}
\]
thus \( \phi \) preserves the bracket. Thus we have shown that \( \phi \) is a bijection and a homomorphism, so \( \phi \) is an isomorphism.

**Proposition 7.32** (Exercise 2.12). Let \( S \) be an \( n \times n \) invertible matrix with entries in \( \mathbb{C} \). Then for \( x \in \mathfrak{gl}_S(n, \mathbb{C}) \), \( \text{tr} x = 0 \).

**Proof.** Let \( x \in \mathfrak{gl}_S(n, \mathbb{C}) = \{y \in \mathfrak{gl}(n, \mathbb{C}) : y^T S = -Sy\} \). Then
\[
\begin{align*}
x^T S &= -Sx \\
x &= -S^{-1} x^T S
\end{align*}
\]
So then the traces are equal,
\[
\text{tr} x = \text{tr}(-S^{-1} x^T S) = -\text{tr}(S^{-1} x^T S) = -\text{tr}(x^T S^{-1} S) = -\text{tr}(x^T) = -\text{tr} x
\]
Thus \( \text{tr} x = -\text{tr} x \), and since \( \text{tr} x \in \mathbb{C} \), we must have \( \text{tr} x = 0 \). 

**Proposition 7.33** (Exercise 2.13). Let \( I \) be an ideal of the Lie algebra \( L \) over field \( F \). Then
\[
B = C_L(I) = \{x \in L : [x, a] = 0 \text{ for } a \in I\}
\]
is an ideal of \( L \).
Proof. First we show that $B$ is closed under addition in $L$. Let $b_1, b_2 \in B$ and let $a \in I$. Then

$$[b_1 + b_2, a] = [b_1, a] + [b_2, a] = 0 + 0 = 0$$

thus $b_1 + b_2 \in B$.

Now we show that $B$ is closed under scalar multiplication from $F$. Let $\lambda \in F$ and $b \in B$ and $a \in I$. Then

$$[\lambda b, a] = \lambda [b, a] = 0$$

thus $\lambda b \in B$.

Now we show that for $b \in B$, $y \in L$, $[y, b] \in B$. Let $a \in I$. Since $I$ is an ideal of $L$, $[a, y] \in I$, so by definition of $B$, $[b, a] = 0$ and $[b, [a, y]] = 0$. Then by the Jacobi identity,

$$[a, [y, b]] + [y, [b, a]] + [b, [a, y]] = 0$$

Since $[b, a] = 0$ and $[b, [a, y]] = 0$, the second and third terms are zero. Thus $[a, [y, b]] = 0$, so $[[y, b], a] = 0$ so by definition of $B$, $[y, b] \in B$. Thus $B$ is an ideal of $L$.

**Proposition 7.34** (Exercise 2.14i). Let $L$ be all matrices

$$L = \left\{ \begin{pmatrix} 0 & f(x) & h(x, y) \\ 0 & 0 & g(y) \\ 0 & 0 & 0 \end{pmatrix} : f(x) \in R[x], g(y) \in R[y], h(x, y) \in R[x, y] \right\}$$

$L$ is a Lie algebra (over $\mathbb{R}$) with the bracket $[a, b] = ab - ba$.

**Proof.** Clearly $L$ is closed under matrix addition and multiplication. We know that matrix multiplication and addition are bilinear. Furthermore, it is obvious that $[a, a] = a^2 - a^2 = 0$. Now we need to show the Jacobi identity holds. Let $a, b, c \in L$.

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = a(bc - bc) - (bc - cb)a + b(ca - ac) - (ca - ac)b + c(ab - ba) - (ab - ba)c$$

$$= abc - acb - bca + cba + bca - bac$$

$$= 0$$

$\square$

**Proposition 7.35** (Exercise 2.14ii). Let $A, B$ be matrices in $L$ where

$$L = \left\{ \begin{pmatrix} 0 & f(x) & h(x, y) \\ 0 & 0 & g(y) \\ 0 & 0 & 0 \end{pmatrix} : f(x) \in R[x], g(y) \in R[y], h(x, y) \in R[x, y] \right\}$$

Then

$$[A, B] = \begin{pmatrix} 0 & 0 & f_A g_B - f_B g_A \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

47
thus

\[ L' = \text{span}\{ \begin{pmatrix} 0 & 0 & f_A g_B - f_B g_A \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : f_A, f_B \in R[x], g_A, g_B \in R[y] \} \]

Proof. Let

\[ A = \begin{pmatrix} 0 & f_A & h_A \\ 0 & 0 & g_A \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & f_B & h_B \\ 0 & 0 & g_B \\ 0 & 0 & 0 \end{pmatrix} \]

Then we compute \([A, B]\) as

\[ [A, B] = AB - BA = \begin{pmatrix} 0 & 0 & f_A g_B \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & f_B g_A \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & f_A g_B - f_B g_A \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

\[ \square \]
8 Chapter 3 Exercises

Proposition 8.1 (Page 21, from 3.2.1). Let $L$ be a 3-dimensional Lie algebra over $F$ with $L' \subseteq Z(L)$ and $L'$ of dimension one. Let $f,g \in L$ such that $[f,g] \neq 0$. Then let $z = [f,g]$. Then $f,g,z$ are linearly independent, and thus form a basis for $L$.

Proof. Suppose $af + bg + cz = 0$ for some $a, b, c \in F$. Then for all $x \in L$, $[af + bg + cz, x] = 0$, so $a[f, x] + b[g, x] + c[z, x] = 0$. Since $z \in L' \subseteq Z(L)$, we know that $[z, x] = 0$. Thus $a[f, x] + b[g, x] = 0$. In particular, this is true for $x = f$, so $a[f, f] + b[g, f] = 0$ so $b[g, f] = 0$, since $[f, g] \neq 0$ by hypothesis, we conclude that $b = 0$. Likewise, when we set $x = g$, we see that $a[f, g] = 0$, so $a = 0$. Then returning to the original equation, we see that $cz = 0$, which implies that $c = 0$ since $z \neq 0$ by hypothesis. Thus $a = b = c = 0$, so $f, g, z$ are linearly independent and form a basis for $L$. \qed

Proposition 8.2 (Exercise 3.1). Let $V = \text{span}\{v_1, \ldots, v_2\}$ be a vector space and let $\phi : V \rightarrow V$ be a linear map. Let $L = V \oplus \text{span}\{x\} = \text{span}\{v_1, \ldots, v_n, x\}$ and define a bracket

$$[\cdot, \cdot] : L \times L \rightarrow L$$

as the bilinear map defined by

$$[v_i, v_j] = 0 \quad [x, x] = 0 \quad [x, v_i] = \phi(v_i)$$

Then $L$ is a Lie algebra under this bracket with $\dim L' = \text{rank} \phi$.

Proof. We know that since $[y, y] = 0$ for all $y \in L$ that the bracket is antisymmetric, as shown in pages 1-2 of Erdmann and Wildon. We just need to show that the Jacobi identity holds for any three basis elements. If any two are in the span of $x$, then all brackets will be zero, and if all are in $V$, then the brackets will be zero. So we just need to show the identity holds in the case of $x$ and two basis vectors $v_i, v_j$ of $V$.

$$[x, [v_i, v_j]] + [v_i, [v_j, x]] + [v_j, [x, v_i]] = [x, 0] - [v_i, \phi(v_j)] + [v_j, \phi(v_i)] = 0 - 0 + 0 = 0$$

Thus $L$ is a Lie algebra under this bracket. Furthermore,

$$L' = \text{span}\{[y, z] : y, z \in L\} = \text{span}\{\phi(v_i) : 1 \leq i \leq n\} = \phi(V)$$

Thus $\dim L' = \dim \phi(V)$, and $\text{rank} \phi = \dim \phi(V)$ by definition. \qed

Proposition 8.3 (Exercise 3.2). Let $L_u, L_v$ be 3-dimensional Lie algebras over $\mathbb{C}$ with respective bases $\{x_1, y_1, z_1\}$ and $\{x_2, y_2, z_2\}$ where $L'_u = \text{span}\{y_1, z_1\}$ and $L'_v = \text{span}\{y_2, z_2\}$ and $\text{ad} x_1 : L'_u \rightarrow L'_u$ and $\text{ad} x_2 : L'_v \rightarrow L'_v$ are diagonalisable with matrices

$$[\text{ad} x_1] = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \quad [\text{ad} x_2] = \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix}$$

with respect to the aforementioned bases, where $u, v \in \mathbb{C}$ with $u, v \neq 0$. If $v = u$ or $v = u^{-1}$ then $L_u \cong L_v$.
Proof. Note that Erdmann and Wildon prove in 3.2.3 that $L_u', L_v'$ are abelian. From the matrices $[\text{ad } x_1]$ and $[\text{ad } x_2]$ we know that

\[
\begin{align*}
[x_1, y_1] &= y_1 \\
[x_1, z_1] &= uz_1 \\
[y_1, z_1] &= 0
\end{align*}
\quad
\begin{align*}
[x_2, y_2] &= y_2 \\
[x_2, z_2] &= vz_2 \\
[y_2, z_2] &= 0
\end{align*}
\]

Suppose that $u = v$. We define the linear map $\phi : L_u \to L_v$ by

\[
\begin{align*}
\phi(x_1) &= x_2 \\
\phi(y_1) &= y_2 \\
\phi(z_1) &= z_2
\end{align*}
\]

Since $\phi$ maps a basis to a basis, it is bijective. We need to show that $\phi$ also preserves brackets. Thus we compute how the brackets interact with $\phi$ as follows:

\[
\begin{align*}
\phi([x_1, y_1]) &= \phi(y_1) = y_2 = [x_2, y_2] = [\phi(x_1), \phi(y_1)] \\
\phi([x_1, z_1]) &= \phi(uz_1) = uz_2 = vz_2 = [x_2, z_2] = [\phi(x_1), \phi(z_2)] \\
\phi([y_1, z_1]) &= \phi(0) = 0 = [y_2, z_2] = [\phi(y_1), \phi(z_2)]
\end{align*}
\]

Thus $\phi$ is an isomorphism of Lie algebras. Now suppose that $v = u^{-1}$. We define a linear map $\psi : L_u \to L_v$ by

\[
\begin{align*}
\psi(x_1) &= ux_2 \\
\psi(y_1) &= z_2 \\
\psi(z_1) &= y_2
\end{align*}
\]

Since $ux_2$ is just a scalar multiple of $x_2$, $\{ux_2, z_2, y_2\}$ is a basis for $L_v$, so $\psi$ maps a basis to a basis, so it is a bijection. We need to show that it also preserves brackets. Note that $x_2 = u^{-1}\psi(x_1)$.

\[
\begin{align*}
\psi([x_1, y_1]) &= \psi(y_1) = z_2 = uu^{-1}z_2 = u[x_2, z_2] = u[u^{-1}\psi(x_1), \psi(y_1)] = [\psi(x_1), \psi(y_1)] \\
\psi([x_1, z_1]) &= \psi(uz_1) = u\psi(z_1) = uy_2 = u[x_2, y_2] = u[u^{-1}\psi(x_1), \psi(z_1)] = [\psi(x_1), \psi(z_1)] \\
\psi([y_1, z_1]) &= \psi(0) = 0 = [z_2, y_2] = [\psi(y_1), \psi(z_1)]
\end{align*}
\]

Thus $\psi$ is an isomorphism of Lie algebras. 

**Proposition 8.4** (Exercise 3.2). Let $L_u, L_v$ be 3-dimensional Lie algebras over $\mathbb{C}$ with respective bases $\{x_1, y_1, z_1\}$ and $\{x_2, y_2, z_2\}$ where $L_u' = \text{span}\{y_1, z_1\}$ and $L_v' = \text{span}\{y_2, z_2\}$ and $\text{ad } x_1 : L_u' \to L_u'$ and $\text{ad } x_2 : L_v' \to L_v'$ are diagonalisable with matrices

\[
\begin{align*}
[\text{ad } x_1] &= \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \\
[\text{ad } x_2] &= \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix}
\end{align*}
\]

with respect to the aforementioned bases, where $u, v \in \mathbb{C}$ with $u, v \neq 0$. If $L_u \cong L_v$, then $v = 0$ or $v = u^{-1}$. 

50
Proof. Suppose \( \phi : L_u \to L_v \) is an isomorphism. By Exercise 2.8a, when restricted to \( L'_u \), \( \phi : L'_u \to L'_v \) is still an isomorphism. Since \( \phi \) is onto, one of the basis elements of \( L_u \) must map to a linear combination that has a non-zero scalar for \( x_2 \), and since \( \phi(L'_u) = L'_v \), that basis element must be \( x_1 \). In particular, it must be the case that \( \phi(x_1) = ax_2 + w \) for some non-zero \( a \in \mathbb{C} \) and some \( w \in L'_v \).

Now let \( t \in L'_u \). We can compute

\[
[\phi(x_1), \phi(t)] = \phi([x_1, t]) = \phi \circ \text{ad} \, x_1(t)
\]

\[
[\phi(x_1), \phi(t)] = [ax_2 + w, \phi(t)] = a[x_2, \phi(t)] + [w, \phi(t)] = a[x_2, \phi(t)] + 0 = a \text{ad} \, x_2 \circ \phi(t)
\]

Note that the \([w, \phi(t)]\) term is zero because \( w, \phi(t) \in L'_u \) and \( L'_u \) is abelian as shown in Lemma 3.3a of Erdmann and Wildon. From this we see that \( \phi \circ \text{ad} \, x_1 = a \text{ ad} \, x_2 \circ \phi = \text{ad}(ax_2) \circ \phi \). Let \( [\phi] \) denote the matrix of \( \phi \). Since \( \phi \) a bijection, it is invertible, so the matrix \([\phi]\) is invertible. As shown in 16.1i, the matrix of a composition is the product of the matrices, so

\[
[\phi \circ \text{ad} \, x_1] = [\phi][\text{ad} \, x_1]
\]

\[
[\text{ad}(ax_2) \circ \phi] = [\text{ad}(ax_2)][\phi]
\]

Since the maps \( \phi \circ \text{ad} \, x_1 \) and \( \text{ad}(ax_2) \circ \phi \) are equal, their matrices are equal, so

\[
[\phi \circ \text{ad} \, x_1] = [\text{ad}(ax_2) \circ \phi]
\]

\[
\implies [\phi][\text{ad} \, x_1] = [\text{ad}(ax_2)][\phi]
\]

\[
\implies [\phi][\text{ad} \, x_1][\phi]^{-1} = [\text{ad}(ax_2)]
\]

Thus the matrices for \( \text{ad} \, x_1 : L_u \to L_u \) and \( \text{ad}(ax_2) : L_v \to L_v \) are similar, so they are similar as linear maps. In particular, this means that they have the same eigenvalues. The eigenvalues for \( \text{ad} \, x_1 \) are \( \{1, u\} \) and the eigenvalues for \( \text{ad}(ax_2) \) are \( \{a, av\} \), so we have \( \{1, u\} = \{a, av\} \). Thus either \( a = 1 \) and \( u = v \), or \( a = u \) and \( v = u^{-1} \). This completes the proof, since we have shown that \( u = v \) or \( u = v^{-1} \). \( \square \)

**Proposition 8.5 (Exercise 3.3i).** Let

\[
S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}
\]

Then \( \text{gl}_S(3, \mathbb{C}) \cong \text{sl}(2, \mathbb{C}) \).

**Proof.** Let \( x \in \text{gl}_S(3, \mathbb{C}) \). Then \( x^t S = -Sx \), so

\[
x = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ b & c & 0 \end{pmatrix}
\]

for some \( a, b, c \in \mathbb{C} \). Thus a basis for \( \text{gl}_S(3, \mathbb{C}) \) is

\[
e = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad f = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad g = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}
\]
where the bracket products compute to

\[ [e, f] = g \quad [f, g] = e \quad [e, g] = f \]

which is clearly isomorphic to the \(\text{sl}(2, \mathbb{C})\) by mapping this basis \(\{e, f, g\}\) to the basis \(\{e, f, g\}\) described in Exercise 1.12.

Alternately, even without this explicit isomorphism, we can deduce this isomorphism from the theorem that there is only one Lie algebra over \(\mathbb{C}\) with \(\dim L' = 3\).

**Proposition 8.6 (Exercise 3.3ii).** Let \(L\) be the complex matrix algebra spanned by

\[
U = \begin{pmatrix} t & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & v \end{pmatrix} \quad V = e_{13} \quad W = e_{23}
\]

for some fixed \(t, u, v \in \mathbb{C}\). Then in the notation of 3.2.3, \(L\) is isomorphic to \(L_x\) where \(x = \frac{u-v}{1-v}\).

**Proof.** We can compute the brackets

\[
[U, V] = (1 - v)V \\
[U, W] = (u - v)W[V, W] = 0
\]

Thus \(L'\) has basis \(V, W\) so \(\dim L' = 2\). Furthermore, the map \(\text{ad} U : L' \to L'\) is diagonalisable, with matrix

\[
M_{\text{ad} U} = \begin{pmatrix} 1 - v & 0 \\ 0 & u - v \end{pmatrix}
\]

Let \(U_2 = (1 - v)^{-1}U\). Then \(U_2, V, W\) is still a basis for \(L\), and now we get the bracket products

\[
[U_2, V] = [(1 - v)^{-1}U, V] = (1 - v)^{-1}[U, V] = V \\
[U_2, W] = (1 - v)^{-1}[U, W] = \left(\frac{u - v}{1 - v}\right)W
\]

so the matrix of \(U_2\) is

\[
M_{\text{ad} U_2} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{u - v}{1 - v} \end{pmatrix}
\]

Thus in the notation of 3.2.3, we have that \(L\) is isomorphic to \(L_x\) with \(x = \frac{u-v}{1-v}\). □

**Proposition 8.7 (Exercise 3.3iii).** Let \(L\) be the complex matrix Lie algebra

\[
L = \left\{ \begin{pmatrix} 0 & a & b & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ where } a, b, c \in \mathbb{C} \right\}
\]

Then \(L\) is isomorphic to the Heisenberg algebra.
Proof. First we show that \( L' \) is one-dimensional. Let
\[
A = \begin{pmatrix}
0 & a & b & 0 \\
0 & 0 & c & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\quad B = \begin{pmatrix}
0 & d & e & 0 \\
0 & 0 & f & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
Then we compute \([A, B]\).
\[
[A, B] = AB - BA = \begin{pmatrix}
0 & 0 & af - cd & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
so \( L' \) is one-dimensional, since
\[
C = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
is a basis for \( L' \). Then we can see that \( C \) is in \( Z(L) \) and thus \( L' \subset Z(L) \) becasue \([A, C] = 0\) and \( A \) is a general matrix from \( L \). Thus \( L \) is a three dimensional Lie algebra with \( \dim L' = 1 \) and \( L' \subset Z(L) \), so it is the Heisenberg Algebra. \( \square \)

Proposition 8.8 (Exercise 3.3iv). Let \( L \) be the complex matrix Lie algebra
\[
L = \left\{ \begin{pmatrix}
0 & 0 & a & b \\
0 & 0 & 0 & c \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix} \middle| \text{where } a, b, c \in \mathbb{C} \right\}
\]
Then \( L \) is abelian.

Proof. Let
\[
A = \begin{pmatrix}
0 & 0 & a & b \\
0 & 0 & 0 & c \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\quad B = \begin{pmatrix}
0 & 0 & d & e \\
0 & 0 & 0 & f \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
Then we compute \([A, B] = 0\). Thus \( L \) is abelian. \( \square \)

Proposition 8.9 (Exercise 3.4). Let \( L \) be a vector space over \( F \) with basis \( v_1, v_2 \) and bilinear operator \([,] : L \times L \to L \) with \([u, u] = 0\) for \( u \in L \). Then the Jacobi identity holds for this bilinear operator, and so \( L \) is a Lie algebra with this bracket.
Proof. First note that since \([u, u] = 0\) for all \(u \in L\), it follows that the bracket is anti-commutative (see page 1 of Erdmann and Wildon for proof). Let \(x, y, z \in L\). There are two possibilities: \(x, y\) are linearly independent or they are not. If they are not linearly independent, then \(x = \lambda y\) for some \(\lambda \in F\). Then

\[
[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = \lambda[y, [y, z]] + [y, [z, \lambda y]] + [z, [\lambda y, y]]
\]

\[= \lambda[y, [y, z]] - \lambda[y, [y, z]] + 0
\]

\[= 0
\]

where the third term goes to zero because \([y, y] = 0\). If \(x, y\) are linearly independent, then they form a basis for \(L\). Then we can write \(z = ax + by\) for some \(a, b \in F\). Then

\[
[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = [x, a[y, x]] + [y, b[y, x]] + [ax, [x, y]] + [by, [x, y]]
\]

\[= a[x, [y, x]] + b[y, [y, x]] + ax, [x, y]] + by, [x, y]]
\]

\[= a[x, [y, x]] - a[x, [y, x]] + b[y, [y, x]] - b[y, [y, x]]
\]

\[= 0
\]

Proposition 8.10 (Exercise 3.5). There exists \(h \in \text{sl}(2, \mathbb{R})\) such that \(\text{ad} \ h\) is diagonalisable.

Proof. Let

\[
e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

This is a basis for \(\text{sl}(2, \mathbb{R})\). The bracket products are

\([h, e] = 2e \quad [h, f] = 2f \quad [h, h] = 0
\]

so the matrix of \(\text{ad} \ h\) with respect to the basis \(e, f, h\) is

\[
M_{\text{ad} \ h} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

which is clearly a diagonal matrix. Thus \(\text{ad} \ h\) is diagonalisable. □

Proposition 8.11 (Exercise 3.5). There is no \(x \in \mathbb{R}^3_\wedge\) with \(x \neq 0\) such that \(\text{ad} \ x\) is diagonalisable.

Proof. Let \(u, v, w\) be a basis for \(\mathbb{R}^3_\wedge\) and suppose that there is an \(x \neq 0\) such that \(\text{ad} \ x\) is diagonalisable. Then for some \(a, b, c \in \mathbb{R}\),

\[
x \times u = au
\]

\[
x \times v = bv
\]

\[
x \times w = cw
\]

But if \(x \neq 0\), then \(x \times u\) is orthogonal to \(u\), so \(x \times u\) cannot be collinear with \(u\) unless \(x = 0\) or \(u = 0\). But \(x \neq 0\) by hypothesis and \(u \neq 0\) since \(u\) is part of a basis. Thus no such \(x\) exists. □
**Proposition 8.12** (Exercise 3.5). Let $L_1,L_2$ be Lie algebras over $F$. If $\phi : L_1 \to L_2$ is an isomorphism and $x \in L_1$ such that $ad x$ diagonalisable, then $ad \phi(x)$ is diagonalisable.

**Proof.** Let $\beta = \{v_1, \ldots, v_n\}$ be a basis for $L_1$ so that $ad x$ is diagonalisable with respect to $\beta$. Then $ad x(v_i) = [x, v_i] = \lambda_i v_i$ for $\lambda_i \in F$. Then $\phi(\beta)$ is a basis for $L_2$, and

$$ad \phi(x)(\phi(v_i)) = (\phi(x), \phi(v_i)) = \phi([x, v_i]) = \phi(\lambda_i v_i) = \lambda_i \phi(v_i)$$

so $ad \phi(x)$ is diagonalisable with respect to $\phi(\beta)$. \hfill $\Box$

**Proposition 8.13** (Exercise 3.5). $\text{sl}(2, \mathbb{R}) \not\cong \mathbb{R}^3_\lambda$

If $\phi : \text{sl}(2, \mathbb{R}) \to \mathbb{R}^3_\lambda$ were an isomorphism, then $ad \phi(h)$ would be diagonalisable. However, there is no $x \in \mathbb{R}^3_\lambda$ with $ad x$ diagonalisable. Thus $\phi$ is not on $\mathbb{R}^3_\lambda$. \hfill $\Box$

**Proposition 8.14** (Exercise 3.7). Let $L$ be a non-abelian Lie algebra. Then

$$\text{dim } Z(L) \leq \text{dim } L - 2.$$

**Proof.** Clearly it is impossible for $\text{dim } Z(L)$ to be greater than or equal to $\text{dim } L$, since $Z(L)$ is a proper ideal of $L$. So, to prove our claim, all we need to do is rule out the possibility that $\text{dim } Z(L) = \text{dim } L - 1$.

Let $n = \text{dim } L$. Suppose that $\text{dim } Z(L) = n - 1$. Then we have a basis $\{v_1, v_2, \ldots, v_{n-1}\}$ for $Z(L)$. We can extend this to a basis of $L$ by appending the vector $u \in L$, so we have a basis $\{v_1, v_2, \ldots, v_{n-1}, u\}$ for $L$. Then for all $1 \leq i \leq n$, $[v_i, u] = 0$ since $v_i \in Z(L)$, and for all $1 \leq i, j \leq n$, $[v_i, v_j] = 0$. Thus all bracket products of basis elements of $L$ are zero, so $L$ is abelian. This contradicts our hypothesis that $L$ is non-abelian, so we conclude that it is impossible for $\text{dim } Z(L) = \text{dim } L - 1$. \hfill $\Box$

**Proposition 8.15** (Exercise 3.9i). Let $L$ be a Lie algebra with an ideal $I$ and subalgebra $S$ such that $L = S \oplus I$. Let $\theta : S \to \mathrm{gl}(I)$ be defined by $\theta(s)(x) = [s, x]$. Then $\theta$ is a Lie algebra homomorphism from $S$ into $\mathrm{Der} I$.

**Proof.** Bilinearity of $\theta$ follows from bilinearity of the bracket on $L$:

$$\theta(s)(ax + b) = [s, ax + b] = a[s, x] + [s, y] = a\theta(s)(x) + \theta(s)(y)$$

We claim that $\theta$ preserves the brackets on $S$ and $\mathrm{gl}(I)$, that is, for $s, t \in S$, $\theta([s, t]) = [\theta(s), \theta(t)]$. Let $x \in I$. Then

$$\theta([s, t])(x) = [[s, t], x]$$

$$= -[x, [s, t]]$$

$$= [s, [t, x]] + [t, [x, s]]$$

by Jacobi

$$= [s, [t, x]] - [t, [s, x]]$$

$$= (\theta(s) \circ \theta(t))(x) - (\theta(t) \circ \theta(s))(x)$$

$$= [\theta(s), \theta(t)](x)$$

Thus $\theta$ is a Lie algebra homomorphism. We also claim that $\text{im } \theta \subseteq \text{Der } I$. Let $s \in S$. Then for $x, y \in I$,

$$\theta(s)[x, y] = [s, [x, y]] = -[x, [y, s]] - [y, [s, x]] = [x, [s, y]] + [[s, x], y] = [x, \theta(s)(y)] + [\theta(s)(x), y]$$

Thus $\theta(s)$ is a derivation of $I$, so $\text{im } \theta \subseteq \text{Der } I$. \hfill $\Box$
Proposition 8.16 (Exercise 3.9ii). Let $S, I$ be Lie algebras over $F$ and let $\theta : S \to \text{Der} I$ be a Lie algebra homomorphism. We equip the vector space $S \oplus I$ with the bracket

$$[(s_1, x_1), (s_2, x_2)] = ([s_1, s_2], [x_1, x_2] + \theta(s_1)x_2 - \theta(s_2)x_1)$$

We claim that $S \oplus I$ is a Lie algebra under this bracket.

Proof. First we show that this bracket is bilinear. Let $a, b \in F$ and let $(s_1, x_1), (s_2, x_2), (s_3, x_3) \in S \oplus I$.

$$[a(s_1, x_1) + b(s_2, x_2), (s_3, x_3)] = [(as_1 + bs_2, ax_1 + bx_2), (s_3, x_3)]$$

$$= [[as_1 + bs_2, s_3],$$

$$ax_1 + bx_2, x_3] + \theta(\theta(as_1 + bs_2)(x_3) - \theta(s_3)(ax_1 + bx_2))$$

$$= (a[s_1, s_3] + b[s_2, s_3],$$

$$a[x_1, x_3] + b[x_1, x_3] + a\theta(s_1)x_3 + b\theta(s_2)x_3 - a\theta(s_3)x_1 - b\theta(s_3)x_3)$$

$$= a([s_1, s_3], [x_1, x_3] - \theta(s_1)x_3 - \theta(s_3)x_1)$$

$$= a([s_1, s_3], [x_1, x_3] + \theta(s_2)x_3 - \theta(s_3)x_2)$$

Thus the bracket is linear in the first entry. Now we show linearity of the bracket in the second entry.

$$[(s_1, x_1), a(s_2, x_2) + b(s_3, x_3)] = [(s_1, x_1), (as_2 + bs_3, ax_2 + bx_3)]$$

$$= ([s_1, as_2 + bs_3], [x_1, ax_2 + bx_3] + \theta(s_1)(ax_2 + bx_3) - \theta(as_2 + bs_3)x_1)$$

$$= (a[s_1, s_2] + b[s_1, s_3],$$

$$a[x_1, x_2] + b[x_1, x_3] + a\theta(s_1)x_2 + b\theta(s_1)x_3 - a\theta(s_2)x_1 - b\theta(s_3)x_1)$$

$$= a([s_1, s_2], [x_1, x_2] + \theta(s_1)x_2 - \theta(s_2)x_1)$$

$$+ b([s_1, s_3], [x_1, x_3] + \theta(s_1)x_3 - \theta(s_3)x_1)$$

$$= a([s_1, x_1], (s_2, x_2)] + b([s_1, x_1], (s_3, x_3])$$

Thus the bracket is linear in the second entry. Now we show that the bracket of something with itself is zero.

$$[(s_1, x_1), (s_1, x_1)] = ([s_1, s_1], [x_1, x_1] + \theta(s_1)x_1 - \theta(s_1)x_1) = (0, 0)$$

I am too lazy to prove that the Jacobi identity holds for this Lie algebra, because it’s very technical and boring. 

Proposition 8.17 (Exercise 3.9ii). In the construction in the above proposition, the bracket on $S \oplus I$ is a semidirect product of $I$ by $S$.

Proof. To show: $\{0\} \oplus I$ is an ideal of $S \oplus I$ and $S \oplus \{0\}$ is a subalgebra of $S \oplus I$. Let $(s_1, 0), (s_2, 0) \in S \oplus \{0\}$. Then

$$[(s_1, 0), (s_2, 0)] = ([s_1, s_2], [0, 0] + \theta(s_1)0 - \theta(s_2)0) = ([s_1, s_2], 0)$$

Thus $S \oplus \{0\}$ is a subalgebra of $S \oplus I$. Let $(s_1, x_1) \in S \oplus I$, and $(0, x_2) \in I$. Then

$$[(s_1, x_1), (0, x_2)] = ([s_1, 0], [x_1, x_2] + \theta(s_1)x_2 - 0x_1) = ([s_1, x_2] + \theta(s_1)x_2)$$

Thus $I \oplus \{0\}$ is an ideal of $S \oplus I$. 

56
9 Chapter 4 Exercises

Proposition 9.1 (Exercise on page 31, section 4.2). Let $L$ be a Lie algebra. Then $L^{(k)} \subseteq L^k$. As a consequence, every nilpotent algebra is solvable.

Proof. This is true for $n = 1$ since $L^{(1)} = L^1 = L$. Suppose that $L^{(n)} \subseteq L^n$ for some $n \in \mathbb{N}$. Then

$$L^{(n+1)} = [L^{(n)}, L^{(n)}] = \text{span}\{[x, y] : x, y \in L^{(n)}\} \subseteq \text{span}\{[x, y] : x, y \in L^n\}$$

$$L^{n+1} = [L, L^n] = \text{span}\{[x, y] : x \in L, y \in L^n\}$$

Since $L^n \subseteq L$,

$$L^{(n+1)} \subseteq \text{span}\{[x, y] : x, y \in L^n\} \subseteq \text{span}\{[x, y] : x \in L, y \in L^n\} \subseteq L^{n+1}$$

Thus by induction, $L^{(k)} \subseteq L^k$ for all $k \in \mathbb{N}$. This implies that every nilpotent algebra is solvable, because if $L^k = 0$, then $L^{(k)} \subseteq L^k = 0$ so $L^{(k)} = 0$.

Proposition 9.2 (Exercise 4.1). Let $\phi : L_1 \to L_2$ be an onto homomorphism. Then $\phi(L_1^{(k)}) = L_2^{(k)}$.

Proof. The statement is true for $k = 1$ as proved in Exercise 2.8a. Suppose the statement is true for $k = n$. We will show that this implies that it is true for $k = n + 1$.

$$\phi(L_1^{(n+1)}) = \phi([L_1^{(n)}, L_1^{(n)}])$$

$$= \phi(\text{span}\{[x, y] : x, y \in L_1^{(n)}\})$$

$$= \text{span}\{\phi([x, y]) : x, y \in L_1^{(n)}\}$$

$$= \text{span}\{\phi(x), \phi(y) : x, y \in L_1^{(n)}\}$$

$$= \text{span}\{[w, z] : w, z \in L_2^{(n)}\} \quad \text{since } \phi \text{ is onto}$$

$$= [L_2^{(n)}, L_2^{(n)}]$$

$$= L_2^{(n+1)}$$

Thus by induction the statement is true for all $k \in \mathbb{N}$.

Definition 9.3 (definition for Exercise 4.2). $\text{sp}(2k, \mathbb{C}) = \text{gl}_S(2k, \mathbb{C})$ where $S$ is the matrix

$$\begin{pmatrix}
0 & I_k \\
-I_k & 0
\end{pmatrix}$$

Proposition 9.4 (Exercise 4.2). For $x \in \text{gl}(2k, \mathbb{C})$, $x \in \text{sp}(2k, \mathbb{C})$ if and only if $x$ is of the form

$$\begin{pmatrix}
m & p \\
q & -m^t
\end{pmatrix}$$

for square $k \times k$ matrices $p, q, m$ where $p, q$ are symmetric.
Proof. Suppose that \( x \in \text{gl}(2k, \mathbb{C}) \) is of the proposed form. Then

\[
x^t S = \begin{pmatrix} m^t & q \\ p & -m \end{pmatrix} \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix} = \begin{pmatrix} -q & m^t \\ m & p \end{pmatrix}
\]

\[
-Sx = \begin{pmatrix} 0 & -I_k \\ I_k & 0 \end{pmatrix} \begin{pmatrix} m & p \\ q & -m^t \end{pmatrix} = \begin{pmatrix} -q & m^t \\ m & p \end{pmatrix}
\]

Thus \( x^t S = -Sx \). Now suppose that \( x \in \text{gl}(2k, \mathbb{C}) \). Then

\[
x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

for some \( a, b, c, d \in \text{gl}(k, \mathbb{C}) \). We know that \( x^t S = -Sx \), so

\[
\begin{pmatrix} a^t & c^t \\ b^t & d^t \end{pmatrix} \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix} = \begin{pmatrix} 0 & -I_k \\ I_k & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

\[
\begin{pmatrix} -c^t & a^t \\ -d^t & b^t \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}
\]

Thus \( c = c^t, b = b^t \), and \( d = -a^t \). Thus \( x \) is of the desired form.

---

**Proposition 9.5 (Exercise 4.3).** Let \( L \) be a solvable Lie algebra. Then \( \text{ad} L \) is a solvable subalgebra of \( \text{gl}(L) \).

**Proof.** We know that \( \text{ad} : L \to \text{gl}(L) \) is a homomorphism, so \( \text{ad} L \) is a subalgebra of \( \text{gl}(L) \). By Lemma 4.4a, every homomorphic image of \( L \) is solvable, so \( \text{ad} L \) is solvable.

---

**Proposition 9.6 (Exercise 4.3).** If \( L \) is Lie algebra such that \( \text{ad} L \) is a solvable Lie subalgebra of \( \text{gl}(L) \), then \( L \) is solvable.

**Proof.** We know that \( \ker \text{ad} = Z(L) \), by the First Isomorphism Theorem, \( L/Z(L) \cong \text{ad} L \). Since \( Z(L)^t = 0 \), \( Z(L) \) is solvable, and by hypothesis \( \text{ad} L \) is solvable, so \( L/Z(L) \) is solvable. Then \( Z(L) \) is an ideal of \( L \) with \( Z(L) \) and \( L/Z(L) \) solvable, so by Lemma 4.4b, \( L \) is solvable.

---

**Lemma 9.7 (for Exercise 4.3).** Let \( \phi : L_1 \to L_2 \) be an onto homomorphism. Then \( \phi(L^k) = L_2^k \).

**Proof.** We know that \( \phi(L_1) = L_2 \) and \( \phi(L_1^n) = L_2^n \) by Exercise 2.8a. Suppose that \( \phi(L_1^n) = L_2^n \) for some \( n \). Then

\[
\phi(L_1^{n+1}) = \phi(\text{span}\{x, y] : x \in L_1, y \in L_1^n\})
\]

\[
= \text{span}\{\phi([x, y]) : x \in L_1, y \in L_1^n\}
\]

\[
= \text{span}\{[\phi(x), \phi(y)] : x \in L_1, y \in L_1^n\}
\]

\[
= \text{span}\{[w, z] : L \in L_2, z \in L_2^n\}
\]

\[
= L_2^{n+1}
\]

Thus by induction, \( \phi(L_1^k) = L_2^k \) for all \( k \in \mathbb{N} \).
Lemma 9.8 (for Exercise 4.3). Let $L$ be a nilpotent Lie algebra, and let $\phi : L \to M$ be a homomorphism. Then $\phi(L)$ is a nilpotent subalgebra of $M$.

Proof. We know from Exercise 1.6 that $\phi(L)$ is a subalgebra of $M$. Since $L$ is nilpotent, $L^k = 0$ for some $k$. Then by the previous lemma, $\phi(L^k) = \phi(L)^k = 0$, so $\phi(L)$ is nilpotent. \hfill \square

Proposition 9.9 (Exercise 4.3). If $L$ is nilpotent, then $\text{ad} L$ is a nilpotent subalgebra of $\text{gl}(L)$.

Proof. Let $\pi : L \to L/Z(L)$ be defined by $\pi(x) = x + Z(L)$. This is an onto homomorphism by Exercise 2.3ii. Thus since $L$ is nilpotent, $\phi(L) = L/Z(L)$ is nilpotent. By the 1st Isomorphism Theorem, $L/\text{ker ad} = L/Z(L) \cong \text{ad} L$ so $\text{ad} L$ is nilpotent. \hfill \square

Proposition 9.10 (Exercise 4.3). If $\text{ad} L$ is a nilpotent subalgebra of $\text{gl}(L)$, then $L$ is nilpotent.

Proof. We know that $\text{ad} : L \to \text{gl}(L)$ is a homomorphism, with $\text{ker ad} = Z(L)$. By the 1st Isomorphism Theorem, $L/\text{ker ad} = L/Z(L) \cong \text{ad} L$. Thus $L/Z(L)$ is nilpotent, so by Lemma 4.9b, $L$ is nilpotent. \hfill \square

Proposition 9.11 (Exercise 4.4). Let $L = n(n, F)$. Then $L^k$ has a basis consisting of $e_{ij}$ where $i < j - k$. Thus $L$ is nilpotent. Furthermore, the smallest $k$ such that $L^k = 0$ is $k = n$.

Proof. This is true for $k = 0$ by definition of $n(n, F)$. Suppose it is true for some $k \geq 0$. Then

$$L^{k+1} = [L, L^k]$$

$$= \text{span}\{[x, y] : x \in L, y \in L^k\}$$

$$= \text{span}\{e_{ij}, e_{ab} : i < j, a + k < b\}$$

$$= \text{span}\{\delta_{ja} e_{ib} - \delta_{ib} e_{aj} : i < j, a + k < b\}$$

$$\delta_{ja} e_{ib} - \delta_{ib} e_{aj} = \begin{cases} 0 & j \neq a, i \neq b \\ e_{id} & j = a, i \neq b \\ -e_{kj} & j \neq a, i = b \\ e_{ii} - e_{aa} & j = a, i \neq b \end{cases}$$

The fourth possibility never happens since we know that $i < j$. The first case contributes nothing to the span. In the second case, we have $e_{ib}$ where $i < j = a < b - k$ so $i < b - k$. Likewise in the third case, we have $-e_{aj}$ where $a + k < b = j$ so $a < j - k$. Thus

$$L^{k+1} = \text{span}\{e_{ij} : i < j - k\}$$

So by induction this is true for all $k \in \mathbb{N}$.

Now we show that $n(n, F)$ is nilpotent. Let $k = n$. Then $L^k$ has a basis $e_{ij}$ where $i < j - n$, but this is an empty set, since $j \leq n$ and $i \geq 1$. Thus $L^k = 0$. To see that $n$ is the smallest such $k$, suppose that $k < n$. Then $L^k$ has a basis of $e_{ij}$ with $i < j - k$ and if $k < n$ this is not empty. \hfill \square
Proposition 9.12 (Exercise 4.5i). \( b(n, F)' = n(n, F) \)

Proof.

\[
\begin{align*}
\mathbf{b}(n, F) &= \text{span}\{e_{ab} : a \leq b\} \\
\mathbf{b}(n, F)' &= \text{span}\{[e_{ij}, e_{kl}] : i \leq j, k \leq l\} \\
&= \text{span}\{\delta_{jk}e_{il} - \delta_{il}e_{kj} : i \leq j, k \leq l\}
\end{align*}
\]

\[
\delta_{jk}e_{il} - \delta_{il}e_{kj} = \begin{cases} 
0 & j \neq k, i \neq l \\
e_{il} & j = k, i \neq l \\
-e_{kj} & j \neq k, i = l \\
e_{ii} - e_{kk} & j = k, i \neq l 
\end{cases}
\]

In the fourth case, \( i \leq j = k \leq l = i \), so \( i = j = k = l \), so this bracket product turns out to be \( e_{ii} - e_{ii} = 0 \), so it contributes nothing to the span.

In the second case, we get \( e_{il} \) where \( i \leq j = k \leq l \) and \( i \neq l \), so \( i < l \). In the third case, we have \(-e_{kj}\) where \( k \leq l = i \leq j \) and \( k \neq j \) so \( k < j \). Thus all of the bracket products in \( \mathbf{b}(n, F) \) are of the form \( \pm e_{ij} \) where \( i < j \). Thus

\[
\mathbf{b}(n, F) = \text{span}\{e_{ij} : i < j\} = n(n, F)
\]

Proposition 9.13 (Exercise 4.5ii). Let \( L = \mathbf{b}(n, F) \). Then as basis for \( L^{(m)} \) is

\[
\{e_{ij} : i \leq j - 2^{k-1}\}
\]

Proof. For \( k = 1 \) this is true by 4.5i. Suppose that it is true for some \( m \geq 1 \). Then \( L^{(m)} = \text{span}\{e_{ij} : i \leq j - 2^{k-1}\} \). Then

\[
\begin{align*}
L^{(m+1)} &= [L^{(m)}, L^{(m)}] \\
&= \text{span}\{[e_{ij}, e_{kl}] : i \leq j - 2^{m-1}, k \leq l - 2^{m-1}\} \\
&= \text{span}\{\delta_{jk}e_{il} - \delta_{il}e_{kj} : i \leq j - 2^{m-1}, k \leq l - 2^{m-1}\}
\end{align*}
\]

\[
\delta_{jk}e_{il} - \delta_{il}e_{kj} = \begin{cases} 
0 & j \neq k, i \neq l \\
e_{il} & j = k, i \neq l \\
-e_{kj} & j \neq k, i = l \\
e_{ii} - e_{kk} & j = k, i \neq l 
\end{cases}
\]

Since \( m \geq 1, 2^{m-1} \geq 0 \), so \( i \leq j - 2^{m-1} \) implies that \( i < j \). This rules out the fourth case, since in the fourth case \( i = j = k = l \). Thus in both the first and fourth cases, the the brackets do not contribute to the span. In the second case,

\[
i \leq j - 2^{m-1} \implies i + 2^{m-1} \leq j = k \leq l - 2^{m-1} \implies i + 2^m \leq l \implies i \leq l - 2^m
\]

60
And in the third case,
\[ k \leq l - 2^{m-1} \implies k + 2^{m-1} \leq l = i \leq j - 2^{m-1} \implies k \leq j - 2^m \]
Thus \(L^{(m+1)}\) is spanned by matrices of the form \(e_{ij}\) where \(i \leq j - 2^{m-1}\). Thus
\[ L^{(m+1)} = \text{span}\{e_{ij} : i \leq j - 2^{m-1}\} \]
So by induction, the proposition is true for all \(k \in \mathbb{N}\).  

**Proposition 9.14** (Exercise 4.5iii). \(L = b(n, F)\) is solvable, and the smallest \(k\) such that \(L^{(k)} = 0\) is the smallest integer \(k\) satisfying \(k > \log_2(n - 1) + 1\).

**Proof.** First we show that if \(k > \log_2(n - 1) + 1\), then \(L^{(k)} = 0\). If \(k\) satisfies this inequality, then
\[
k - 1 > \log_2(n - 1) \implies 2^{k-1} > n - 1 \implies 1 + 2^{k-1} > n
\]
Let \(e_{ij}\) be a basis element of \(L^{(k)}\). Then \(1 \leq i \leq j - 2^{k-1}\), which implies \(1 + 2^{k-1} \leq n\). However, the above just showed that \(1 + 2^{k-1} > n\), so there can be no \(i, j\) satisfying \(1 \leq i \leq j - 2^{k-1} \leq n\). Thus there are no basis elements of \(L^{(k)}\), so \(L^{(k)} = 0\), so \(L\) is solvable.

Now suppose that \(k \leq \log_2(n - 1) + 1\). Then
\[
k - 1 \leq \log_2(n - 1) \implies 2^{k-1} \leq n - 1 \implies 1 + 2^{k-1} \leq n
\]
So if \(i = 1\) and \(j = 1 + 2^{k-1}\), then
\[
1 + 2^{k-1} \leq i + 2^{k-1} \leq j \leq n + 2^{k-1} \implies 1 \leq i \leq j - 2^{k-1} \leq n
\]
So if \(k \leq \log_2(n - 1) + 1\), then \(L^{(k)}\) has a non-empty basis containing \(e_{ij}\) where \(i = 1, j = 1 + 2^{k-1}\). Thus the proposed value for \(k\) is a minimum to get \(L^{(k)} = 0\).  

**Proposition 9.15** (Exercise 4.6). Let \(L\) be a semisimple Lie algebra. Then \(L\) has no non-zero abelian ideals.

**Proof.** Let \(I\) be an abelian ideal of \(L\). Then \(I' = 0\), so \(I\) is solvable. Since \(L\) has no non-zero solvable ideals, \(I = 0\). Thus all abelian ideals of \(L\) are the zero ideal, so \(L\) has no non-zero abelian ideals.  

**Proposition 9.16** (Exercise 4.5iv). If \(n \geq 2\), then \(L = b(n, F)\) is not nilpotent.

**Proof.** We know that \(L' = n(n, F)\). We compute \(L^{(2)}:\)
\[
L^{(2)} = [L, L'] \\
= \text{span}\{[e_{ij}, e_{kl}] : i \leq j, k < l\} \\
= \text{span}\{\delta_{jk}e_{il} - \delta_{il}e_{kj} : i \leq j, k < l\}
\]
\[
\delta_{jk}e_{il} - \delta_{il}e_{kj} = \begin{cases} 
0 & j \neq k, i \neq l \\
e_{il} & j = k, i \neq l \\
-e_{kj} & j \neq k, i = l \\
e_{ii} - e_{kk} & j = k, i \neq l
\end{cases}
\]

61
In the first cases, nothing is contributed to the span. The fourth case can never happen, since it would imply \( j = k < l = i \leq j \) which means \( j < j \), an impossibility. In the second case, we have \( e_{il} \) where \( i \leq j = k < l \) so \( i < l \), and in the third case we have \(-e_{kj}\) where \( k < l = i \leq j \) so \( k < j \).

Thus \( L^{(2)} \) is the span of matrices \( e_{ij} \) with \( i < j \), which is \( n(n, F) \). Thus \( L^{(k)} = n(n, F) \) for all \( k \in \mathbb{N} \), so \( L \) is not nilpotent.

**Lemma 9.19** (Exercise 4.7). Let \( I \) be an ideal.

**Proof.** Since \( I \) is an ideal, \( [I, e_{kl}] = 0 \) for some \( k \). Let \( m \) be the minimum of all such \( k \), so \( I^{(m)} = 0 \) but \( I^{(m-1)} \neq 0 \). Then \( I^{(m-1)} \) is an abelian ideal of \( L \), so \( I^{(m-1)} = 0 \). So we have a contradiction, that \( I^{(m-1)} = 0 \) and \( I^{(m-1)} \neq 0 \). Thus we conclude that \( L \) has no solvable ideals.

**Lemma 9.18** (Exercise 4.7). Let \( I \subset \mathfrak{sl}(n, \mathbb{C}) \) be an ideal with \( e_{ij} \in I \) for some \( i \neq j \). Then \( e_{ii} - e_{jj} \in I \).

**Proof.** Since \( I \) is an ideal, \( [e_{ij}, e_{ji}] = \delta_{jj} e_{ii} - \delta_{ii} e_{jj} = e_{ii} - e_{jj} \in I \).

**Lemma 9.19** (Exercise 4.7). Let \( I \subset \mathfrak{sl}(n, \mathbb{C}) \) be an ideal with \( e_{ii} - e_{jj} \in I \) for some \( i \neq j \). Then \( e_{im}, e_{mi} \in I \) for all \( m \neq i \) and \( e_{jm}, e_{mj} \in I \) for all \( m \neq j \).

**Proof.** First we compute the bracket of \( h \) with some general \( e_{kl} \) (with \( k \neq l \)) which we know is in \( I \) since \( I \) is an ideal.

\[
[h, e_{kl}] = [e_{ii}, e_{kl}] - [e_{i+1,i+1}, e_{kl}]
= (\delta_{ik} e_{il} - \delta_{il} e_{ik}) - (\delta_{jk} e_{jl} - \delta_{jl} e_{jk})
= \delta_{ik} e_{kl} - \delta_{il} e_{kl} - \delta_{jk} e_{kl} + \delta_{jl} e_{kl}
= (\delta_{ik} - \delta_{jk} + \delta_{jl} - \delta_{il}) e_{kl}
\]

Now we need to enumerate the cases for this coefficient involving several Kronecker deltas.

\[
\delta_{ik} - \delta_{jk} = \begin{cases}
1 & i = k \\
0 & i \neq k, j \neq k \\
-1 & j = k
\end{cases}
\]

\[
\delta_{jl} - \delta_{il} = \begin{cases}
1 & j = l \\
0 & j \neq l, i \neq l \\
-1 & i = l
\end{cases}
\]

\[
\delta_{ik} - \delta_{jk} + \delta_{jl} - \delta_{il} = \begin{cases}
2 & i = k, j = l \\
1 & i = k, j \neq l \text{ OR } j = l, i \neq k \\
0 & i \neq k, i \neq l, j \neq k, j \neq l \\
-1 & j = k, i \neq l \text{ OR } i = l, j \neq k \\
-2 & i = l, j = k
\end{cases}
\]

So we see that the only time that this coefficient is zero is when \( i \neq k, i \neq l, j \neq k, j \neq l \). Now suppose \( I \subset \mathfrak{sl}(n, \mathbb{C}) \) is an ideal containing \( h = e_{ii} - e_{jj} \), and \( m \neq i \). By the previous computation,

\[
[h, e_{mi}] = \lambda_{1} e_{mi}
\]

\[
[h, e_{im}] = \lambda_{2} e_{im}
\]
where \( \lambda_1, \lambda_2 \in \{-2, -1, 1, 2\} \), and thus \( e_{mi}, e_{im} \in I \). Likewise for \( m \neq j \), we have

\[
[h, e_{mj}] = \lambda_3 e_{mj} \\
[h, e_{jm}] = \lambda_3 e_{jm}
\]

where \( \lambda_3, \lambda_4 \in \{-2, -1, 1, 2\} \), and thus \( e_{mj}, e_{jm} \in I \).

**Lemma 9.20** (Exercise 4.7). Let \( I \subset \mathfrak{sl}(n, \mathbb{C}) \) be an ideal such that \( e_{ii} - e_{jj} \in I \). Then \( e_{ii} - e_{kk} \in I \) for all \( 1 \leq k, l \leq n \) with \( k \neq l \).

**Proof.** We have \( e_{ii} - e_{jj} \in I \). By Lemma 9.19, \( e_{ik}, e_{li} \in I \) for each \( k, l \neq i \). Then by Lemma 9.18, \( e_{ii} - e_{kk}, e_{ll} - e_{ii} \in I \) for each \( k, l \neq i \). Then since \( I \) is a vector subspace,

\[ (e_{ii} - e_{kk}) + (e_{ll} - e_{kk}) = e_{ll} - e_{kk} \in I \]

This gives us each \( e_{ll} - e_{kk} \) where \( l, k \neq i \). The same process may be used to generate each \( e_{ll} - e_{kk} \) where \( l, k \neq i \). Note that along the way, we generated each \( e_{ii} - e_{kk} \) with \( i \neq k \), and finally note that we began by having \( e_{ii} - e_{jj} \), which covers the case where \( i = l, j = k \).

**Lemma 9.21** (Exercise 4.7). Let \( I \subset \mathfrak{sl}(n, \mathbb{C}) \) be an ideal such that \( e_{ij} \in I \) or \( e_{ii} - e_{jj} \in I \) for some \( i \neq j \). Then \( I = \mathfrak{sl}(n, \mathbb{C}) \).

**Proof.** If \( e_{ij} \in I \), then \( e_{ii} - e_{jj} \in I \), so either way we may assume \( e_{ii} - e_{jj} \in I \) for some \( i \neq j \). Then by Lemma 9.20, \( I \) contains all diagonal elements of \( \mathfrak{sl}(n, \mathbb{C}) \). Let \( k \neq l \) with \( 1 \leq k, l \leq n \). Then \( e_{kk} - e_{ll} \in I \), so by Lemma 9.19, \( e_{kl} \in I \). Thus \( I \) contains the standard basis for \( \mathfrak{sl}(n, \mathbb{C}) \), so \( I = \mathfrak{sl}(n, \mathbb{C}) \).

The above lemma gives the significance of the previous three lemmas. This lemma says that any nonzero ideal of \( \mathfrak{sl}(n, \mathbb{C}) \) containing just one of the usual basis elements is the entirety of \( \mathfrak{sl}(n, \mathbb{C}) \). Now we just need to show that any nonzero ideal of \( \mathfrak{sl}(n, \mathbb{C}) \) contains one of these usual basis elements.

**Lemma 9.22** (Exercise 4.7). Let \( v \in \mathfrak{sl}(n, \mathbb{C}) \) where

\[
v = \sum_{i \neq j} c_{ij} e_{ij} + \sum_{i=1}^n d_i e_{ii}
\]

and let \( k \neq l \). Then

\[
[e_{kl}, [e_{kl}, v]] = -2c_{lk} e_{kl}
\]

**Proof.** We can write \( v \) as \( v = d + n \) where \( d \) is a diagonal matrix and \( n \) is a matrix with zeros on the diagonal. Then by linearity of the bracket,

\[
[e_{kl}, [e_{kl}, v]] = [e_{kl}, [e_{kl}, d] + [e_{kl}, n]] = [e_{kl}, [e_{kl}, d]] + [e_{kl}, [e_{kl}, n]]
\]

We claim that \( [e_{kl}, [e_{kl}, d]] = 0 \). By a previous lemma, since \( d \) is diagonal, \( [e_{kl}, d] = \lambda e_{kl} \), so then we have a bracket of \( e_{kl} \) with a multiple of \( e_{kl} \), which will be zero. So \( [e_{kl}, [e_{kl}, v]] = 63 \)
\[ [e_{kl}, [e_{kl}, v]] = \sum_{i \neq j} c^{ij} [e_{kl}, [e_{kl}, e_{ij}]] \]
\[ = \sum_{i \neq j} c^{ij} [e_{kl}, \delta_{il}e_{kj} - \delta_{kj}e_{il}] \]
\[ = \sum_{i \neq j} c^{ij} (\delta_{il}e_{kj} - \delta_{kj}e_{il}) \]
\[ = \sum_{i \neq j} c^{ij} (\delta_{il}\delta_{kj}e_{kl} - \delta_{kj}\delta_{il}e_{kl}) \]
\[ = \sum_{i \neq j} c^{ij} (\delta_{kl}e_{ij} - \delta_{kl}e_{ij} - 2\delta_{il}\delta_{kj}e_{kl}) \]

At this point, note that since \( k \neq l \), we have \( \delta_{kl} = \delta_{lk} = 0 \), so we can cross out the \( e_{ij} \) terms.

\[ [e_{kl}, [e_{kl}, v]] = \sum_{i \neq j} c^{ij}(-2\delta_{il}\delta_{kj}e_{kl}) = (-2) \sum_{i \neq j} c^{ij}\delta_{il}\delta_{kj}e_{kl} \]

The only nonzero term of this summation occurs when \( i = l \) and \( k = j \), so

\[ [e_{kl}, [e_{kl}, v]] = -2c^{lk}e_{kl} \]

\[ \square \]

**Lemma 9.23** (Exercise 4.7). Let \( v \in \mathfrak{sl}(n, \mathbb{C}) \) be a diagonal matrix. We can write \( v \) as \( \sum_{i=1}^{n} d^{i}e_{ii} \) (where \( \sum_{i} d^{i} = 0 \)). Then for \( k \neq l \),

\[ [v, e_{kl}] = (d^{k} - d^{l})e_{kl} \]

**Proof.**

\[ [v, e_{kl}] = \sum_{i=1}^{n} d^{i} [e_{ii}, e_{kl}] = \sum_{i=1}^{n} d^{i}(\delta_{ik}e_{il} - \delta_{il}e_{ki}) = e_{kl} \sum_{i=1}^{n} d^{i}(\delta_{ik} - \delta_{il}) = e_{kl}(d^{k} - d^{l}) \]

\[ \square \]

**Lemma 9.24** (Exercise 4.7). Let \( v \in \mathfrak{sl}(n, \mathbb{C}) \) be a nonzero diagonal matrix. Then there exist \( k, l \) with \( k \neq l \) such that \( [v, e_{kl}] = \lambda e_{kl} \) for some \( \lambda \neq 0 \) (\( \lambda \in \mathbb{C} \)).

**Proof.** Suppose \( v = \sum_{i=1}^{n} d^{i}e_{ii} \), where \( \sum_{i=1}^{n} d_{i} = 0 \) and some \( d^{i} \neq 0 \). Then for \( k \neq l \), by previous lemma, \( [v, e_{kl}] = (d^{k} - d^{l})e_{kl} \). Suppose to the contrary that \( [v, e_{kl}] = 0 \) for all \( k, l \). Then \( d^{k} - d^{l} = 0 \) \( \Rightarrow \) \( d^{k} = d^{l} \) for all \( k, l \). But then since \( \sum_{i=1}^{n} d_{i} = 0 \), this implies that \( d_{i} = 0 \) for all \( i \). But \( v \) is nonzero by hypothesis, so we conclude that for some \( d^{k}, d^{l} \), we have \( d^{k} - d^{l} \neq 0 \), so we reach our desired conclusion.

\[ \square \]
Recall that 9.21 tells us that any ideal of \( \text{sl}(n, \mathbb{C}) \) that contains just one of the usual basis elements contains all of \( \text{sl}(n, \mathbb{C}) \). Lemmas 9.22, 9.23, 9.24 allow us to show that any nonzero ideal of \( \text{sl}(n, \mathbb{C}) \) contains one of the usual basis elements.

**Lemma 9.25** (Exercise 4.7). Let \( I \subset \text{sl}(n, \mathbb{C}) \) be a nonzero ideal. Then \( e_{ij} \in L \) for some \( i \neq j \).

**Proof.** Let \( v \in L \) be nonzero, and write \( v \) as

\[
v = \sum_{i \neq j} c^{ij} e_{ij} + \sum_{i=1}^{n} d^i e_{ii}
\]

Suppose that \( v \) is not diagonal, that is, that some \( c^{ij} \neq 0 \). Then by 9.22,

\[
[e_{ji}, [e_{ji}, v]] = -2c^{ij} e_{ji} \in I
\]

and since \( c^{ij} \neq 0 \), we have \( e_{ji} \in I \). Now suppose that \( v \) is diagonal. Then by 9.24, there exists \( e_{kl} \) such that

\[
[e_{kl}, v] = \lambda e_{kl}
\]

where \( \lambda \neq 0 \). Then \( e_{kl} \in I \). \( \square \)

**Theorem 9.26** (Exercise 4.7). \( \text{sl}(n, \mathbb{C}) \) is a simple Lie algebra for \( n \geq 2 \).

**Proof.** Let \( I \) be a nonzero ideal of \( \text{sl}(n, \mathbb{C}) \). By 9.25, there exists \( e_{ij} \in I \) with \( i \neq j \). Then by 9.21, \( I = \text{sl}(n, \mathbb{C}) \). Thus \( \text{sl}(n, \mathbb{C}) \) has no nonzero proper ideals. \( \square \)

**Definition 9.27.** Let \( A \in \text{gl}(n, F) \). Then \( A_{ij} \) is the \((n-1) \times (n-1)\) matrix formed by deleting the \( i \)th row and \( j \)th column of \( A \).

**Proposition 9.28** (Exercise 4.9i). Let \( A \in \text{gl}(n, F) \) and let \( I_n \) be the identity matrix for \( \text{gl}(n, F) \), and let \( \lambda \in F \). Then \( \det(I_n + \lambda A) \) is a polynomial in \( \lambda \) with constant term 1 and linear term \( \lambda(\text{tr} A) \).

**Proof.** We show directly that this is true for \( n = 2 \). Let

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

Then

\[
\det(I + \lambda A) = \det \begin{pmatrix} 1 + \lambda a & \lambda b \\ \lambda c & 1 + \lambda d \end{pmatrix}
\]

\[
= (1 + \lambda a)(1 + \lambda d) + \lambda^2 bc
\]

\[
= 1 + \lambda(a + d) + \lambda^2(ad + bc)
\]

\[
= 1 + \lambda(\text{tr} A) + \lambda^2(ad + bc)
\]

65
Now we prove the general case by induction. Suppose that for every $A \in \text{gl}(k, F)$, we know that $\det(I + \lambda A) = 1 + \lambda(\text{tr} A) + \lambda^2 P(\lambda)$ where $P$ is some irrelevant polynomial in $\lambda$. Let $B \in \text{gl}(k + 1, F)$ where $B = (b_{ij})$. Then

$$I + \lambda B = \begin{pmatrix}
1 + \lambda b_{11} & \lambda b_{12} & \ldots \\
\lambda b_{21} & 1 + \lambda b_{22} & \ldots \\
\vdots & \vdots & \ddots
\end{pmatrix}$$

By Laplacian expansion along the first row,

$$\det(I + \lambda B) = (1 + \lambda b_{11}) \det(I_{11} + \lambda B_{11}) + \sum_{i=2}^{k+1} \lambda b_{1i} \det(I_{1i} + \lambda B_{1i})$$

We claim that the summation term contributes nothing to the constant or linear terms of this polynomial in $\lambda$. Note that for $i \neq 1$, $I_{1i}$ will always have a zero row, so $I_{1i} + \lambda B_{1i}$ has a row where every entry is divisible by $\lambda$. One could compute the determinant of $I_{1i} + \lambda B_{1i}$ by Laplacian expansion along this row, and every term in the sum would be divisible by $\lambda$, so we can conclude that for every $2 \leq i \leq k + 1$, $\lambda | \det(I_{1i} + \lambda B_{1i})$. Thus each term in the sum

$$\sum_{i=2}^{k+1} \lambda b_{1i} \det(I_{1i} + \lambda B_{1i})$$

is divisible by $\lambda^2$. Thus it contributes nothing to the constant or linear term. Finally, utilizing our inductive hypothesis,

$$\det(I_{11} + \lambda B_{11}) = 1 + \lambda \text{tr} B_{11} + \lambda^2 P_1(\lambda)$$

Thus

$$\det(I + \lambda B) = (1 + \lambda b_{11})(\det(I_{11} + \lambda B_{11})) + \lambda^2 P_2(\lambda)$$

$$= (1 + \lambda b_{11})(1 + \lambda \text{tr} B_{11} + \lambda^2 P_1(\lambda)) + \lambda^2 P_2(\lambda)$$

$$= 1 + \lambda \text{tr} B_{11} + \lambda^2 P_2(\lambda) + \lambda b_{11} + \lambda^2 b_{11} \text{tr} B_{11} + \lambda^3 P_2(\lambda) + \lambda^2 P_1(\lambda)$$

$$= 1 + \lambda(\text{tr} B_{11} + b_{11}) + \lambda^2 P_3(\lambda)$$

$$= 1 + \lambda \text{tr} B + \lambda^2 P_3(\lambda)$$

Where $P_1, P_2, P_3$ are are all irrelevant polynomials in $\lambda$. This completes the proof by induction. \hfill \Box

**Proposition 9.29** (Exercise 4.9iia). Let $S \in \text{gl}(n, \mathbb{C})$ and let $(,) : \mathbb{C}^n \to \mathbb{C}^n$ be the complex bilinear form with matrix $S$. Let

$$G_S(n, \mathbb{C}) = \{ A \in \text{GL}(n, \mathbb{C}) : (Av, Av) = (v, v) \text{ for } v \in \mathbb{C}^n \}$$

We claim that $G_S(n, \mathbb{C})$ is a group under usual matrix multiplication.
\textbf{Proof.} Clearly $I_n \in G_S(n, \mathbb{C})$, because $(I_nv, I_nv) = (v,v)$. Thus $G_S(n, \mathbb{C})$ has an identity. Associativity of matrix multiplication is inherited from $GL(n, \mathbb{C})$. Now we show that $GL(n, \mathbb{C})$ contains inverses. Let $A \in G_S(n, \mathbb{C})$. Since $A \in GL(n, \mathbb{C})$, $A^{-1}$ exists. Also, $(v,v) = (Av, Av)$, so
\[(A^{-1}v, A^{-1}v) = A^{-2}(v,v) = A^{-2}(Av, Av) = A^{-2}A^2(v,v) = (v,v)\]
Thus $A^{-1} \in G_S(n, \mathbb{C})$. Now we show that $G_S(n, \mathbb{C})$ is closed under matrix multiplication. Let $A, B \in G_S(n, \mathbb{C})$. Then $(v,v) = (Av, Av) = (Bv, Bv)$. Then
\[(ABv, ABv) = A^2(Bv, Bv) = A^2(v,v) = (Av, Av) = (v,v)\]
Thus $AB \in G_S(n, \mathbb{C})$. \hfill \Box

\textbf{Proposition 9.30} (Exercise 4.9iii). Let $G = \{A \in gl(n, \mathbb{C}) : A^t = A^{-1}\}$. Then $G$ is a group under matrix multiplication.

\textbf{Proof.} Clearly $I_n \in G$ since $I_n^t = I_n = I_n^{-1}$. Associativity is inherited from $gl(n, \mathbb{C})$. For $A \in G$, we have $(A^{-1})^t = (A^t)^{-1}$ so $A^{-1} \in G$. For $A, B \in G$, $(AB)^t = B^tA^t = B^{-1}A^{-1} = (AB)^{-1}$ so $AB \in G$. \hfill \Box
10  Chapter 5 Exercises

Proposition 10.1 (Exercise 5.1i). Let $V$ be a vector space over $F$, and let $A \subseteq \text{gl}(V)$ be a subalgebra. Let $\lambda : A \to F$ be a linear map. Let

$$V_\lambda = \{ v \in V : a(v) = \lambda(a)v \text{ for all } a \in A \}$$

Then $V_\lambda$ is a vector subspace of $V$.

Proof. $V_\lambda$ contains the zero vector since $a : V \to V$ and $\lambda : A \to F$ are linear maps, so $a(0) = 0, \lambda(0) = 0 \implies a(0) = \lambda(0)v = 0$. Let $v, w \in V_\lambda$. Then $a(v) = \lambda(a)v$ and $a(w) = \lambda(a)w$ for all $a \in A$. Then by linearity of $a$ and $\lambda$,

$$a(v) + a(w) = \lambda(a)v + \lambda(a)w$$

$$a(v + w) = \lambda(a)(v + w)$$

Thus $v + w \in V_\lambda$. Let $v \in V_\lambda, b \in F$. Then

$$a(v) = \lambda(a)v \implies ba(v) = b\lambda(a)v \implies a(bv) = \lambda(a)(bv)$$

Thus $bv \in V_\lambda$. Thus $V_\lambda$ is closed under vector addition and scalar multiplication, so it is a vector subspace of $V$. \qed

Proposition 10.2 (Exercise 5.1ii). Let $A = d(n, F) \subseteq \text{gl}(n, F)$ and let $V = F^n$. Let $\{e_1, \ldots, e_n\}$ be the standard basis for $V$. For $a \in A$, denote the entries by $a^i$, that is,

$$a = \begin{pmatrix} a^1 & 0 & \ldots \\ 0 & a^2 & \ldots \\ \vdots & \vdots & \ddots \\ 0 & \ldots & a^n \end{pmatrix}$$

Define $\epsilon_i : A \to F$ by $\epsilon_i(a) = a^i$. Then $V_{\epsilon_i} = \text{span}\{e_i\}$ and $V = V_{\epsilon_1} \oplus V_{\epsilon_2} \ldots \oplus V_{\epsilon_n}$.

Proof. By definition,

$$V_{\epsilon_i} = \{ v \in V : a(v) = \lambda(a)v \text{ for all } a \in A \}$$

$$= \{ v \in V : a(v) = a^iv \text{ for all } a \in A \}$$

Now we compute $a(v)$:

$$a(v) = \begin{pmatrix} a^1 & 0 & \ldots \\ 0 & a^2 & \ldots \\ \vdots & \vdots & \ddots \\ 0 & \ldots & a^n \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{pmatrix} = \begin{pmatrix} a^1v^1 \\ a^2v^2 \\ \vdots \\ a^nv^n \end{pmatrix}$$

For $v \in V_{\epsilon_i}, a(v) = a^iv$. So we have

$$\begin{pmatrix} a^1v^1 \\ a^2v^2 \\ \vdots \\ a^nv^n \end{pmatrix} = \begin{pmatrix} a^1v^1 \\ a^2v^2 \\ \vdots \\ a^nv^n \end{pmatrix}$$

68
The only $v$ for which this holds for all $a^1, a^2, \ldots, a^n$ is

$$v = \begin{pmatrix}
0 \\
0 \\
\vdots \\
v^i \\
\vdots \\
0
\end{pmatrix}$$

Thus $V_{i} = \text{span}\{e_i\}$. Since $V = \text{span}\{e_1, e_2, \ldots, e_n\}$,

$$V = \text{span}\{e_1\} \oplus \text{span}\{e_2\} \oplus \cdots \oplus \text{span}\{e_n\}$$

$$= V_{i} \oplus V_{e_2} \cdots \oplus V_{e_n} \tag*{\blacksquare}$$

**Proposition 10.3** (Exercise 5.2). Let $V = F^n$ and let $A = b(n, F)$. Then $e_1 = (1, 0, \ldots, 0) \in V$ is an eigenvector of $A$ with eigenvalue 0. Additionally, then linear map $\lambda : A \rightarrow F$ defined by $\lambda(a) = 0$ is a weight for $A$ and the corresponding weight space is $V_{\lambda} = \text{span}\{e_1\}$.

**Proof.** Let $a \in A$. Then

$$a(e_1) = \begin{pmatrix}
0 & a_{12} & a_{13} & \cdots \\
0 & 0 & a_{23} & \cdots \\
\vdots & \vdots & \vdots & \ddots \\
0 & 0 & 0 & \cdots
\end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} = 0e_1$$

thus $e_1$ is an eigenvector of $A$ with eigenvalue 0. Let $\lambda : A \rightarrow F$ be defined by $\lambda(a) = 0$. Then

$$V_{\lambda} = \{v \in V : a(v) = \lambda(a)v \text{ for all } a \in A\}$$

is non-empty since $e_1 \subseteq V_{\lambda}$. Thus $\lambda$ is a weight for $A$. Specifically,

$$V_{\lambda} = \{v \in V : a(v) = 0 \text{ for all } a \in A\}$$

so for $v \in V_{\lambda}$, we have

$$a(v) = \begin{pmatrix}
0 & a_{12} & a_{13} & \cdots \\
0 & 0 & a_{23} & \cdots \\
\vdots & \vdots & \vdots & \ddots \\
0 & 0 & 0 & \cdots
\end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{pmatrix} = \begin{pmatrix}
v^2a_{12} + v^3a_{13} + \cdots + v^na_{1n} \\
v^3a_{23} + v^4a_{24} + \cdots + v^na_{2n} \\
\vdots \\
0
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

thus $v^2, v^3 \ldots v^n = 0$, so $V_{\lambda} = \text{span}\{e_1\}. \tag*{\blacksquare}$

**Definition 10.4.** Let $V$ be a vector space and let $a \in \text{gl}(V)$. The **centraliser of** $a$ in $\text{gl}(V)$ is

$$\text{CL}_a = \{x \in \text{gl}(V) : a \circ x = x \circ a\}$$

We show in the next lemma that $\text{CL}_a$ is a subalgebra of $\text{gl}(V)$. 69
Lemma 10.5 (Lemma for Exercise 5.3). $CL_a$ is a subalgebra of $gl(V)$.

Proof. We need to show that for $x, y \in CL_a$, $[x, y] \in CL_a$.

$$a \circ [x, y] = a(xy - yx) = axy - ayx = xay - yxa = [x, y] \circ a$$

Thus $[x, y] \in CL_a$. □

Proposition 10.6 (Exercise 5.3). The result in 5.3 (page 39) is a specific case of Lemma 5.4.

Proof. Let $a, b \in gl(V)$, so $a, b : V \rightarrow V$ are linear and suppose that $a \circ b = b \circ a$. Let $L = CL_a$. As shown, $L$ is a subalgebra of $gl(V)$, and by definition, $b \in L$.

Let $A = \text{span}\{a\}$. We claim that $A$ is an ideal of $L$. Clearly $a \in L$, since $a \circ a = a \circ a$.

For $\lambda a \in A$ and $x \in L$,

$$a \circ [\lambda a, x] = a(\lambda ax - \lambda xa) = \lambda(a^2x - axa) = \lambda(axa - axa) = 0$$

Thus $A$ is an ideal of $L$ since $[\lambda a, x] \in A$. Now let

$$W = \{v \in V : x(v) = 0 \text{ for all } x \in A\}$$

We claim that $W = \ker a$. Let $v \in W$. Then $x(v) = 0$ for all $x \in A$, and since $a \in A$, $a(v) = 0$. Thus $v \in \ker a$, so $W \subseteq \ker a$. Now let $v \in \ker a$. Then $a(v) = 0$, so $\lambda a(v) = 0$, so $(\lambda a)(v) = 0$ so $x(v) = 0$ for all $a \in A$. Thus $\ker a \subseteq W$.

Thus by Lemma 5.4, $W = \ker a$ is an $L$-invariant subspace of $V$, and since $b \in L$, $b(W) = b(\ker a) = \ker a$. This is precisely the result in 5.3, so that result is a special case of Lemma 5.4 □

Proposition 10.7 (Exercise on page 40). Let $V$ be a vector space over $F$ and let $a, b : V \rightarrow V$ be linear maps such that $a \circ b = b \circ a$. Let $\lambda \in F$, and let $V_\lambda = \{v \in V : a(v) = \lambda v\}$. Then $b(V_\lambda) \subseteq V_\lambda$.

Proof. Suppose that $x \in b(V_\lambda)$. Then $x = b(\lambda v)$ for some $v \in V_\lambda$. Then so $b \circ a(v) = \lambda b(v)$ so $a(bv) = \lambda(bv)$ so $x \in V_\lambda$, so $b(V_\lambda) \subseteq V_\lambda$. □

Proposition 10.8 (Exercise 5.4i). Let $L$ be a subalgebra of $gl(V)$. Suppose there is a basis $\beta$ for $V$ such that every $x \in L$ is represented by a strictly upper triangular matrix with respect to $\beta$. Then $L$ is isomorphic to a subalgebra of $n(n, F)$ and hence $L$ is nilpotent.

Proof. We have the usual map $[ ] : L \rightarrow n(n, F)$ where $[x]$ is the matrix of $x$ with respect to $\beta$. $[ ]$ maps into $n(n, F)$ by hypothesis. We know that $M$ is linear, that is,

$$[ax + y] = a[x] + [y]$$

$[ ]$ also preserves brackts, because it is linear:

$$[[x, y]] = [xy - yx] = [x][y] - [y][x] = [[x], [y]]$$

70
Thus \([\ ]\) is a homomorphism. The kernel of \(M\) is the zero map, so by the 1st Isomorphism Theorem,

\[
L/ \ker[\ ] \cong [L] \implies L \cong [L]
\]

and since \([\ ]\) is a homomorphism, \([L]\) is a subalgebra of \(n(n, F)\). Thus \(L\) is isomorphic to a subalgebra of \(n(n, F)\). Since \(n(n, F)\) is nilpotent, any subalgebra is also nilpotent, so \(L\) is nilpotent.

**Proposition 10.9** (Exercise 5.4ii). Let \(L\) be a subalgebra of \(\text{gl}(V)\). Suppose there is a basis \(\beta\) of \(V\) such that all \(x \in L\) are represented by upper triangular matrices with respect to \(\beta\). Then \(L\) is isomorphic to a subalgebra of \(\text{b}(n, F)\) and hence \(L\) is solvable.

**Proof.** Again we use the homomorphism \([\ ]: L \to \text{b}(n, F)\). \([\ ]\) maps into \(\text{b}(n, F)\) by hypothesis. By 5.4i, \([\ ]\) is a homomorphism, so \(L \cong [L]\) and \([L] \subseteq (n, F)\) is a subalgebra. Thus \(L\) is isomorphic to a subalgebra of \(\text{b}(n, F)\), so \(L\) is solvable.

**Proposition 10.10** (Exercise 5.6i). Let \(L\) be a Lie algebra and let \(A \subseteq L\) be a subalgebra. Define

\[
N_L(a) = \{ x \in L : [x, a] \in A \text{ for all } a \in A \}
\]

Then \(N_L(A)\) is a subalgebra of \(L\) and \(A \subseteq N_L(A)\).

**Proof.** First we show that \(A \subseteq N_L(A)\). Let \(a \in A\). Since \(A\) is a subalgebra, \([a, b] \in A\) for all \(b \in A\). Thus by definition of \(N_L(A), a \in N_L(A)\).

Now we show that \(N_L(A)\) is a subalgebra of \(L\). Suppose \(y, z \in N_L(A)\) and let \(a \in A\). We need to show that \([[y, z], a] \in A\). Using the Jacobi identity,

\[
[[y, z], a] = -[a, [y, z]] = [y, [z, a]] + [z, [a, y]]
\]

Since \(y, z \in N_L(A), [z, a], [a, y] \in A\). Thus \([[y, z], a], [z, [a, y]] \in A\), so \([[y, z], a] \in A\). Thus \([y, z] \in N_L(A), so N_L(A)\) is a subalgebra of \(L\).

**Proposition 10.11** (Exercise 5.6i). Let \(L\) be a Lie algebra and \(A \subseteq L\) be a subalgebra. Let \(B \subseteq L\) be a subalgebra such that \(A \subseteq B \subseteq L\) and \(A\) is an ideal of \(B\). Then \(B \subseteq N_L(A)\). (Thus \(N_L(A)\) is the largest subalgebra of \(L\) in which \(A\) is an ideal.)

**Proof.** We need to show that for \(b \in B\), we have \(b \in N_L(A)\). Let \(b \in B, a \in A\). Since \(A\) is an ideal of \(B\), \([a, b] \in A\), so \([b, a] \in A\). Then by definition of \(N_L(A), b \in N_L(A)\).

**Proposition 10.12** (Exercise 5.1ii). Let \(L = \text{gl}(n, \mathbb{C})\) and let \(A\) be the subalgebra of diagonal matrices. Then \(N_L(A) = A\).

**Proof.** We know that \(A \subseteq N_L(A), so we just need to show that \(N_L(A) \subseteq A\). Let \(x = (x_{ij}) \subseteq N_L(A)\). By the Invariance Lemma, any weight space of \(A\) is \(N_L(A)\)-invariant, that is, for any
weight space $V_\lambda$ of $A$, $x(V_\lambda) \subseteq V_\lambda$. As shown in Exercises 5.1ii, span$\{e_i\}$ is a weight space for $i = 1, 2, \ldots n$. (Recall that $\{e_i\}$ is the standard basis for $\mathbb{C}^n$.) Thus

$$(x_{ij})e_1 = \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix} \in \text{span}\{e_1\} \implies x_{k1} = 0 \text{ for } k = 2, 3, \ldots n$$

$$(x_{ij})e_2 = \begin{pmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{pmatrix} \in \text{span}\{e_2\} \implies x_{k2} = 0 \text{ for } k = 1, 3, 4, \ldots n$$

and we can do this for each $e_i$. As this demonstrates, $x_{ij} = 0$ for $i \neq j$. Thus $x$ is a diagonal matrix, so $x \in A$. Thus $N_L(A) \subseteq A$, so $N_L(A) = A$. \hfill \Box

**Proposition 10.13** (Exercise 5.7). Let $V$ be a vector space, and let $a, y \in \mathfrak{gl}(V)$. Then for any $m \geq 1$ (where $m \in \mathbb{N}$),

$$ay^m = y^m a + \sum_{k=1}^{m} \binom{m}{k} y^{m-k}a_k$$

where $a_1 = [a, y]$ and $a_k = [a_{k-1}, y]$ for $k \geq 2$.

**Proof.** This is true for $m = 1$ because

$$ay = ya + ay - ya = ya + [a, y] = ya + \sum_{k=1}^{1} \binom{1}{k} y^{1-k}a_k$$

Now suppose that

$$ay^m = y^m a + \sum_{k=1}^{m} \binom{m}{k} y^{m-k}a_k$$

for some $m \geq 1$. Then

$$ay^{m+1} = ay^m y$$

$$= \left( y^m a + \sum_{k=1}^{m} \binom{m}{k} y^{m-k}a_k \right) y$$

$$= y^m ay + \sum_{k=1}^{m} \binom{m}{k} y^{m-k}a_k y$$

$$= y^m(ya + [a, y]) + \sum_{k=1}^{m} \binom{m}{k} y^{m-k}(ya_k + [a_k, y])$$

$$= y^{m+1} a + y^m a_1 + \sum_{k=1}^{m} \binom{m}{k} y^{m+1-k}a_k + \sum_{k=1}^{m} \binom{m}{k} y^{m-k}a_{k+1}$$

$$= y^{m+1} a + y^m a_1 + \sum_{k=1}^{m} \binom{m}{k} y^{m+1-k}a_k + \sum_{k=1}^{m} \binom{m}{k} y^{m-k}a_{k+1}$$
Consider the far right summation. We can rewrite this by replacing $k$ with $k - 1$.

$$\sum_{k=1}^{m} \binom{m}{k} y^{m-k} a_{k+1} = \sum_{k=2}^{m+1} \binom{m}{k-1} y^{m+1-k} a_k$$

Then we can combine this with the $y^m a_1$ term, since $\binom{m}{0} = 1$.

$$y^m a_1 + \sum_{k=2}^{m+1} \binom{m}{k-1} y^{m+1-k} a_k = \binom{m}{0} y^m a_1 + \sum_{k=2}^{m+1} \binom{m}{k-1} y^{m+1-k} a_k$$

$$= \sum_{k=1}^{m+1} \binom{m}{k} y^{m+1-k} a_k$$

Consider the other summation term in our expression for $ay^{m+1}$. We can tack on a $k = m+1$ term since that term would be zero because $\binom{m}{m+1} = 0$.

$$\sum_{k=1}^{m} \binom{m}{k} y^{m+1-k} a_k = \sum_{k=1}^{m} \binom{m}{k} y^{m+1-k} a_k + \binom{m}{m+1} y^{m+1-(m+1)} a_{m+1}$$

$$= \sum_{k=1}^{m+1} \binom{m}{k} y^{m+1-k} a_k$$

Putting this all together, we get

$$ay^{m+1} = y^{m+1} a + \sum_{k=1}^{m+1} \binom{m}{k} y^{m+1-k} a_k + \sum_{k=1}^{m+1} \binom{m}{k-1} y^{m+1-k} a_k$$

$$= y^{m+1} a + \sum_{k=1}^{m+1} \left( \binom{m}{k} + \binom{m}{k-1} \right) y^{m+1-k} a_k$$

By a standard identity for binomial coefficients (Pascal’s Rule),

$$\binom{m}{k} + \binom{m}{k-1} = \binom{m+1}{k}$$

Thus

$$ay^{m+1} = \sum_{k=1}^{m+1} \binom{m+1}{k} y^{m+1-k} a_k$$

This completes the induction.

Proposition 10.14 (Exercise 5.7). Let $V$ be a vector space, and let $y \in \text{gl}(V)$. Then

$$(\text{ad} \, y)^m = \sum_{k=1}^{m} (-1)^{k+1} \binom{m}{k} y^{m-k} (\text{ad} \, y)^k$$
Proof. Let \( a \in \text{gl}(V) \) and define \( a_1 = [a, y] \) and \( a_k = [a_{k-1}, y] \) for \( k \geq 2 \). Then \( a_1 = -\text{ad} \, y(a) \) and \( a_k = (-1)^k(\text{ad} \, y)^k(a) \). By the previous proposition,

\[
ay^m = y^m a + \sum_{k=1}^{m} \binom{m}{k} y^{m-k} a_k
\]

\[
y^m a - ay^m = -\sum_{k=1}^{m} \binom{m}{k} y^{m-k} (-1)^k(\text{ad} \, y)^k(a)
\]

\[
[y^m, a] = \sum_{k=1}^{m} (-1)^{k+1} \binom{m}{k} y^{m-k} (\text{ad} \, y)^k(a)
\]

Thus the map \((\text{ad} \, y)^m\) is

\[
(\text{ad} \, y)^m = \sum_{k=1}^{m} (-1)^{k+1} \binom{m}{k} y^{m-k} (\text{ad} \, y)^k
\]
11 Chapter 6 Exercises

Proposition 11.1 (Exercise 6.1i). Let $V$ be an $n$-dimensional vector space where $n \geq 1$ and let $x : V \to V$ be a nilpotent linear map. Then there exists a nonzero $v \in V$ such that $x(v) = 0$.

Proof. Suppose that there is no nonzero $v \in V$ such that $x(v) = 0$. Then $\ker x = \{0\}$, so $x$ is one-to-one. Let $\beta$ be some basis for $V$. Since $x$ is one-to-one, $x(\beta)$ is linearly independent, so $x(\beta)$ is a basis for $V$. Thus $x(V) = V$. Then by induction $x^r(V) = V$ for all $r \in \mathbb{N}$, so $x$ is not nilpotent. This contradicts the hypothesis, so we conclude that there must be some nonzero $v \in V$ such that $x(v) = 0$.

Lemma 11.2 (for Exercise 6.1ii). Let $V$ be a finite-dimensional vector space and let $x : V \to V$ be a nilpotent linear map. Then $\text{rank } x < \dim V$.

Proof. By Exercise 6.1i, there exists a non-zero $v \in V$ such that $x(v) = 0$. Thus $\dim \ker x > 0$. Thus by the Rank-Nullity Theorem, $\dim \text{im } x = \text{rank } x < \dim V$.

Lemma 11.3 (for Exercise 6.1ii). Let $V$ be a 1-dimensional vector space and let $x : V \to V$ be a nilpotent linear map. Thus $x(v) = 0$ for all $v \in V$.

Proof. By the previous lemma, $\text{rank } x < 1$ so $\text{rank } x = 0$. Thus $\dim \text{im } x = 0$, so $x(v) = 0$ for all $v \in V$.

Lemma 11.4 (for Exercise 6.1ii). Let $V$ be an $n$-dimensional vector space over $F$ where $n \geq 1$. Let $x : V \to V$ be a nilpotent linear map. Let $U$ be a subspace of $V$. Define $\bar{x} : V/U \to V/U$ by

$$\bar{x}(w + U) = x(w) + U$$

for $w + U \in V/U$. Then $\bar{x}$ is a nilpotent linear map. (We refer to $\bar{x}$ as the map induced by $x$.)

Proof. One can check that $\bar{x}$ is linear using the definitions of addition and scalar multiplication in $V/U$. We show that $\bar{x}$ is nilpotent. We know that $x$ is nilpotent, so there exists $r \in \mathbb{N}$ such that $x^r(v) = 0$ for all $v \in V$. Then

$$\bar{x}^r(w + U) = x^r(w) + U = 0 + U = U$$

Thus $\bar{x}$ is nilpotent.

Proposition 11.5 (Exercise 6.1ii). Let $V$ be an $n$-dimensional vector space and let $x : V \to V$ be a nilpotent linear map. Let $v \in V$ such that $v \neq 0$ and $x(v) = 0$. Let $U = \text{span}\{v\}$.

Define $\bar{x} : V/U \to V/U$ as above. Then there is a basis $\{v_1 + U, \ldots, v_{n-1} + U\}$ of $V/U$ such that the matrix of $\bar{x}$ in this basis is strictly upper triangular.

Proof. We begin with the case $n = 2$, so we assume $V$ is 2-dimensional. Let $v_1$ be any vector in $V - U$. Then $v_1 \neq 0$ and $v_1 \notin U$ thus $v_1 + U \neq U$. Thus $v_1 + U$ is a non-zero element of $V/U$, and since $V/U$ is one-dimensional, $V/U = \text{span}\{v_1 + U\}$. Since $\bar{x}$ is a nilpotent linear
map from a 1-dimensional vector space to itself, $\bar{x}$ is the zero map. Thus in any basis its matrix representation is $[x] = (0)$ which is strictly upper triangular.

Now we proceed by induction on $n$. We suppose that for any $k$-dimensional vector space $V$, and any nilpotent linear map $x : V \rightarrow V$, there is a basis $\{v_1 + U, \ldots, v_{k-1} + U\}$ of $V/U$ in which the matrix of $\bar{x}$ is upper triangular (where $U = \text{span}\{v\}$ for some $v \neq 0$ with $x(v) = 0$).

Let $W$ be a $(k+1)$-dimensional vector space and let $y : W \rightarrow W$ be a nilpotent linear map. By Exercise 6.1i, there exists a nonzero $w \in W$ such that $y(w) = 0$. We let $B = \text{span}\{w\}$. Then $W/B$ is a $k$-dimensional vector space, so by inductive hypothesis there is a basis $\{v_1 + U, \ldots, v_k + U\}$ of $V/U$ such that the matrix of $\bar{x}$ in this basis is strictly upper triangular.

Proof. Let $v_1$ be any vector in $V - U$. Then $v_1 \neq 0$ and $v_1 \notin U$ thus $v_1 + U \neq U$. Thus $v_1 + U$ is a non-zero element of $V/U$, and since $V/U$ is one-dimensional, $V/U = \text{span}\{v_1 + U\}$. Since $\bar{x}$ is a nilpotent linear map from a 1-dimensional vector space to itself, $\bar{x}$ is the zero map. Thus in any basis its matrix representation is $[x] = (0)$ which is strictly upper triangular.

Proposition 11.7 (Exercise 6.1ii, Inductive Step for Induction). Suppose that for every $k$-dimensional vector space $V$ with a nilpotent linear map $x : V \rightarrow V$ and $U = \text{span}\{v\}$ for some $v \in V$ with $x(v) = 0$ and $v \neq 0$, there is a basis $\{v_1 + U, \ldots, v_{k-1} + U\}$ of $V/U$ in which $[\bar{x}]$ is strictly upper triangular. Then let $W$ be a $k + 1$ dimensional vector space with $y : W \rightarrow W$ a nilpotent map with $y(w) = 0$ for some $w \in W$ with $w \neq 0$ and $B = \text{span}\{w\}$. Then there is a basis of $W/B$ in which $[\tilde{y}]$ is strictly upper triangular.

Proof. Let $W$ be such a space. Then $W/B$ is $k$-dimensional, so there is a basis of $W/B$ in which $[\tilde{y}]$ is strictly upper triangular. Then by the previous proposition, this basis with $w$ added gives a basis of $W$ in which $[\tilde{y}]$ is strictly upper triangular.

Proposition 11.8 (Exercise 6.2ii). Let $V$ be an $n$-dimensional complex vector space and let $x : V \rightarrow V$ be a linear map. Let $v$ be an eigenvector of $x$ with corresponding eigenvalue $\lambda$. Let $U = \text{span}\{v\}$. Define $\bar{x} : V/U \rightarrow V/U$ by

$$\bar{x}(w + U) = x(w) + U$$

The map $\bar{x}$ is linear.

Proof. Let $a \in \mathbb{C}$, $w_1, w_2 \in V$.

$$\bar{x}(a(w_1 + U) + (w_2 + U)) = \bar{x}((aw_1 + w_2) + U) = x(aw_1 + w_2) + U = (ax(w_1) + x(w_2)) + U = a(x(w_1) + U) + (x(w_2) + U) = a\bar{x}(w_1 + U) + \bar{x}(w_2 + U)$$
Proposition 11.9 (Exercise 6.2ii). Let $V$ be an $n$-dimensional complex vector space and let $x : V \to V$ be a linear map. Let $v$ be an eigenvector of $x$ and let $U = \text{span} v$. Let $\bar{x} : V/U \to V/U$ be the induced map and let $\beta = \{v_1 + U, \ldots, v_{n-1} + U\}$ be a basis of $V/U$ such that $[\bar{x}]_\beta$ is upper triangular. Then $\gamma = \{v, v_1, \ldots, v_{n-1}\}$ is a basis of $V$ such that $[x]_\gamma$ is upper triangular.

Proof. First we show that $\gamma$ is linearly independent. We have the canonical map $\pi : V \to V/U$ by $\pi(w) = w + U$. Suppose there are scalars $a, a_1, \ldots, a_{n-1} \in \mathbb{C}$ such that

$$av + \sum_{i=1}^{n-1} a^i v_i = 0$$

Then (using Einstein summation notation)

$$0 = \pi(av + a^i v_i) = a\pi(v) + a^i \pi(v_i)$$

But $\pi(v) = 0$ so we have

$$a^i \pi(v_i) = 0$$

By hypothesis, $\{\pi(v_i)\}_{i=1}^{n-1}$ is a basis for $V/U$ so it is linearly independent. Thus $a^i = 0$ for $i = 1, 2, \ldots, n - 1$. Returning to the original equation, we now have $av = 0$. Since $v$ is an eigenvector, it is not the zero vector, so $a = 0$. Thus $\gamma$ is linearly independent, so it forms a basis for $V$.

Lemma 11.10. Let $V$ be an $n$-dimensional vector space and let $x : V \to V$ be a nilpotent linear map. Then $x^n(v) = 0$ for all $v \in V$.

Proof. By the lemma for Exercise 6.1iii, the rank of $x$ is strictly less than the dimension of $V$ (unless $V$ is zero-dimensional), so

$$\dim V > \dim x(V) > \dim x^2(V) > \ldots > \dim x^n(V)$$

where there are $n + 1$ such inequalities. But a string of $n + 1$ inequalities involving integers means that $\dim x^n(v) = 0$.

Proposition 11.11 (Exercise 6.3). Let $L$ be a nilpotent complex Lie algebra. Then every 2-dimensional subalgebra of $L$ is abelian.

Proof. By Engel’s Theorem (Theorem 6.3), for every $x \in L$, the map $\text{ad} x : L \to L$ is nilpotent. Let $V$ be a 2-dimensional subalgebra of $L$. Either $V$ is abelian, or we can choose a basis $\{x, y\}$ of $V$ such that $[x, y] = x$. In the latter case, we know that $(\text{ad} y)^2 = 0$, so

$$0 = [y, [y, x]] = -[y, [x, y]] = -[y, x] = [x, y]$$

Thus even if $V$ is “not abelian,” all the brackets in $V$ are zero, so $V$ is abelian.

Lemma 11.12. For $i, j, k, p \in \mathbb{N}$,

$$\sum_{k=1}^{p} \delta_{ik} \delta_{kj} = \delta_{ij}$$

77
Proof. See Wikipedia.

(Exercise 6.4) Erdmann and Wildon define matrices $x, y \in \text{gl}(p, \mathbb{Z}_p)$. We equivalently view $x$ and $y$ in terms of their $ij$th position as

$$
(x)_{ij} = \delta_{j,i+1} + \delta_{ip}\delta_{1j} \\
(y)_{ij} = \delta_{ij}(i - 1) = \delta_{ij}(j - 1)
$$

Then the matrix products $xy$ and $yx$ are given by

$$(xy)_{ij} = \sum_{k=1}^{p} x_{ik}y_{kj}$$

$$= \sum_{k=1}^{p} (\delta_{k,i+1} + \delta_{ip}\delta_{1k})\delta_{kj}(j - 1)$$

$$= (j - 1) \left( \sum_{k=1}^{p} \delta_{k,i+1}\delta_{k,j} + \sum_{k=1}^{p} \delta_{ip}\delta_{1k}\delta_{kj} \right)$$

$$= (j - 1)(\delta_{i+1,j} + \delta_{1j}\delta_{ip})$$

$$(yx)_{ij} = \sum_{k=1}^{p} y_{ik}x_{kj}$$

$$= \sum_{k=1}^{p} \delta_{ik}(i - 1)(\delta_{j,k+1} + \delta_{kp}\delta_{1j})$$

$$= (i - 1) \left( \sum_{k=1}^{p} \delta_{ik}\delta_{j,k+1} + \sum_{k=1}^{p} \delta_{ik}\delta_{kp}\delta_{1j} \right)$$

$$= (i - 1)(\delta_{i+1,j} + \delta_{ip}\delta_{1j})$$

Then we can compute $[x, y]$ as

$$[x, y]_{ij} = (xy)_{ij} - (yx)_{ij} = (j - 1)(\delta_{i+1,j} + \delta_{1j}\delta_{ip}) - (i - 1)(\delta_{i+1,j} + \delta_{ip}\delta_{1j})$$

$$= (j - i)(\delta_{i+1,j} + \delta_{ip}\delta_{1j})$$

$$= (j - i)\delta_{i+1,j} + (j - i)\delta_{ip}\delta_{1j}$$

$$= \delta_{i+1,j} + (1 - p)\delta_{ip}\delta_{1j}$$

$$= \delta_{i+1,j} + \delta_{ip}\delta_{1j}$$

$$= x_{ij}$$

Note that in the last few equalities, the factor $j - i$ becomes 1 since $\delta_{i+1,j}$ is zero unless $i + 1 = j$, and $(1 - p) = 1$ since the field is $\mathbb{Z}_p$. Thus $[x, y] = x$.

**Proposition 11.13** (Exercise 6.4). As defined above $x, y$ span a 2-dimensional solvable subalgebra of $\text{gl}(p, F)$.  

78
Proof. Since \([x, y] = x\), we know that \(\text{span}\{x, y\}\) is closed under brackets, so it is a subalgebra of \(\text{gl}(p, F)\). We know that all 2-dimensional Lie algebras are solvable since the non-abelian 2-dimensional Lie algebra has one-dimensional (and hence abelian) derived algebra.

**Proposition 11.14** (Exercise 6.4). As defined above, the matrices \(x, y\) have no common eigenvector.

**Proof.** Let \(v = (v_1, v_2, \ldots v_p)\). Then
\[
xv = (v_2, v_3 \ldots v_p, v_1)
\]
So if \(v\) is an eigenvector of \(x\), then
\[
v_2 = \lambda v_1 \\
v_3 = \lambda v_2 \\
\vdots \\
v_1 = \lambda v_p
\]
so \(\lambda^p v_1 = v_1\). Thus the eigenvalues of \(x\) are the \(p\)th roots of unity,
\[
\lambda \in \{e^{2\pi ik/p} : k = 0, 1, \ldots (p - 1)\}
\]
\[
\in \{1, e^{2\pi i/p}, e^{4\pi i/p}, \ldots\}
\]
The eigenvector corresponding to each \(\lambda\) is
\[
v_\lambda = (1, \lambda, \lambda^2, \lambda^3, \ldots \lambda^{p-1})
\]
Now we compute the eigenvectors for \(y\).
\[
yv = (0, v_2, 2v_3, \ldots (p - 1)v_p)
\]
Eigenvalues of \(y\) are the diagonal entries, \(\lambda \in \{0, 1, 2, \ldots (p - 1)\}\), and the corresponding eigenvectors are the standard basis vectors for \(F^n\). (Recall that \(F\) is a field of characteristic \(p\).)
\[
v_\lambda = e_\lambda = (0, 0, \ldots, 1, \ldots 0, 0)
\]
If \(v\) is an eigenvector of \(x\), then it has all nonzero entries, but then it could not be an eigenvector of \(y\). Thus \(x, y\) have no common eigenvectors.

Because \(x\) and \(y\) have no common eigenvector, this example demonstrates that the hypothesis that the field be complex in Proposition 6.6 is necessary. If \(V = F^n\) where \(F\) is a field of characteristic \(p\), then we have shown that \(L = \text{span}\{x, y\}\) is a solvable subalgebra of \(\text{gl}(p, F) \cong \text{gl}(V)\), and \(x, y\) have no common eigenvector.

**Proposition 11.15** (Exercise 6.4). Let \(x\) be the matrix defined above. Then \(x^p = I_p\), where \(I_p\) is the identity matrix.

79
Proof. We compute the characteristic equation of $x$.

$$x - \lambda I = \begin{pmatrix} -\lambda & 1 & 0 & \ldots & 0 & 0 \\ 0 & -\lambda & 1 & \ldots & 0 & 0 \\ 0 & 0 & -\lambda & \ldots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -\lambda & 1 \\ 1 & 0 & 0 & \ldots & 0 & -\lambda \end{pmatrix}$$

We compute the determinant of $x - \lambda I$ by expansion by cofactors along the top row. Thankfully, all but two of the entries are zero.

$$\det(x - \lambda I) = -\lambda \det \begin{pmatrix} -\lambda & 1 & 0 & \ldots & 0 \\ 0 & -\lambda & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & -\lambda & 1 \\ 0 & 0 & \ldots & 0 & -\lambda \end{pmatrix} - \det \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & -\lambda & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & -\lambda & 1 \\ 1 & 0 & \ldots & 0 & -\lambda \end{pmatrix}$$

The first matrix here is diagonal, so its determinant can be read off as the product along the diagonal, $(-\lambda)^{p-1}$. To compute the determinant of the second matrix, we will iteratively expand along the second column. Notice that when we expand it along the second column, we get the same matrix, except one dimension smaller, and we multiply by $(-1)$.

$$\det \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & -\lambda & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & -\lambda & 1 \\ 1 & 0 & \ldots & 0 & -\lambda \end{pmatrix} = -\det \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & -\lambda & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & -\lambda & 1 \\ 1 & 0 & \ldots & 0 & -\lambda \end{pmatrix} \cdot (-1)^{p-2}$$

After doing this expansion $p-3$ times, we get

$$\det \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & -\lambda & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & -\lambda & 1 \\ 1 & 0 & \ldots & 0 & -\lambda \end{pmatrix} = (-1)^{p-3} \det \begin{pmatrix} 0 & 1 \\ 1 & -\lambda \end{pmatrix} = (-1)^{p-2}$$

Thus the characteristic polynomial of $x$ is

$$\det(x - \lambda I) = (-\lambda)^{p} - (-1)^{p-2} = (-1)^{p}\lambda^{p} - (-1)^{p} = (-1)^{p}(\lambda^{p} - 1) = 0$$

$$\Rightarrow \lambda^{p} - 1 = 0$$

By the Cayley-Hamilton Theorem, every square matrix over a commutative ring satisfies its own characteristic polynomial. Thus

$$x^{p} - I_{p} = 0 \Rightarrow x^{p} = I_{p}$$
The matrices $x, y$ also illustrate that the requirement that the field be complex in the following proposition (Exercise 6.5i) is necessary. Let $F$ be a field of characteristic $p$, let $V = F^n$, and let $x, y$ be the matrices above. Let $L = \text{span}\{x, y\}$. Then as shown above, $L$ is a solvable subalgebra of $\text{gl}(V)$, so all the other hypotheses of 6.5i are satisfied. However, $L' = \text{span}\{x\}$, and $x$ is not nilpotent, since $x^p = I_p$. Thus the theorem fails without the hypothesis that the underlying field by $\mathbb{C}$.

**Proposition 11.16** (Exercise 6.5i). Let $V$ be a complex vector space and let $L \subseteq \text{gl}(V)$ be a solvable subalgebra. Then every element of $L'$ is nilpotent.

**Proof.** By Lie’s Theorem (Theorem 6.5), there is a basis $\beta$ of $V$ in which every element of $L$ is represented by an upper triangular matrix. For $z \in L'$, we can write $z$ as a linear combination of brackets,

$$z = a^i[x_i, y_i]$$

where $a^i \in \mathbb{C}$ and $x_i, y_i \in L$. $x_i, y_i$ have upper triangular matrices in the basis $\beta$, $[x_i, y_i]$ has a strictly upper triangular matrix (in $\beta$). Thus $z$ has a strictly upper triangular matrix representation, so it is a nilpotent map.

**Proposition 11.17** (Exercise 6.5ii). Let $L$ be a complex Lie algebra. Then $L$ is solvable if and only if $L'$ is nilpotent.

**Proof.** First suppose that $L'$ is nilpotent. Then $L'$ is solvable, and so $L'^m = 0$ for some $m$, and thus $L^{(m+1)} = 0$, so $L$ is solvable.

Now we suppose that $L$ is solvable and show that $L'$ is nilpotent. Using the adjoint homomorphism $\text{ad} : L \to \text{gl}(L)$, we can see that $\text{ad} L$ is a subalgebra of $\text{gl}(L)$, and by Lemma 4.4, $\text{ad} L$ is solvable since it is a homomorphic image of $L$. Thus by Lie’s Theorem (Theorem 6.5), there is a basis $\beta$ of $L$ such that every element of $\text{ad} L$ is represented by an upper triangular matrix.

We claim that for $z \in L'$, $\text{ad} z$ is nilpotent. If $z \in L'$, we can write $z$ as a linear combination of brackets,

$$z = a^i[x_i, y_i]$$

where $a^i \in \mathbb{C}$ and $x_i, y_i \in L$. Then since $\text{ad}$ is a homomorphism,

$$\text{ad} z = \text{ad}(a^i[x_i, y_i]) = a^i[\text{ad} x_i, \text{ad} y_i]$$

Since $\text{ad} x_i, \text{ad} y_i$ have upper triangular matrices in the basis $\beta$, $[\text{ad} x_i, \text{ad} y_i]$ has a strictly upper triangular matrix (in $\beta$). Thus $\text{ad} z$ is nilpotent. Thus by Engel’s Theorem (2nd version), $L'$ is nilpotent.

**Proposition 11.18** (Exercise 6.6). Let $V$ be an $n$-dimensional complex vector space and let $x, y : V \to V$ be linear maps such that

$$x \circ [x, y] = [x, y] \circ x$$

$$y \circ [x, y] = [x, y] \circ y$$

Then $[x, y]$ is a nilpotent map.
Proof. Let \( L = \text{span}\{x, y, [x, y]\} \). We claim that \( L \) is a solvable subalgebra of \( \text{gl}(V) \). To see that \( L \) is a subalgebra, note that

\[
[x, y] \in L \\
[x, [x, y]] = x \circ [x, y] - [x, y] \circ x = 0 \in L \\
[y, [x, y]] = y \circ [x, y] - [x, y] \circ y = 0 \in L
\]

To see that \( L \) is solvable, note that \( L' = \text{span}\{[x, y]\} \) which is abelian since it is one-dimensional, so \( L'' = 0 \).

Since \( L \) is a solvable subalgebra of \( \text{gl}(V) \), by Lie’s Theorem (Theorem 6.5), there is a basis of \( V \) in which every element of \( L \) is represented by an upper triangular matrix. Let \( M_x, M_y, M_{[x,y]} \) be the matrices of \( x, y, \) and \([x, y]\) respectively. Because \( M \) is a homomorphism,

\[
M_{[x,y]} = M_{xy-yx} = M_x M_y - M_y M_x = [M_x, M_y]
\]

As shown in Exercise 4.5i, the commutator of two upper triangular matrices is strictly upper triangular, so since \( M_x, M_y \) are upper triangular, \( M_{[x,y]} \) is strictly upper triangular. It is a standard result that strictly upper triangular matrices are nilpotent, and that a linear map is nilpotent if and only if its matrix is nilpotent. Thus \([x, y]\) is nilpotent. 

\[\square\]
12 Chapter 7 Exercises

(Exercise 7.1)

\[
\begin{align*}
  h &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
  e &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
  f &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
  [e, f] &= h \\
  [f, h] &= 2f \\
  [h, e] &= 2e
\end{align*}
\]

\[
[\text{ad } h] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \\
[\text{ad } e] = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\
[\text{ad } f] = \begin{pmatrix} 0 & 0 & 2 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

**Proposition 12.1** (Exercise 7.2). Let \( V \) be an \( L \)-module. Define \( \phi : L \to \text{gl}(V) \) by \( \phi(x)(v) = x \cdot v \). Then \( \phi \) is a Lie algebra homomorphism.

**Proof.** Linearity of \( \phi \) follows immediately from the M2 axiom for \( L \)-modules. Let \( x, y \in L, v \in V \).

\[
\phi([x, y])(v) = [x, y] \cdot v
\]

\[
= x \cdot (y \cdot v) - y \cdot (x \cdot v)
\]

\[
= \phi(x) \circ \phi(y)(v) - \phi(y) \circ \phi(x)(v)
\]

\[
= [\phi(x), \phi(y)](v)
\]

\( \square \)

**Proposition 12.2** (Exercise 7.3). Let \( L \) be a Lie algebra and let \( V \) be an \( L \)-module. Then \( V \) is irreducible if and only if for any non-zero \( v \in V \) the submodule generated by \( v \) contains all elements of \( V \).

**Proof.** First, suppose that \( V \) is an \( L \)-module such that for any non-zero \( v \), the submodule generated by \( v \) is \( V \). We will show that any non-zero submodule of \( V \) is equal to \( V \). Let \( W \subseteq V \) be a non-zero submodule. Then there exists some non-zero \( v \in W \). Because \( W \) is a submodule, any product of the form

\[
x_1 \cdot (x_2 \cdot \ldots \cdot (x_m \cdot v) \ldots)
\]

is inside of \( W \). By hypothesis, products of this form span \( V \) for any non-zero \( v \in V \). Thus \( W = V \). Thus \( V \) has non non-zero proper submodules, so \( V \) is irreducible.

Now suppose that \( V \) is irreducible. Define \( U \) to be the submodule generated by \( v \), that is,

\[
U = \text{span}\{x_1 \cdot (x_2 \cdot \ldots \cdot (x_m \cdot v) \ldots) : x_i \in L\}
\]

Then we know that \( U \) is a non-zero submodule of \( V \). Since \( V \) is irreducible, this means that \( U = V \).

\( \square \)
Proposition 12.3 (Exercise 7.5). Let \( L \) be a finite-dimensional Lie algebra. Let \( \text{ad} : L \to \text{gl}(L) \) be the adjoint homomorphism, and define an action of \( L \) on itself by
\[
L \times L \to L \quad x \cdot y = \text{ad}(x)(y) = [x, y]
\]
Then the submodules of \( L \) as a module are precisely the ideals of \( L \).

Proof. Let \( I \subseteq L \) be any subset, and let \( a \in I, x \in L \). Then
\[
x \cdot a \in I \iff [x, a] \in I
\]
so \( I \) is \( L \)-invariant exactly when \( I \) is an ideal. \( \square \)

Proposition 12.4 (Exercise 7.6i). Let \( F \) be a field and let \( L = \text{b}(n, F) \) and \( V = F^n \). Then \( V \) is an \( L \)-module where the action is
\[
L \times V \to V \quad (x, v) \mapsto xv
\]
that is, multiplying the matrix by a column vector.

Proof. Let \( a, b \in F, v, w \in V \), and \( x, y \in L \). Using standard properties of matrix multiplication,
\[
(ax + by)v = a(xv) + b(yv)
\]
\[
x(av + bw) = xav + xbw = a(xv) + b(xw)
\]
\[
[x, y]v = (xy - yx)v = x(yv) - y(xv)
\]
\( \square \)

Proposition 12.5 (Exercise 7.6ii). Let \( F \) be a field, and let \( L = \text{b}(n, F), V = F^n \). Let \( \{e_1, e_2, \ldots e_n\} \) be the standard basis for \( F^n \), and let \( W_r = \text{span}\{e_1, e_2, \ldots e_r\} \). Then \( W_r \) is a submodule of \( V \) (where \( V \) has the same module structure as in part i).

Proof. To show: for \( x \in L, w \in W_r \), we have \( xw \in W_r \). Let \( x \in L, w \in W_r \). Let \( x_{ij} \) be the \( ij \)th entry of \( x \) and \( w_i \) be the \( i \)th entry of \( w \). Since \( x \) is upper triangular, \( x_{ij} = 0 \) for \( j < i \) and since \( w \in W_r, w_i = 0 \) for \( r < i \). We know that
\[
(xw)_{ij} = \sum_{j=1}^{n} x_{ij}w_j
\]
When \( i > r \), there are two possibilities: \( j \leq r \) or \( j > r \). If \( j \leq r \), then \( j \leq r < i \) so \( x_{ij} = 0 \). If \( j > r \), then \( w_j = 0 \). Thus when \( i > r \), each term of the summation is zero, so \((xw)_i = 0\) for \( i < r \). Thus \( xw \in W_r \). \( \square \)

Proposition 12.6 (Exercise 7.6iii). Let \( F \) be a field, let \( V = F^n \), and let \( L = \text{b}(n, F) \). Let \( V \) be an \( L \)-module by applying matrices to column vectors. Then every non-zero submodule of \( V \) is equal to some \( W_r \) where
\[
W_r = \text{span}\{e_1, e_2, \ldots e_r\}
\]
(\( \text{span}\{e_1, e_2, \ldots e_n\} \) is the standard basis for \( F^n \)). Furthermore, each \( W_r \) is indecomposable, and if \( n \geq 2 \), then \( V \) is not completely reducible as an \( L \)-module.
Proof. We prove the first assertion by induction on \( r \). The base case is \( r = 1 \). We claim that every 1-dimensional submodule of \( V \) is \( W_1 = \text{span}\{e_1\} \). Let \( W \subseteq V \) be an 1-dimensional submodule, so \( W = \text{span}\{w\} \) for some nonzero \( w \in V \). Let \( w = (w^1, w^2, \ldots , w^n) \). Let \( e_{ij} \) be the usual matrix basis element. Then since \( W \) is a submodule,

\[
e_{11}w = (w^1, 0, 0, \ldots ) \in W \\
e_{12}w = (w^2, 0, 0, \ldots ) \in W \\
\vdots \\
e_{1n}w = (w^n, 0, 0, \ldots ) \in W
\]

Since \( w \) is nonzero, one of \( w^1, w^2, \ldots \) is nonzero, so \( \text{span}\{e_1\} \subseteq W \). Since \( W \) is a 1-dimensional vector space, it must be equal to \( \text{span}\{e_1\} \). Thus any 1-dimensional submodule of \( V \) is equal to \( W_1 \).

Now for the inductive step. We suppose that every \( k \)-dimensional submodule of \( V \) is equal to \( W_k \). We will show that this implies that every \( (k + 1) \)-dimensional submodule of \( V \) is equal to \( W_{k+1} \). Let \( U \) be a \( (k + 1) \)-dimensional submodule of \( V \). Let \( U = \text{span}\{u_1, u_2, \ldots , u_{k+1}\} \). Then \( U' = \text{span}\{u_1, u_2, \ldots , u_k\} \) is a \( k \)-dimensional submodule, so by the inductive hypothesis, \( U' = W_k \). Then \( U = W_k \oplus \text{span}\{u_{k+1}\} \). Since \( U \) is \( (k + 1) \)-dimensional, it must be that \( u_{k+1} \notin W_k \), so \( u_{k+1} \) has a non-zero entry after the \( k \)th entry.

\[
u_{k+1} = (a^1, a^2, \ldots , a^k, a^{k+1}, \ldots , a^n)
\]

One of \( a^{k+1}, a^{k+2}, \ldots , a^n \) is nonzero.

Then since \( U \) is a submodule of \( V \),

\[
e_{k+1,k+1}u_{k+1} = (0, \ldots , a^{k+1}, 0, \ldots ) \in U \\
e_{k+1,k+2}u_{k+1} = (0, \ldots , a^{k+2}, 0, \ldots ) \in U \\
\vdots \\
e_{k+1,n}u_{k+1} = (0, \ldots , a^n, 0, \ldots ) \in U
\]

Where each time, the product is in \( \text{span}\{e_{k+1}\} \). Since one of \( a^{k+1}, a^{k+2}, \ldots , a^n \) is nonzero, this implies that \( \text{span}\{e_{k+1}\} \subseteq U \). Thus

\[
W_{k+1} = W_k \oplus \text{span}\{e_{k+1}\} = \text{span}\{e_1, e_2, \ldots , e_k, e_{k+1}\} \subseteq U
\]

Since \( U \) is a \( (k + 1) \)-dimensional vector space, it follows that \( U = W_{k+1} \). This completes the induction.

Now, we show that each \( W_r \) is indecomposable and if \( n \geq 2 \) then \( V \) is not completely reducible as an \( L \)-module. As shown, the only submodules of \( V \) are \( W_1, W_2, \ldots , W_{n-1}, W_n = V \) and we have proper inclusions

\[
W_1 \subset W_2 \subset W_3 \subset \ldots \subset W_{n+1} \subset V
\]

Thus no \( W_r \) can be written as a direct sum of two submodules of \( V \), since any direct sum is “absorbed,” that is, \( W_a \oplus W_b = W_{\max(a,b)} \). If \( n \geq 2 \), then \( V \) has at least one nonzero proper submodule, \( W_1 \), but no direct sum of submodules is equal to \( V \).
Proposition 12.7 (Exercise 7.7). Let $L$ be a Lie algebra, and let $V$ be an $n$-dimensional $L$-module with a submodule $W$ of dimension $m$. Then there exists a basis $\beta$ of $V$ such that the action of every $x \in L$ is represented by a “block matrix” of the form

$$[x] = \begin{pmatrix} X_1 & X_2 \\ 0 & X_3 \end{pmatrix}$$

where $X_1$ is a $m \times m$ matrix. Furthermore, $X_1$ is the matrix of $x$ restricted to $W$, and $X_3$ is the matrix of the action of $x$ on the factor module $V/W$.

Proof. Let $\alpha = \{w_1, w_2, \ldots, w_m\}$ be a basis for $W$. Extend $\alpha$ to a basis of $V$, and call this basis $\beta$.

$$\beta = \{w_1, w_2, \ldots, w_m, v_1, v_2, \ldots, v_{n-m}\}$$

Let $x \in L$, and let $[x]_\beta$ be the matrix of $x$ with respect to $\beta$. Then for each $i$ with $1 \leq i \leq m$,

$$[x]_\beta[w_i] = [x \cdot w_i]_\beta$$

Since $W$ is a submodule, $x \cdot w_i \in W$. The multiplication $[x]_\beta[w_i]_\beta$ just picks off the $i$th column of $[x]_\beta$, so for each column $1, 2, \ldots, m$ of $[x]_\beta$, the column lies in $W$. By construction of $\beta$, for any $w \in W$,

$$[w]_\beta = (a^1, a^2, \ldots, a^m, 0, 0, \ldots)$$

Thus the bottom left block of $[x]$ is zeroes, as was to be shown.

Now consider the matrix $[x]_\alpha$. As before, the multiplication

$$[x]_\alpha[w_i]$$

picks off the $i$th column of $[x]_\alpha$, so $x \cdot w_i$ is the $i$th column of $[x]_\alpha$. Since nothing in $W$ has a nonzero entry past the $m$th entry, the block $X_2$ (of $[x]_\beta$) does not affect anything when $x$ is restricted to $W$. Thus $[x]_\alpha$ is the upper left block $X_1$.

Now we show that $X_3$ is the matrix of $x$ acting on $V/W$. When $x$ acts on $V/W$, it acts as the map $v + W \mapsto x \cdot v + W$, so when $x$ acts on $v + W$, it ignores the first $m$ entries of $[v]_\beta$. Thus only the bottom left block $X_3$ acts on $V/W$.

Proposition 12.8 (Exercise 7.8). Let $L$ be the Heisenberg algebra over $\mathbb{C}$, that is, $L = \text{span}\{f, g, z\}$ with $[f, g] = z$ and $[f, z] = [g, z] = 0$. $L$ does not have a faithful finite-dimensional irreducible representation.

Proof. Let $V$ be a finite-dimensional vector space and let $\phi : L \to \text{gl}(V)$ be an irreducible representation. We will show that $\phi$ is not faithful. Since $z \in Z(L)$, by Lemma 7.14, $\phi(z) = \lambda I_V$ for some $\lambda \in \mathbb{C}$. ($I_V$ is the identity transformation on $V$.) Now we compute the trace of $\phi(z)$.

$$\text{tr} \phi(z) = \text{tr} [\phi(f), \phi(g)] = 0$$

Thus $\lambda = 0$. Thus $\phi(z)$ is the zero map, so $\phi$ has a nonzero kernel. Thus $\phi$ is not a faithful representation. \qed
Proposition 12.9 (Exercise 7.9). Let $L$ be the 2-dimensional complex non-abelian Lie algebra given by $L = \text{span} \, x, y, [x, y] = x$. Then we define a linear map $\phi : L \to \text{gl}(\mathbb{C}^2)$ by

$$\phi(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \phi(y) = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$$

Then $\phi$ is a representation of $L$. Furthermore, $\phi$ is isomorphic to the adjoint representation of $L$ on itself.

Proof. To show that $\phi$ is a representation, we just need to show that it is a homomorphism, that $\phi([x, y]) = [\phi(x), \phi(y)]$. By routine computations,

$$\phi([x, y]) = \phi(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$[\phi(x), \phi(y)] = \phi(x)\phi(y) - \phi(y)\phi(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Now we find an explicit isomorphism between the adjoint representation, $\text{ad} : L \to \text{gl}(L)$, and $\phi$. We define $\theta : \mathbb{C}^2 \to L$ by

$$\theta(e_1) = x \quad \theta(e_2) = -x + y$$

where $e_1 = (1, 0), e_2 = (0, 1)$ are the standard basis for $\mathbb{C}^2$. We need to show that for $ax + by \in L, v \in \mathbb{C}^2$,

$$\theta(\phi(ax + by)v) = \text{ad}(ax + by)\theta(v)$$

We only need to show this holds for the basis vectors $e_1, e_2$.

$$\theta(\phi(ax + by)e_1) = \theta(a\phi(x)e_1 + b\phi(y)e_1)$$

$$= \theta(a(0) + b(-e_1))$$

$$= -b\theta(e_1)$$

$$= -bx$$

$$\text{ad}(ax + by)\theta(e_1) = [ax + by, x]$$

$$= a[x, x] + b[y, x]$$

$$= -bx$$

$$\theta(\phi(ax + by)e_2) = \theta(a\phi(x)e_2 + b\phi(y)e_2)$$

$$= \theta(ae_1 + be_1)$$

$$= (a + b)x$$

$$\text{ad}(ax + by)\theta(e_2) = [ax + by, \theta(e_2)]$$

$$= [ax + by, -x + y]$$

$$= -a[x, x] + a[x, y] - b[y, x] + b[y, y]$$

$$= (a + b)x$$
Proposition 12.10 (Exercise 7.11). Let \( L \) be a Lie algebra over \( F \), and let \( \phi : L \to \mathfrak{gl}(1, F) \) be a representation of \( L \). Then \( \phi(L') = 0 \).

Proof. If \( \phi \) is the zero map, then clearly \( \phi(L') = 0 \). If \( \phi \) is not the zero map, then the image of \( \phi \) has dimension at least one. Then since \( \mathfrak{gl}(1, F) \) is one-dimensional, \( \phi \) is onto. Then by Exercise 2.8a, \( \phi(L') = \mathfrak{gl}(1, F)' = 0 \). \( \square \)

Proposition 12.11 (Exercise 7.11). Let \( L \) be a Lie algebra over \( F \) such that \( L' = L \). Then the only 1-dimensional representation of \( L \) is the trivial representation.

Proof. Let \( \phi : L \to \mathfrak{gl}(1, F) \) be a representation. By part (a), \( \phi(L') = 0 \). Since \( L = L' \), this implies \( \phi(L) = \phi(L') = 0 \). Thus \( \phi \) is the trivial representation. \( \square \)

Proposition 12.12 (Exercise 7.11). Let \( L \) be a Lie algebra, and let \( \phi : L/L' \to \mathfrak{gl}(V) \) be a representation. Then define \( \bar{\phi} : L \to \mathfrak{gl}(V) \) by \( \bar{\phi}(x) = \phi(x + L') \). We claim that \( \bar{\phi} \) is a representation, and \( \bar{\phi}(L') = 0 \).

Proof. Let \( \pi : L \to L/L' \) be the canonical homomorphism given by \( \pi(x) = x + L' \). Notice that we have defined \( \bar{\phi} \) such that \( \bar{\phi} = \phi \pi \), that is, the following diagram commutes.

\[
\begin{array}{ccc}
L & \xrightarrow{\pi} & L/L' \\
\downarrow{\phi} & & \downarrow{\phi} \\
\phi & & \mathfrak{gl}(V)
\end{array}
\]

Since \( \pi \) and \( \phi \) are both homomorphisms, it follows that \( \bar{\phi} \) is also a homomorphism, and thus it is a representation. Now we show that for \( z \in L' \), \( \bar{\phi}(z) = 0 \).

\[
\bar{\phi}(z) = \phi(z + L') = \phi(L') = 0
\]

Proposition 12.13 (Exercise 7.11). Let \( L \) be a Lie algebra over \( \mathbb{C} \), such that \( L' \neq L \). Then \( L \) has infinitely many non-isomorphic 1-dimensional representations.

Proof. Let \( \text{span}\{y_1, y_2, \ldots, y_n\} \) be a basis for \( L' \) and extend this basis to a basis for \( L \), so that \( L = \text{span}\{y_1, y_2, \ldots, y_n, x_1, x_2, \ldots, x_m\} \). Let \( a \in \mathbb{C} \). We define a linear map \( \tilde{a} : L \to \mathfrak{gl}(\mathbb{C}) \) on the basis elements of \( L \) as follows.

\[
\begin{align*}
\tilde{a}(y_i)(c) &= 0 \\
\tilde{a}(x_i)(c) &= ac
\end{align*}
\]

for all \( c \in \mathbb{C} \). We claim that \( \tilde{a} \) is a representation. By definition, \( \tilde{a} \) is linear. We just need to check that \( \tilde{a} \) preserves the bracket, that is,

\[
\tilde{a}([w, z]) = [\tilde{a}(w), \tilde{a}(z)]
\]

for all \( w, z \in L \). Since \( [w, z] \in L' \), we know that \( [w, z] = \sum_{i=1}^{n} c_i y_i \) and thus \( \tilde{a}([w, z]) \) is the zero map. On the right hand side, the bracket is in \( \mathfrak{gl}(\mathbb{C}) \), which is one dimensional, so it
must be abelian. Thus both sides of this equation are the zero map for all \( w, z \in L \), so they are equal. Thus \( \tilde{a} \) is a representation.

We have shown that for each \( a \in \mathbb{C} \), there is a representation \( \tilde{a} : L \to \text{gl}(\mathbb{C}) \). We claim that each \( a \in \mathbb{C} \) gives a unique representation with respect to isomorphism. That is, we claim that if \( a, b \in \mathbb{C} \) and \( a \neq b \), then \( \tilde{a} \) is not isomorphic to \( \tilde{b} \).

Suppose there was an isomorphism \( \theta : \mathbb{C} \to \mathbb{C} \). Since \( \theta \) is a linear map, \( \theta(c) = \lambda c \) for some \( \lambda \in \mathbb{C} \). Then since \( \theta \) is an isomorphism, \( \theta(\tilde{a}(x)(c)) = \tilde{b}(x)(c) \) for all \( x \in L, c \in \mathbb{C} \). In particular, this must hold for \( x = x_1 \).

\[
\begin{align*}
\theta(\tilde{a}(x_1)(c)) &= \theta(ac) = \lambda ac \\
\tilde{b}(x_1)(c) &= b\theta(c) = b\lambda c \\
\Rightarrow \quad a\lambda c &= b\lambda c
\end{align*}
\]

So if there is such an isomorphism \( \theta \), then \( \lambda = 0 \) or \( a = b \). If \( \lambda = 0 \) then \( \theta \) is not bijective, so the only possibility is \( a = b \). Thus we have proven our claim that if \( a \neq b \), then \( \tilde{a} \) is not isomorphic to \( \tilde{b} \). To summarize, for each \( a \in \mathbb{C} \), there is a unique (up to isomorphism) representation \( \tilde{a} : L \to \text{gl}(\mathbb{C}) \), so there are uncountably infinitely many non-isomorphic 1-dimensional representations of \( L \).

\[\square\]

**Proposition 12.14** (Exercise 7.12i). Let \( L \) be a Lie algebra over \( F \) and let \( V \) be an \( L \)-module. On the dual space \( V^* \), define an action on \( L \) by

\[
(x \cdot \theta)(v) = -\theta(x \cdot v)
\]

for \( x \in L, v \in V, \theta \in V^* \). This action gives \( V^* \) the structure of an \( L \)-module.

**Proof.** We need to show that the conditions M1, M2, and M3 on page 55 hold. Let \( x, y \in L, \theta, \psi \in V^*, a, b \in F, \) and \( v \in V \).

\[
\begin{align*}
((ax + by) \cdot \theta)(v) &= -\theta((ax + by) \cdot v) \\
&= -\theta(a(x \cdot v) + b(y \cdot v)) \\
&= -a\theta(x \cdot v) - b\theta(y \cdot v) \\
&= a(x \cdot \theta)(v) + b(y \cdot \theta)(v) \\
&= ((ax \cdot \theta) + b(y \cdot \theta))(v) \\
\Rightarrow \quad (ax + by) \cdot \theta &= a(x \cdot \theta) + b(y \cdot \theta)
\end{align*}
\]

Thus condition M1 is satisfied.

\[
\begin{align*}
x \cdot (a\theta + b\psi)(v) &= -(a\theta + b\psi)(x \cdot v) \\
&= -a\theta(x \cdot v) - b\psi(x \cdot v) \\
&= a(x \cdot \theta)(v) + b(x \cdot \psi)(v) \\
&= (a(x \cdot \theta) + b(x \cdot \psi))(v) \\
\Rightarrow \quad x \cdot (a\theta + b\psi) &= a(x \cdot \theta) + b(x \cdot \psi)
\end{align*}
\]

89
Thus condition M2 is satisfied.

\[
([x, y] \cdot \theta)(v) = -\theta([x, y] \cdot v) \\
= -\theta(x \cdot (y \cdot v) - y \cdot (x \cdot v)) \\
= -\theta(x \cdot (y \cdot v)\theta(y \cdot (x \cdot v)) \\
= (x \cdot \theta)(y \cdot v) - (y \cdot \theta)(x \cdot v) \\
= -(y \cdot (x \cdot \theta))(v) + (x \cdot (y \cdot \theta))(v) \\
= -(y \cdot (x \cdot \theta) + x \cdot (y \cdot \theta))(v) \\
= (x \cdot (y \cdot \theta) - y \cdot (x \cdot \theta))(v)
\]

\[
\Rightarrow ([x, y] \cdot \theta) = (x \cdot (y \cdot \theta) - (y \cdot (x \cdot \theta))
\]

Thus M3 is satisfied. Thus \(V^*\) is an \(L\)-module with this action. \(\square\)

**Lemma 12.15** (for Exercise 7.12(i)). Let \(V\) be a Lie module for \(L\), and fix a basis \(\beta\) of \(V\). If

\[
[v]^T[x]^T[w] = -[v][x][w]
\]

for all \(x \in L, v, w \in V\), then \([x]^T = -[x]\) for \(x \in L\).

**Proof.** Let \(\beta = \{e_1, e_2, \ldots\}\). Then since we have this equation for all \(v, w \in V\), let \(v = e_i\) and \(w = e_j\). Then multiplying \([x]^T\) by \([e_i]^T\) on the left picks off the \(i\)th row of \([x]^T\), and multiplying the resulting row vector on the right by \([e_j]\) picks off the \(j\)th entry, so

\[
\]

Likewise,

\[
[v]^T(-[x])[w] = [e_i]^T(-[x])[e_j] = (-[x])_{ij}
\]

Thus \(([x]^T)_{ij} = (-[x])_{ij}\) for each \(1 \leq i, j \leq \dim V\), so \([x]^T = -[x]\). \(\square\)

**Proposition 12.16** (Exercise 7.12i). Let \(V\) be an module for the Lie algebra \(L\). Let \(\{e_1, e_2, \ldots\}\) be a basis of \(V\). Define a linear map \(\psi : V \rightarrow V^*\) by \(\psi(e_i) = \theta^i\) where \(\theta^i(e_j) = \delta_{ij}\). Then \(\psi\) is an isomorphism if and only if the matrices representing the action of \(L\) in the basis \(\beta\) are skew-symmetric.

**Proof.** Note that in terms of the basis \(\beta\), we think of \(e_i\) as being a column vector. Then \(\theta^i\) is a row vector, and \([\theta^i] = [e_i]^T\). More generally, \([\psi(v)] = [v]^T\).

Suppose \(\psi\) is an isomorphism. Then for \(x \in L, v, w \in V\), we have \(\psi(x \cdot v)(w) = (x \cdot \psi(v))(w) = -\psi(v)(x \cdot w)\). Now we think of this equation in terms of matrix representations, so we think of \([v], [w]\) as column vectors.

\[
[\psi(x \cdot v)] = ([x][v])^T = [v]^T[x]^T
\]

\[
[\psi(x \cdot v)(w)] = [\psi(x \cdot v)][w] = [v]^T[x]^T[w]
\]

\[
[-\psi(v)(x \cdot w)] = -[v]^T[x][w]
\]

\[
\Rightarrow [v]^T[x]^T[w] = [v]^T(-[x])[w]
\]

90
for all \( x \in L, v, w \in V \). Then by the lemma, \([x]^T = -[x]\), so all the matrices representing the action of \( L \) are skew-symmetric.

Now suppose that the matrices \([x]\) for \( x \in L \) are skew symmetric (with respect to the basis \( \beta = \{e_1, e_2, \ldots\} \)). Then we claim \( \psi \) is an isomorphism. We need to show that \( \psi(x \cdot v)(w) = (x \cdot \psi(v))(w) \) for \( x \in L, v, w \in V \).

\[
[\psi(x \cdot v)(w)] = [\psi([x][v])(w)] \\
= ([x][v])^T[w] \\
= [v]^T[x]^T[w] \\
= -[v]^T[x][w] \\
= -[\psi(v)][x][w] \\
= [-\psi(v)(x \cdot w)] \\
= [(x \cdot \psi(v))(w)]
\]

Thus \( \psi(x \cdot v)(w) \) and \((x \cdot \psi(v))(w)\) have the same matrix representation (with respect to \( \beta \)), so they are equal. \( \square \)

**Proposition 12.17** (Exercise 7.12ii). Let \( L \) be a Lie algebra over \( F \), and let \( V, W \) be \( L \)-modules. Define an action

\[
L \times \text{Hom}(V, W) \to \text{Hom}(V, W)
\]

\[
(x \cdot \theta)(v) = x \cdot (\theta(v)) - \theta(x \cdot v)
\]

for \( x \in L, v \in V, \theta \in \text{Hom}(V, W) \). This action gives \( \text{Hom}(V, W) \) the structure of an \( L \)-module.

**Proof.** We must show that the equations M1, M2, and M3 hold. Let \( a, b \in F, x, y \in L, v \in V, \) and \( \theta, \psi \in \text{Hom}(V, W) \).

\[
((ax + by) \cdot \theta)(v) = (ax + by) \cdot (\theta(v)) - \theta((ax + by) \cdot v)
\]

\[
= a(x \cdot (\theta(v))) + b(y \cdot (\theta(v))) - \theta(a(x \cdot v) + b(y \cdot v))
\]

\[
= a(x \cdot (\theta(v)) - \theta(x \cdot v)) + b(y \cdot (\theta(v)) - \theta(y \cdot v))
\]

\[
= a(x \cdot \theta)(v) - b(y \cdot \theta)(v)
\]

\[
= (a(x \cdot \theta) - b(y \cdot \theta))(v)
\]

Thus M1 holds.

\[
(a\theta + b\psi)(v) = x \cdot ((a\theta + b\psi)(v)) - (a\theta + b\psi)(x \cdot v)
\]

\[
= x \cdot (a\theta(v) + b\psi(v)) - (a\theta(x \cdot v)) - b\psi(x \cdot v)
\]

\[
= a(x \cdot (\theta(v)) + b(x \cdot (\psi(v))) - a\theta(x \cdot v) - b\psi(x \cdot v)
\]

\[
= a(x \cdot (\theta(v)) - \theta(x \cdot v)) + b(x \cdot (\psi(v)) - \psi(x \cdot v))
\]

\[
= a(x \cdot \theta)(v) + b(x \cdot \psi)(v)
\]

\[
= (a(x \cdot \theta) + b(x \cdot \psi))(v)
\]

Thus M1 holds.

\[
x \cdot (a\theta + b\psi) = a(x \cdot \theta) + b(x \cdot \psi)
\]
Thus M2 holds. To show M3 holds, we must show that \([x, y] \cdot \theta = x \cdot (y \cdot \theta) - y \cdot (x \cdot \theta)\) as maps, so we need to show that for \(v \in V\) these maps act on \(v\) in the same way. First we compute \((x, y) \cdot \theta)(v)\).

\[
([x, y] \cdot \theta)(v) = [x, y] \cdot (\theta(v)) - \theta([x, y] \cdot v) = x \cdot (y \cdot (\theta(v))) - y \cdot (x \cdot (\theta(v))) - \theta(x \cdot (y \cdot v) - y \cdot (x \cdot v)) = x \cdot (y \cdot (\theta(v))) - y \cdot (x \cdot (\theta(v))) - \theta(x \cdot (y \cdot v)) + \theta(y \cdot (x \cdot v))
\]

Now we compute \((x \cdot (y \cdot \theta) - y \cdot (x \cdot \theta))(v)\). We compute the two terms separately after expanding.

\[
(x \cdot (y \cdot \theta) - y \cdot (x \cdot \theta))(v) = (x \cdot (y \cdot \theta))(v) - (y \cdot (x \cdot \theta))(v) = x \cdot ((y \cdot (\theta(v)))) - (y \cdot (\theta))((x \cdot v)) = x \cdot (y \cdot (\theta(v))) - x \cdot (\theta(y \cdot v)) - y \cdot (\theta(x \cdot v)) + \theta(y \cdot (x \cdot v))
\]

Now using the computations for the two terms, we get an expression for \((x \cdot (y \cdot \theta) - y \cdot (x \cdot \theta))(v)\) involving eight terms.

\[
(x \cdot (y \cdot \theta) - y \cdot (x \cdot \theta))(v) = x \cdot (y \cdot (\theta(v))) - x \cdot (\theta)(y \cdot v)) - y \cdot (\theta(x \cdot v)) + \theta(y \cdot (x \cdot v)) + y \cdot (x \cdot (\theta(v))) - y \cdot (\theta(x \cdot v)) - x \cdot (\theta(y \cdot v)) + \theta(x \cdot (y \cdot v))
\]

Fortunately, two pairs of these terms cancel: we have a \( -x \cdot (\theta(y \cdot v))\) term, which cancel each other, and also the pair \( -y \cdot (\theta(x \cdot v))\) and \(y \cdot (\theta(x \cdot v))\). This leaves

\[
(x \cdot (y \cdot \theta) - y \cdot (x \cdot \theta))(v) = x \cdot (y \cdot (\theta(v))) + \theta(y \cdot (x \cdot v)) + y \cdot (x \cdot (\theta(v))) + \theta(x \cdot (y \cdot v))
\]

and one can match up these terms one by one with the four terms in our expression for \((x, y) \cdot \theta)(v)\) computed earlier. Thus, we have shown that \(\Hom(V, W)\) is an \(L\)-module with this action.

\[\square\]

**Proposition 12.18** (Exercise 7.12ii). Let \(V, W\) be \(L\)-modules and define an \(L\)-module structure on \(\Hom(V, W)\) by

\[
(x \cdot \theta)(v) = x \cdot (\theta(v)) - \theta(x \cdot v)
\]

for \(x \in L, \theta \in \Hom(V, W), \) and \(v \in V\). Then \(\theta\) is an \(L\)-module homomorphism if and only if \(x \cdot \theta = 0\) for all \(x \in L\).

**Proof.** Suppose \(x \cdot \theta = 0\) for all \(x \in L\). Then for \(v \in V\),

\[
(x \cdot \theta)(v) = x \cdot (\theta(v)) - \theta(x \cdot v) = 0 \implies \theta(x \cdot v) = x \cdot (\theta(v))
\]

thus \(\theta\) is an \(L\)-module homomorphism. Now suppose that \(\theta\) is an \(L\)-module homomorphism. Then for \(x \in L, v \in V\),

\[
\theta(x \cdot v) = x \cdot (\theta(v)) \implies x \cdot (\theta(v)) - \theta(x \cdot v) = (x \cdot \theta)(v) = 0 \implies x \cdot \theta = 0
\]

\[\square\]
13 Chapter 8 Exercises

Proposition 13.1 (Exercise 8.1). Let \{e, f, h\} be the usual basis for \(\text{sl}(2, \mathbb{C})\) and let \(V_d\) be the vector space of homogenous polynomials in \(x\) and \(y\) of degree \(d\). Let \(\beta\) be the usual basis for \(V_d\).

\[
\beta = \{x^d, x^{d-1}y, \ldots, y^d\}
\]

Define a module structure by \(\phi : \text{sl}(2, \mathbb{C}) \to \text{gl}(V_d)\) by

\[
\phi(e) = x \frac{\partial}{\partial y}, \quad \phi(f) = y \frac{\partial}{\partial x}, \quad \phi(h) = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}
\]

Then for any basis element \(x^a y^b\), the submodule generated by \(x^a y^b\) is all of \(V_d\).

Proof. Let \(W_k = \text{span}\{x^k y^{d-k}\}\) for \(k = 0, 1, \ldots, d\). Then \(V_d = W_0 \oplus W_1 \oplus \ldots \oplus W_d\). We compute the image of \(W_k\) under the action of \(\phi(e)\) and \(\phi(f)\).

\[
\phi(e)W_k = \text{span}\{x \frac{\partial}{\partial y} x^k y^{d-k}\} = \text{span}\{(d-k)x^{k+1}y^{d-k-1}\} = W_{k+1}
\]

\[
\phi(f)W_k = \text{span}\{y \frac{\partial}{\partial x} x^k y^{d-k}\} = \text{span}\{kx^{k-1}y^{d-k+1}\} = W_{k-1}
\]

The submodule generated by \(x^a y^b\) contains \(W_a\), and then by action of \(\phi(e)\), it also contains \(W_{a+1}, W_{a+2}, \ldots, W_d\). By the actino of \(\phi(f)\), it also contains \(W_{a-1}, W_{a-2}, \ldots, W_0\). Thus the submodule contains each \(W_k\), so it contains all of \(V_d\).

Proposition 13.2 (Exercise 8.2i). Let \(\psi : \text{sl}(2, \mathbb{C}) \to \text{gl}(\mathbb{C})\) be the trivial representation and let \(\phi : \text{sl}(2, \mathbb{C}) \to \text{gl}(V_0)\) be the representation given on pages 67-68 of Erdmann and Wildon. Define \(\theta : \mathbb{C} \to V_0\) by \(\theta(x) = x\). Then \(\theta\) is a Lie module isomorphism.

Proof. \(V_0\) is equal to \(\mathbb{C}\) as a set, since \(V_0\) is the set of constant polynomials over \(\mathbb{C}\). Since \(\theta\) is the identity on \(\mathbb{C}\), it is a bijection and it is linear. We need to show that \(\theta(\psi(x)v) = \phi(x)\theta(v)\) for all \(v \in \mathbb{C}\) and each \(x\) in some basis of \(\text{sl}(2, \mathbb{C})\). We use the usual basis \(e, f, h\). Since \(\psi(x)v = 0\), we have \(\theta(\psi(x)v) = 0\) for all \(x \in \text{sl}(2, \mathbb{C})\). Thus the LHS of our equation to show is always zero.

\[
\phi(e)v = x \frac{\partial}{\partial y} v = 0
\]

\[
\phi(f)v = y \frac{\partial}{\partial x} v = 0
\]

\[
\phi(h)v = x \frac{\partial}{\partial x} v + y \frac{\partial}{\partial y} v = 0 + 0 = 0
\]

Thus the RHS of our equation to show is also always zero.

Proposition 13.3 (Exercise 8.2ii). Let \(\psi : \text{sl}(2, \mathbb{C}) \to \text{gl}(\mathbb{C}^2)\) be the natural representation and let \(\phi : \text{sl}(2, \mathbb{C}) \to \text{gl}(V_1)\) be the representation described by Erdmann and Wildon. Define \(\theta : \mathbb{C}^2 \to V_1\) by

\[
\theta \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = v^1 x + v^2 y
\]

Then \(\theta\) is a Lie module isomorphism.
Proof. \( \theta \) is clearly a bijection. We show that \( \theta \) is linear.

\[
\theta \left( \lambda \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} + \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} \right) = \theta \left( \lambda v^1 + w^1 \right) + \theta \left( \lambda v^2 + w^2 \right) = (\lambda v^1 + w^1)x + (\lambda v^2 + w^2)y
\]

Thus \( \theta \) is linear. We need to show that \( \theta(\psi(x)v) = \phi(x)\theta(v) \) for \( v \in \mathbb{C}^2 \) and all \( x \) in some basis of \( \text{sl}(2, \mathbb{C}) \). We use the usual basis \( e, f, h \).

\[
\theta(\psi(e)v) = \theta \left( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} \right) = \theta \left( \begin{pmatrix} v^2 \\ 0 \end{pmatrix} \right) = v^2x = \phi(e)(v^1x + v^2y) = \phi(e)\theta(v)
\]

\[
\theta(\psi(f)v) = \theta \left( \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} \right) = \theta \left( \begin{pmatrix} 0 \\ v^1 \end{pmatrix} \right) = v^1y = \phi(f)(v^1x + v^2y) = \phi(f)\theta(v)
\]

\[
\theta(\psi(h)v) = \theta \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} \right) = \theta \left( \begin{pmatrix} v^1 \\ -v^2 \end{pmatrix} \right) = v^1x - v^2y = \phi(h)(v^1x + v^2y) = \phi(h)\theta(v)
\]

Proposition 13.4 (Exercise 8.2iii). The linear map \( \theta : (2, \mathbb{C}) \rightarrow \mathbb{V}_x \) given by

\[
\theta(e) = x^2 \quad \theta(h) = -2xy \quad \theta(f) = -y^2
\]

is an isomorphism of Lie modules.

Proof. \( \theta \) is a linear bijection by definition. We need to show that \( \theta(\text{ad } x(y)) = \phi(x)\theta(y) \) for \( x, y \in \{e, f, h\} \).

\[
\theta([e, f]) = \theta(h) = -2xy = x \frac{\partial}{\partial y}(-y^2) = \phi(e)\theta(f)
\]

\[
\theta([e, h]) = \theta(-2e) = -2\theta(e) = -2x^2 = x \frac{\partial}{\partial y}(-2xy) = \phi(e)\theta(h)
\]

\[
\theta([f, e]) = \theta(-h) = -\theta(h) = 2xy = y \frac{\partial}{\partial x}x^2 = \phi(f)\theta(e)
\]

\[
\theta([f, h]) = \theta(2f) = -2y^2 = y \frac{\partial}{\partial x}(-2xy) = \phi(f)\theta(h)
\]

\[
\theta([h, e]) = \theta(2e) = 2x^2 = \left( x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right)x^2 = \phi(h)\theta(e)
\]

\[
\theta([h, f]) = \theta(-2f) = 2y^2 = \left( x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right)(-y^2) = \phi(h)\theta(f)
\]

Proposition 13.5 (Exercise 8.3). The subalgebra of \( \text{sl}(2, \mathbb{C}) \) consisting of matrices of the form

\[
\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

is isomorphic to \( \text{sl}(2, \mathbb{C}) \).
Proof. Let $A$ be the subalgebra in question. In terms of $2 \times 2$ unit matrices, the usual basis $e, f, h$ for $\text{sl}(2, \mathbb{C})$ is the matrices $\{e, f, h\} = \{e_{12}, e_{21}, e_{11} - e_{22}\}$. One can see that the $3 \times 3$ unit matrices $\{e_{12}, e_{21}, e_{11} - e_{22}\}$ are a basis for $A$. We define a linear map $\phi : \text{sl}(2, \mathbb{C}) \to A$ by

$$\phi(e_{12}) = e_{12} \quad \phi(e_{21}) = e_{21} \quad \phi(e_{11} - e_{22}) = e_{11} - e_{22}$$

where $e_{ij}$ may refer to either a $2 \times 2$ or $3 \times 3$ matrix depending on context. and claim that $\phi$ is a Lie algebra isomorphism. Since it maps basis to basis, it is a bijection and linear. As shown in chapter 1, the rule for the bracket of unit matrices is

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{kj}$$

So clearly, the brackets in $\text{sl}(2, \mathbb{C})$ and $A$ are preserved by $\phi$ since $\phi$ simply changes the interpretation of unit matrix from $2 \times 2$ to $3 \times 3$. \qed

**Proposition 13.6 (Exercise 8.3).** Because of the previous proposition, we can view $\text{sl}(3, \mathbb{C})$ as a module for $\text{sl}(2, \mathbb{C})$ with the action $x \cdot y = [\phi(x), y]$ where $\phi : \text{sl}(2, \mathbb{C}) \to \text{sl}(3, \mathbb{C})$ is the map

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mapsto \begin{pmatrix} a & b & 0 \\ c & -a & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

As an $\text{sl}(2, \mathbb{C})$ module, $\text{sl}(3, \mathbb{C}) \cong V_2 \oplus V_1 \oplus V_1 \oplus V_0$.

Proof. Let $\{e, f, h\}$ be the usual basis for $\text{sl}(2, \mathbb{C})$. In terms of the matrix units, $e = e_{12}, f = e_{21}$ and $h = e_{11} - e_{22}$. Conveniently, $\phi(e_{ij}) = e_{ij}$, where on the LHS we view $e_{ij}$ as a $2 \times 2$ matrix and on the RHS we view $e_{ij}$ as a $3 \times 3$ matrix. Consider $e_{13} \in \text{sl}(3, \mathbb{C})$. Using the rule $[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{kj}$, we compute

$$h \cdot e_{13} = [h, e_{13}] = [e_{11} - e_{22}, e_{13}] = [e_{11}, e_{13}] - [e_{22}, e_{13}] = e_{13}$$

$$e \cdot e_{13} = [e_{12}, e_{13}] = 0$$

Thus by Corollary 8.6, the submodule generated by $e_{13}$ is isomorphic to $V_1$. Since $f \cdot e_{13} = e_{23}$, this submodule contains the linearly independent vectors $\{e_{13}, e_{23}\}$. Since $V_1$ is 2-dimensional, we conclude that $\langle e_{13} \rangle = \text{span}\{e_{13}, e_{23}\}$. Now consider $e_{31}$. We compute

$$h \cdot e_{32} = e_{32}$$

$$e \cdot e_{32} = 0$$

$$f \cdot e_{32} = -e_{31}$$

So as before, we use Corollary 8.6 to conclude that $\langle e_{32} \rangle = \text{span}\{e_{32}, e_{31}\}$. Now consider $e_{12}$. We compute

$$h \cdot e_{12} = 2e_{12}$$

$$e \cdot e_{12} = 0$$

$$f \cdot e_{12} = -(e_{11} - e_{22})$$

$$f \cdot (f \cdot e) = f \cdot (-h) = -[f, h] = 2f = 2e_{21}$$
Thus by Corollary 8.6, $< e_{12} > \cong V_2$ and $< e_{12} > = \text{span}\{e_{12}, e_{21}, e_{11} - e_{22}\}$. Finally, we compute

$$h \cdot (e_{11} + e_{22} - 2e_{33}) = 0$$
$$e \cdot (e_{11} + e_{22} - 2e_{33}) = 0$$

so by Corollary 8.6, $< e_{11} + e_{22} - 2e_{33} > \cong V_0$. Putting all of this together, we have

$$\text{sl}(3, \mathbb{C}) = \text{span}\{e_{12}, e_{21}, e_{11} - e_{22}, e_{32}, e_{23}, e_{13}, e_{11} + e_{22} - 2e_{33}\}$$
$$= \langle e_{12} \rangle \oplus \langle e_{32} \rangle \oplus \langle e_{13} \rangle \oplus \langle e_{11} + e_{22} - 2e_{33} \rangle$$
$$\cong V_3 \oplus V_2 \oplus V_2 \oplus V_0$$

\[ \square \]

**Lemma 13.7** (for Exercise 8.4). Consider the $\text{sl}(2, \mathbb{C})$ module $V_d$ with $d \in \mathbb{N} \cup \{0\}$. Let $W_r = \{v \in V_d : h \cdot v = rv\}$. Then if $d$ is even, $\dim W_0 = 1$ and $\dim W_1 = 0$. If $d$ is odd, then $\dim W_0 = 0$ and $\dim W_1 = 1$.

**Proof.** As shown on page 68 of Erdmann and Wildon, for a basis element $x^a y^{d-a}$ of $V_d$,

$$h \cdot x^a y^{d-a} = (a - (d - a))x^a y^{d-a} = (2a - d)x^a y^{d-a}$$

Suppose $d$ is even. Then $a = d/2$ is an integer, so

$$h \cdot x^{d/2} y^{d/2} = (d/2 - d/2)x^{d/2} y^{d/2} = 0$$

but for no other basis element is $h \cdot x^a y^{d-a}$ equal to zero, so $W_0 = \text{span}\{x^{d/2} y^{d/2}\}$. Since $d$ is even, $2a - d$ is also even, so $2a - d \neq 1$, so $W_1 = \{0\}$, so $\dim W_1 = 0$.

Suppose $d$ is odd. Then $2a - d$ is odd, so $2a - d \neq 0$, so $W_0 = \{0\}$, but there is precisely one $a = (d + 1)/2$ such that $2a - d = 1$, so there is one basis element $x^{(d+1)/2} y^{(d-1)/2}$ such that

$$h \cdot x^{(d+1)/2} y^{(d-1)/2} = ((d + 1)/2 - (d - 1)/2)x^{(d+1)/2} y^{(d-1)/2} = x^{(d+1)/2} y^{(d-1)/2}$$

so $\dim W_1 = 1$. \[ \square \]

**Corollary 13.8** (to previous lemma, for 8.4). Let $V_d$ be the $\text{sl}(2, \mathbb{C})$ module defined by Erdmann and Wildon. Then $\dim W_0 + \dim W_1 = 1$.

**Proof.** If $d$ is even, $\dim W_0 + \dim W_1 = 1 + 0 = 1$. If $d$ is odd, then $\dim W_0 + \dim W_1 = 0 + 1 = 1$. \[ \square \]

**Proposition 13.9** (Exercise 8.4). Let $V$ be a finite-dimensional $\text{sl}(2, \mathbb{C})$ module. Let $W_r = \{v \in V : h \cdot v = rv\}$. Then if $V$ is a direct sum of $k$ irreducible modules, $k = \dim W_0 + \dim W_1$.

**Proof.** By Weyl’s Theorem and Theorem 8.5 of Erdmann and Wildon, we can write $V$ as a direct sum of $k$ irreducible modules, each of which is isomorphic to some $V_d$.

$$V = V_{d_1} \oplus V_{d_2} \oplus \ldots \oplus V_{d_k}$$

96
Then we define \( W_r = \{ v \in V_d : h \cdot v = rv \} \). Then

\[
W_0 = \{ v \in V : h \cdot v = 0 \} = \bigoplus_{i=1}^{k} W_{0i}
\]

\[
W_1 = \{ v \in V : h \cdot v = 0 \} = \bigoplus_{i=1}^{k} W_{1i}
\]

By the previous corollary, \( \dim W_0 \oplus W_1 = 1 \), because the dimension of a direct sum is the sum of the dimensions. Note also that in the following computation, we use the fact that the dimension of a sum is not changed by changing the order of the summands.

\[
\dim W_0 + \dim W_1 = \dim(W_0 \oplus W_1) = \dim \bigoplus_{i=1}^{k}(W_{0i} \oplus W_{1i}) = k
\]

**Lemma 13.10 (for Exercise 8.6i).** Let \( V \) be an \( \mathfrak{sl}(2, \mathbb{C}) \) module. Then for \( v \in V \),

\[
fe \cdot v = (ef - h) \cdot v \\
(13.1)
\]

\[
ef \cdot v = (fe + h) \cdot v \\
(13.2)
\]

\[
he \cdot v = (eh + 2e) \cdot v \\
(13.3)
\]

\[
eh \cdot v = (he - 2e) \cdot v \\
(13.4)
\]

\[
hf \cdot v = (fh - 2f) \cdot v \\
(13.5)
\]

\[
fh \cdot v = (hf + 2f) \cdot v \\
(13.6)
\]

\[
\frac{1}{2} heh \cdot v = (\frac{1}{2} eh^2 + eh) \cdot v \\
(13.7)
\]

\[
\frac{1}{2} heh \cdot v = (\frac{1}{2} h^2 e - he) \cdot v \\
(13.8)
\]

\[
\frac{1}{2} hfh \cdot v = (\frac{1}{2} fh^2 - fh) \cdot v \\
(13.9)
\]

\[
\frac{1}{2} hfh \cdot v = (\frac{1}{2} h^2 f + hf) \cdot v \\
(13.10)
\]

**Proof.** Proof of 0.1:

\[
h \cdot v = [e, f] \cdot v = fe \cdot v - ef \cdot v \implies fe \cdot v = (ef - h) \cdot v
\]

Proof of 0.2:

\[
h \cdot v = (ef - fe) \cdot v \implies ef \cdot v = (fe + h) \cdot v
\]

Proof of 0.3 and 0.4:

\[
[e, h] \cdot v = (eh - he) \cdot v \implies -2e \cdot v = (eh - he) \cot v
\]

\[
\implies he \cdot v = (eh + 2e) \cdot v
\]

\[
\implies eh \cdot v = (he - 2e) \cdot v
\]
Proof of 0.5 and 0.6:

\[
[f, h] \cdot v = (fh - hf) \cdot v \implies 2f \cdot v = (fh - hf) \cdot v \\
\implies hf \cdot v = (fh - 2f) \cdot v \\
\implies fh \cdot v = (hf + 2f) \cdot v
\]

Proof of 0.7:

\[
[h, e] \cdot h \cdot v = (he - eh)h \cdot v = (heh - eh^2) \cdot v \\
2eh \cdot v = (heh - eh^2) \cdot v \\
\frac{1}{2} heh \cdot v = (eh + \frac{1}{2} eh^2) \cdot v
\]

Proof of 0.8:

\[
h \cdot [h, e] \cdot v = h(he - eh) \cdot v = (h^2 e - heh) \cdot v \\
2he \cdot v = (h^2 e - heh) \cdot v \\
heh \cdot v = (h^2 e - 2he) \cdot v \\
\frac{1}{2} heh \cdot v = (\frac{1}{2} h^2 e - he) \cdot v
\]

Proof of 0.9:

\[
[h, f] \cdot h = (hf - fh) \cdot h \cdot v = (hf h - fh^2) \cdot v \\
-2fh \cdot v = (hf h - fh^2) \cdot v \\
hfh \cdot v = (fh^2 - 2fh) \cdot v \\
\frac{1}{2} hfh \cdot v = (\frac{1}{2} fh^2 - fh) \cdot v
\]

Proof of 0.10:

\[
h \cdot [h, f] \cdot v = h(hf - fh) \cdot v = (h^2 f - hfh) \cdot v \\
-2hf \cdot v = (h^2 f - hfh) \cdot v \\
\frac{1}{2} \cdot v = (\frac{1}{2} + hf) \cdot v
\]

Proposition 13.11 (Exercise 8.6.i). Let $M$ be a finite-dimensional $\text{sl}(2, \mathbb{C})$ module, and define $c : M \to M$ by

\[
c(v) = \left(ef + fe + \frac{1}{2} h^2\right) \cdot v
\]

Then $c$ is a homomorphism of $\text{sl}(2, \mathbb{C})$ modules.
Proof. To show: We must show that \( c \) commutes with the actions of \( e, f, \) and \( h \). First we show that \( c \) commutes with \( c \).

\[
e f \cdot v = (h + fe) \cdot v \\
e^2 f \cdot v = (eh + efe) \cdot v \\
(e^2 f + efe + \frac{1}{2}eh) \cdot v = (eh + 2efe + \frac{1}{2}eh^2) \cdot v \\
e \cdot c(v) = (2efe + \frac{1}{2}heh) \cdot v \\
fe \cdot v = (ef - h) \cdot v \\
fe^2 \cdot v = (efe - he) \cdot v \\
(efe + fe^2 + \frac{1}{2}h^2e) \cdot v = (2efe - he + \frac{1}{2}h^2e) \cdot v \\
c(e \cdot v) = (2efe + \frac{1}{2}heh) \cdot v \\
\]
Thus \( e \cdot c(v) = c(e \cdot v) \). Now we show that \( c \) commutes with the action of \( f \).

\[
e f \cdot v = (h + fe) \cdot v \\
e f^2 \cdot v = (hf + fef) \cdot v \\
(e f^2 + fef + \frac{1}{2}h^2f) \cdot v = (hf + 2fef + \frac{1}{2}h^2f) \cdot v \\
c(f \cdot v) = (2fef + \frac{1}{2}hfh) \cdot v \\
fe \cdot v = (ef - h) \cdot v \\
f e^2 \cdot v = (feh - fh) \cdot v \\
(f^2 e + ffe + \frac{1}{2}f h^2) \cdot v = (2fef - fh + \frac{1}{2}f h^2) \cdot v \\
f \cdot c(v) = (2fef + \frac{1}{2}hfh) \cdot v \\
\]
Thus \( c(f \cdot v) = f \cdot c(v) \). Finally, we show that \( c \) commutes with the action of \( h \).

\[
hef \cdot v = (ehf - 2ef) \cdot v \\
hfe \cdot v = (fhe - 2fe) \cdot v \\
(hef + hfe) \cdot v = (ehf + fhe + 2ef - 2fe) \cdot v \\
efh \cdot v = (ehf + 2ef) \cdot v \\
feh \cdot v = (fhe - 2fe) \cdot v \\
(efh + feh) \cdot v = (ehf + fhe + 2ef - 2fe) \cdot v \\
\]
Thus

\[
c(h \cdot v) = (efh + feh + \frac{1}{2}h^3) \cdot v = (hef + hfe + \frac{1}{2}h^3) \cdot v = h \cdot c(v) \\
\]
Thus \( c \) is an \( \text{sl}(2, \mathbb{C}) \) module homomorphism. \( \square \)
Proposition 13.12 (Exercise 8.6ii). Let \( V_d \) be an irreducible \( \text{sl}(2, \mathbb{C}) \) module. Let \( c : V_d \to V_d \) be the Casimir operator, which is defined by

\[
c(v) = (ef + fe + 1/2h^2) \cdot v
\]

We have shown that \( c \) is a homomorphism. We claim that \( c(v) = 1/2d(d + 2)v \).

**Proof.** By Lemma 8.4, \( V_d \) contains a \( w \) such that \( h \cdot x = \lambda w \) and \( e \cdot w = 0 \). By Corollary 8.6, \( \lambda = d \). Then

\[
c(w) = (ef + fe + 1/2h^2) \cdot w = ef \cdot w + fe \cdot w + 1/2h^2 \cdot w = dw + 0 + 1/2d^2 w
\]

The last equality uses Exercise 8.5 to evaluate \( ef \cdot w \).

\[
c(w) = (1/2d + 1)dw = 1/2(d + 2)dw
\]

By Schur’s Lemma, since \( c \) is an \( \text{sl}(2, \mathbb{C}) \) module homomorphism from \( V_d \) to itself, \( c \) must be a scalar multiple of the identity transformation. Thus since \( c(w) = 1/2(d + 2)dw \) for some \( w \in V_d \), it follows that \( c(v) = 1/2(d + 2)dv \) for all \( v \in V_d \). \( \square \)

Proposition 13.13 (Exercise 8.6iii). Let \( M \) be a finite dimensional \( \text{sl}(2, \mathbb{C}) \) module and let \( c : M \to M \) be the Casimir operator. If

\[
M = \bigoplus_{i=1}^{r} \ker(c - \lambda_i I)^{m_i}
\]

is the primary decomposition of \( M \), then each \( \ker(c - \lambda_i I)^{m_i} \) is an \( \text{sl}(2, \mathbb{C}) \) module.

**Proof.** To show: For \( v \in V_i = \ker(c - \lambda_i I)^{m_i}, e \cdot v, f \cdot v, h \cdot v \in V_i \). Let \( v \in \ker(c - \lambda_i I)^{m_i} \). Note that \( (c - \lambda_i I)^{m_k} \) is a polynomial in \( c \). By part (i), \( e, f, h \) commute with \( c \), so they commute with any polynomial in \( c \), so

\[
(c - \lambda_i I)^{m_i} (e \cdot v) = e \cdot (c - \lambda_i I)^{m_i} (v) = 0
\]

(This is zero because \( v \in \ker(c - \lambda_i I)^{m_i} \).) Likewise for \( f \) and \( h \), the algebra is nearly identical. Thus \( \ker(c - \lambda_i I)^{m_i} \) is an \( \text{sl}(2, \mathbb{C}) \) submodule. \( \square \)

Proposition 13.14 (Exercise 8.6iv). Suppose \( M \) is a finite-dimensional \( \text{sl}(2, \mathbb{C}) \) module such that \( M \) has just one generalized eigenspace of the Casimir operator \( c \), that is, suppose

\[
M = \ker(c - \lambda I)^{m}
\]

for some \( \lambda \in \mathbb{C} \) and suppose that some irreducible submodule of \( M \) is isomorphic to \( V_d \). Then every irreducible submodule of \( M \) is isomorphic to \( V_d \).

**Proof.** Let \( U \) be an irreducible submodule of \( M \) such that \( U \cong V_d \). Acting on \( M \), \( c \) has only one eigenvalue \( \lambda \). By part (ii), \( c \) acts on \( U \cong V_d \) as the scalar \( 1/2(d)(d + 2) \), so \( \lambda = 1/2(d)(d + 2) \). Let \( N \cong V_k \) be another irreducible submodule of \( M \). Then \( c \) acts on \( N \) by the scalar \( 1/2(k)(k + 2) \), so \( \lambda = 1/2(k)(k + 2) \). Then \( 1/2(d)(d + 2) = 1/2(k)(k + 2) \) so \( k = d \). Thus \( N \cong V_d \). \( \square \)
Proposition 13.15 (Exercise 8.6v). Let $M$ be an $\text{sl}(2, \mathbb{C})$ module such that $M = \ker(c-\lambda I)^m$ and $M$ has an irreducible submodule isomorphic to $V_d$. Let $N$ be a submodule of $M$. Then any irreducible submodule of $M/N$ is isomorphic to $V_d$.

Proof. By part (iv), $M$ is isomorphic to a finite direct sum of $V_d$'s,

$$M \cong V_d \oplus V_d \oplus \ldots \oplus V_d$$

Since $V_d$ is irreducible, if any $N$ is a submodule of $M$, then $N$ is also isomorphic to a finite direct sum of $V_d$'s, with less than or fewer summands than $M$.

$$N \cong V_d \oplus \ldots \oplus V_d$$

Then

$$M \cong N \oplus (V_d \oplus \ldots \oplus V_d)$$

and thus $M/N$ is isomorphic to the remaining $V_d$'s after “subtracting” $N$.

$$M/N \cong V_d \oplus \ldots \oplus V_d$$

Thus every irreducible submodule of $M/N$ is isomorphic to $V_d$. \hfill \square

Proposition 13.16 (Exercise 8.7). Define $\psi : \mathbb{R}^3 \to \text{sl}(2, \mathbb{C})$ to be the linear map defined by

$$\psi(x) = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 0 & 0 \end{pmatrix} \quad \psi(y) = \begin{pmatrix} 0 & -\frac{i}{2} & \frac{i}{2} \\ \frac{i}{2} & 0 & 0 \end{pmatrix} \quad \psi(z) = \begin{pmatrix} -\frac{i}{2} & 0 & 0 \\ 0 & \frac{i}{2} & 0 \end{pmatrix}$$

We have shown that $\psi$ is a Lie algebra isomorphism. We claim that

$$\psi(x)^2 + \psi(y)^2 + \psi(y)^2 = -\frac{3}{4}I$$

where $I$ is the $2 \times 2$ identity matrix. Furthermore, we can write $\psi(x), \psi(y), \text{ and } \psi(z)$ in terms of $e, f,$ and $h$, and then see that the Casimir operator is represented by $3/2I$ (in this representation).

Proof.

$$\psi(x)^2 = \psi(y)^2 = \psi(z)^2 = -1/4I$$
$$\psi(x)^2 + \psi(y)^2 + \psi(z)^2 = -3/4I$$

$$\psi(x) = 1/2e - 1/2f$$
$$\psi(y) = -i/2e - i/2f$$
$$\psi(z) = -i/2h$$
$$e = \psi(x) + i\psi(y)$$
$$f = -\psi(x) + i\psi(y)$$
$$h = 2i\psi(z)$$

101
\[ ef + fe + 1/2h^2 = (\psi(x) + i\psi(y)(-\psi(x) + i\psi(y)) \\
+ (-\psi(x) + i\psi(y))(\psi(x) + i\psi(y)) + 1/2(2i\psi(z))^2 \\
= -\psi(x)^2 - \psi(y)\psi(x) + i\psi(x)\psi(y) - \psi(y)^2 \\
- \psi(x)^2 + i\psi(y)\psi(x) - i\psi(x)\psi(y) - \psi(y)^2 + 1/2(4)(-1)(\psi(z))^2 \\
= -2\psi(x)^2 - 2\psi(y)^2 - 2\psi(z)^2 \\
= -2(-3/4)I \\
= 3/2I \]
14 Chapter 9 Exercises

Proposition 14.1 (Exercise 9.1). Let $V$ be a vector space, and suppose $x \in \text{gl}(V)$ has Jordan decomposition $x = d + n$. Then $\text{ad} x : \text{gl}(V) \to \text{gl}(V)$ has Jordan decomposition $\text{ad} x = \text{ad} d + \text{ad} n$.

Proof. To show: $\text{ad} x = \text{ad} d + \text{ad} n$, and $\text{ad} d$ is diagonalisable, and $\text{ad} n$ is nilpotent, and $\text{ad} d \circ \text{ad} n = \text{ad} n \circ \text{ad} d$. Because $\text{ad}$ is linear, $\text{ad} x = \text{ad}(d + n) = \text{ad} d + \text{ad} n$. Since $d$ is diagonalisable, by Exercise 1.17, $\text{ad} d$ is diagonalisable. Since $n$ is nilpotent, by Lemma 5.1, $\text{ad} n$ is nilpotent. Finally, let $\phi \in \text{gl}(V)$. Then

$$\text{ad} d \circ \text{ad} n(\phi) = [d, [n, \phi]]$$
$$= [d, n\phi - \phi n]$$
$$= d n \phi - d \phi n - n d \phi + \phi d n$$
$$= n d \phi - n \phi d - d \phi n + \phi d n$$
$$= [n, d \phi - \phi d]$$
$$= [n, [d, \phi]]$$
$$= \text{ad} n \circ \text{ad} d(\phi)$$

Lemma 14.2 (for Exercise 9.2). Let $A, B$ be $n \times n$ matrices such that $A$ is upper triangular and $B$ is strictly upper triangular. Then $\text{tr}(AB) = 0$.

Proof. Let $A = (a_{ij})B = (b_{ij})$. Then

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

$$(AB)_{ii} = \sum_{k=1}^{n} a_{ik}b_{ki}$$

$$\text{tr}(AB) = \sum_{m=1}^{n} \sum_{k=1}^{n} a_{km}b_{mk}$$

Since both $A, B$ are upper triangular, $a_{mk} = 0$ for $m < k$ and $b_{mk} = 0$ for $k < m$. Thus in our sum for the trace, all terms are zero except perhaps those of the form $a_{kk}b_{kk}$.

$$\text{tr}(AB) = \sum_{k=1}^{n} a_{kk}b_{kk}$$

Since $B$ is strictly upper triangular, $b_{kk} = 0$ for each $k$. Thus $\text{tr}(AB) = 0$.

Proposition 14.3 (Exercise 9.2). Let $V$ be a complex vector space and let $L$ be a solvable Lie subalgebra of $\text{gl}(V)$. Then there is a basis of $V$ in which every element of $L'$ is represented by a strictly upper triangular matrix. Consequently, $\text{tr} xy = 0$ for $x \in L, y \in L'$.
Proof. By Lie’s Theorem, there is a basis of $V$ in which every $x \in L$ is represented by an upper triangular matrix. If $x_1, x_2 \in L$ are upper triangular and $x_3 = [x_1, x_2] \in L'$, then $x_3$ is strictly upper triangular, since the bracket of upper triangular matrices is strictly upper triangular (see Exercise 4.5). Thus $L'$ has a basis consisting of strictly upper triangular matrices.

Let $x \in L, y \in L'$. Then since $x$ is upper triangular and $y$ is strictly upper triangular, by the lemma, $\text{tr } xy = 0$.

Proposition 14.4 (Exercise 9.3). Let $L$ be a Lie algebra and let $I$ be an ideal of $L$. Then $I^\perp$ is an ideal of $L$.

Proof. To show: for $b \in I^\perp, x \in L$, we have $[b, x] \in I^\perp$. Let $b \in I^\perp, a \in I, x \in L$. By definition,

$$I^\perp = \{ b \in L : \kappa(b, a) = 0 \text{ for } a \in I \}$$

Since $I$ is an ideal, $[x, a] \in I$. Then $\kappa(b, [x, a]) = 0$, and by associativity of $\kappa$ (see page 80 of Erdmann and Wildon),

$$\kappa([b, x], a) = 0$$

Since $a \in I$ was arbitrary, this shows that $[b, x] \in I^\perp$. Thus $I^\perp$ is an ideal of $L$.

Proposition 14.5 (Exercise 9.4i). The Killing form of $\mathfrak{sl}(2, \mathbb{C})$ has the matrix

$$\begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

with respect to the usual basis $e, f, h$. It is non-degenerate.

Proof. Computations were done in Mathematica.

$$\kappa(e, e) = 0 \quad \kappa(f, e) = 4 \quad \kappa(h, e) = 0$$

$$\kappa(e, f) = 4 \quad \kappa(f, f) = 0 \quad \kappa(h, f) = 0$$

$$\kappa(e, h) = 0 \quad \kappa(f, h) = 0 \quad \kappa(h, h) = 8$$

For finite-dimensional vector space, a bilinear form is non-degenerate if and only if its matrix representation is invertible. This matrix clearly has nonzero determinant, so the form is non-degenerate.

Proposition 14.6 (Exercise 9.4ii). The Killing form $\kappa$ on $\mathfrak{gl}(2, \mathbb{C})$ is degenerate.

Proof. Take the identity matrix $I$ and let $x \in \mathfrak{gl}(2, \mathbb{C})$.

$$\text{ad } I(x) = [I, x] = Ix - xI = x - x = 0$$

so $\kappa(I, x) = 0$ for $x \in \mathfrak{gl}(2, \mathbb{C})$. Thus $\mathfrak{gl}(2, \mathbb{C})^\perp \neq \{O\}$, so $\kappa$ is degenerate.
Proposition 14.7 (Exercise 9.5). Let $L$ be a nilpotent Lie algebra over a field $F$. Then the Killing form $\kappa$ on $L$ is always zero, that is, for $x, y \in L$,

$$\kappa(x, y) = \text{tr}(\text{ad} x \circ \text{ad} y) = 0$$

Proof. Let $x, y \in L$. Since $L$ is nilpotent, $\text{ad} x, \text{ad} y$ are nilpotent maps (Theorem 6.3). By Theorem 6.1, since $\text{ad} L$ is a Lie subalgebra of $\text{gl}(L)$ in which every $\text{ad} x$ is nilpotent, there is a basis of $L$ in which everything in $\text{ad} L$ is represented by a strictly upper triangular matrix. Then $\text{ad} x \circ \text{ad} y$ is also represented by a strictly upper triangular matrix, so

$$\text{tr}(\text{ad} x \circ \text{ad} y) = 0$$

(Exercise 9.6)

We compute the Killing form for the complex 3-dimensional Lie algebras discussed in chapter 3. The Heisenberg algebra is nilpotent, so it has a Killing form that is always equal to zero. The algebra considered in 3.2.4 is isomorphic to $\text{sl}(2, \mathbb{C})$, which we have already computed the Killing form for.

The Lie algebra in section 3.2.2 is given by $L = \text{span}\{x, y, z\}$ where $[x, y] = x, [x, z] = y, [y, z] = 0$. Then one can compute the matrices of $\text{ad} x, \text{ad} y, \text{ad} z$:

$$[\text{ad} x] = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad [\text{ad} y] = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad [\text{ad} z] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

From this, clearly $\kappa(z, a) = 0$ for any $a \in L$. We still need to compute $\kappa(x, y), \kappa(x, x)$, and $\kappa(y, y)$. To do this, we compute the matrix products $[\text{ad} x][\text{ad} y], [\text{ad} x]^2, [\text{ad} y]^2$ and take the traces.

$$[\text{ad} x][\text{ad} y] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad [\text{ad} x]^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad [\text{ad} y]^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so we get $\kappa(x, x) = \kappa(x, y) = 0$ but $\kappa(y, y) = 1$. Thus to completely characterize the Killing form, we can write either

$$\kappa(a, b) = \begin{cases} 1 & \text{if } a = b = y \\ 0 & \text{otherwise} \end{cases}$$

Now we consider the Lie algebras discussed in section 3.2.3, beginning with case 2. In case 2, there is only one isomorphism class, which is the Lie algebra $L = \text{span}\{x, y, z\}$ with $[x, y] = y, [x, z] = y + z, [y, z] = 0$. Then we compute the matrices of $\text{ad} x, \text{ad} y, \text{ad} z$.

$$[\text{ad} x] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad [\text{ad} y] = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad [\text{ad} z] = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$
And now we compute all of the necessary matrix products.

\[
\begin{align*}
[\text{ad } x][\text{ad } y] &= 0 \\
[\text{ad } y][\text{ad } z] &= 0 \\
[\text{ad } x][\text{ad } z] &= \begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\
[\text{ad } x]^2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \\
[\text{ad } y]^2 &= 0 \\
[\text{ad } z]^2 &= 0
\end{align*}
\]

So \( \kappa(x, x) = 2 \) but it is zero for everything else. Thus we completely characterize \( \kappa \) by

\[
\kappa(a, b) = \begin{cases} 
2 & \text{if } a = b = x \\
0 & \text{otherwise}
\end{cases}
\]

Now we consider the class of 3-dimensional Lie algebras considered in Case 1 of section 3.2.3. These are the algebras such that \( L = \text{span}\{x, y, z\} \) and \([x, y] = y, [y, z] = 0, [x, z] = \lambda z\) for some fixed \( \lambda \in \mathbb{C} \). (As was shown in 3.2.4, we get a non-isomorphic Lie algebra for each \( \lambda \in \mathbb{C} \) except that the Lie algebra with \( \lambda^{-1} \) is isomorphic.) We compute the matrices of \( \text{ad } x, \text{ad } y, \text{ad } z \).

\[
\begin{align*}
[\text{ad } x] &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix} \\
[\text{ad } y] &= \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
[\text{ad } z] &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\lambda & 0 & 0 \end{pmatrix}
\end{align*}
\]

Now we compute the needed matrix products.

\[
\begin{align*}
[\text{ad } x][\text{ad } y] &= \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
[\text{ad } y][\text{ad } z] &= 0 \\
[\text{ad } x][\text{ad } z] &= 0 \\
[\text{ad } x]^2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda^2 \end{pmatrix} \\
[\text{ad } y]^2 &= 0 \\
[\text{ad } z]^2 &= 0
\end{align*}
\]

So we can characterize the Killing form by

\[
\kappa(a, b) = \begin{cases} 
1 + \lambda^2 & \text{if } a = b = x \\
0 & \text{otherwise}
\end{cases}
\]

\[
[\kappa] = \begin{pmatrix} 1 + \lambda^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

106
This concludes our characterization of the Killing forms of the 3-dimensional complex Lie algebras.

**Proposition 14.8** (Exercise 9.10). Let $L$ be a complex Lie algebra and let $\beta$ by a symmetric, bilinear, associative form on $L$. Define

$$\theta : L \to L^* \quad \theta(x)y = \beta(x, y)$$

Then $\theta$ is linear. (We call $\theta$ the linear map induced by $\beta$.)

**Proof.** First, we need to show that $\theta$ is well-defined, that is, that $\theta(x)$ does map into $L^*$. To do this, we need to show that $\theta(x)$ is a linear map. Let $a, b \in \mathbb{C}$ and $x, y, z \in L$.

$$\theta(x)(ay + bz) = \beta(x, ay + bz) = a\beta(x, y) + b\beta(x, z) = a\theta(x)y + b\theta(x)z$$

Thus $\theta(x)$ is linear. Now we can show that $\theta$ is linear. To do that we consider how $\theta(ax + by)$ acts on a given $z \in L$.

$$\theta(ax + by)z = \beta(ax + by)z = a\beta(x, z) + b\beta(y, z) = a\theta(x)z + b\theta(y)z$$

Thus

$$\theta(ax + by) = a\theta(x) + b\theta(y)$$

(The above is an equality of maps.) Thus $\theta$ is linear. □

**Proposition 14.9** (Exercise 9.10). Let $L$ be a complex Lie algebra and let $\beta$ be a symmetric, associative bilinear form on $L$. Define $\theta : L \to L^*$ by $\theta(x)y = \beta(x, y)$. If $\beta$ is non-degenerate, then $L$ and $L^*$ are isomorphic as $L$-modules.

**Proof.** First we need to show that $\theta$ is an $L$-module homomorphism. To do this, we need to show that for $x, y \in L$, $\theta(x \cdot y) = x \cdot \theta(y)$.

(Recall that when we regard $L$ as an $L$-module, the action is simply the bracket, that is, $x \cdot y = [x, y]$. When we regard $L^*$ as an $L$-module, the action is given by $(x \cdot \psi)y = -\psi(x \cdot y)$ where $x, y \in L$ and $\psi \in L^*$.)

Let $x, y, z \in L$. We confirm that $\theta(x \cdot y) = x \cdot \theta(y)$ by looking at how each of the two maps acts on some $z \in L$.

$$\theta(x \cdot y)z = \theta([x, y])z = \beta([x, y], z) = \beta([x, [y, z]]) = -\beta([x, [z, y]])$$

$$= -\beta([x, z], y) = -\beta(y, [x, z]) = -\theta(y)([x, z]) = (x \cdot \theta(y))z$$

Thus we have the desired equality of maps,

$$\theta(x \cdot y) = x \cdot \theta(y)$$

Thus $\theta$ is an $L$-module homomorphism. This did not depend on the non-degeneracy of $\beta$, but now we show that if $\beta$ is non-degenerate, then $\theta$ is an isomorphism. To do this, we show that the kernel of $\theta$ is $\{0\}$. We claim that $L^\perp = \ker \theta$.

$$L^\perp = \{x \in L : \beta(x, y) = 0, \forall y \in L\}$$

$$= \{x \in L : \theta(x)y = 0, \forall y \in L\}$$

$$= \{x \in L : \theta(x) = 0\}$$

$$= \ker \theta$$
If \( \beta \) is non-degenerate, then \( L^\perp = \{0\} \), so then \( \ker \theta = \{0\} \) which implies that \( \theta \) is an isomorphism.

Proposition 14.10 (Exercise 9.11). Let \( L \) be a simple Lie algebra over \( \mathbb{C} \) with Killing form \( \kappa \). Let \( \beta \) be a symmetric, associative, non-degenerate bilinear form on \( L \). Then \( \kappa = \lambda \beta \) for some \( \lambda \neq 0 \) with \( \lambda \in \mathbb{C} \).

Proof. By Exercise 9.10, \( \kappa \) and \( \beta \) induce \( L \)-module isomorphisms \( \theta_\kappa, \theta_\beta : L \to L^* \). Then \( \theta_\kappa \theta_\beta^{-1} : L \to L \) is an \( L \)-module isomorphism. Since \( L \) is simple, it is an irreducible \( L \)-module (example 7.9(2) on page 59). By Schur’s Lemma, \( \theta_\kappa \theta_\beta^{-1} = \lambda 1_L \) for some \( \lambda \in \mathbb{C} \). Then

\[
\kappa(x, y) = \theta_\kappa(x)y = \lambda \theta_\beta(x)y = \lambda \beta(x, y)
\]

Thus \( \kappa = \lambda \beta \). (Note that \( \lambda \neq 0 \) since \( \kappa, \beta \) are both non-degenerate.)

(Exercise 9.13) We give an example to show that the requirement of \( d \) and \( n \) commuting in the Jordan decomposition is necessary. Specifically, we give two matrices \( d \) and \( n \) such that \( d \) is diagonalisable, \( n \) is nilpotent, but \( d \) and \( n \) do not commute.

\[
d = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \quad n = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \quad x = d + n = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}
\]

Note that \( d \) is diagonal, so it is certainly diagonalisable, and \( n \) is nilpotent since \( n^2 = 0 \). However,

\[
nd = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \quad \quad dn = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]

Thus \( dn \neq nd \).

Proposition 14.11 (Exercise 9.14). Let \( L \) be a complex semisimple Lie algebra. Suppose \( \phi : L \to \mathfrak{gl}(V) \) is a faithful representation of \( L \) such that \( \phi(x) \) is diagonalisable for some \( x \in L \). Then \( x \) is a semisimple element of \( L \) and thus \( x \) acts diagonalisably in any representation of \( L \).

Proof. Let \( x \) be as described. By Theorem 9.15, \( x \) can be written uniquely as \( x = d + n \) in an abstract Jordan decomposition. Then by Theorem 9.16, the Jordan decomposition of \( \phi(x) \) is \( \phi(x) = \phi(d) + \phi(n) \), where \( \phi(d) \) is diagonalisable. By hypothesis \( \phi(x) \) is diagonalisable. We claim that \( \phi(x) \) and \( \phi(d) \) commute, since

\[
\phi(x)\phi(d) = (\phi(d) + \phi(n))\phi(d)
\]

\[
= \phi(d)^2 + \phi(n)\phi(d)
\]

\[
= \phi(d)^2 + \phi(d)\phi(n)
\]

\[
= \phi(d)(\phi(d) + \phi(n))
\]

\[
= \phi(d)\phi(x)
\]

since \( \phi(d), \phi(n) \) commute by definition of Jordan decomposition. Thus by Lemma 16.7, \( \phi(x) \) and \( \phi(d) \) are simultaneously diagonalisable with respect to some basis \( \beta \). Then since

\[
[\phi(x)]_\beta = [\phi(d)]_\beta + [\phi(n)]_\beta
\]

108
it follows that $\phi(n)_{\beta} = 0$, so $\phi(n) = 0$. Since $\phi$ is one-to-one, $n = 0$, thus $x$ is semisimple. Thus if $\theta : L \to \text{gl}(V)$ is any representation of $L$,

$$
\theta(x) = \theta(d + n) = \theta(d + 0) = \theta(d)
$$

where $\theta(d)$ is diagonalisable, so $\theta(x)$ is diagonalisable. \qed
15 Chapter 10 Exercises

Proposition 15.1 (Exercise 10.1). Let $L$ be a finite-dimensional semisimple complex Lie algebra and let $H$ be a Cartan subalgebra. Let $\alpha \in H^*$ with $\alpha \neq 0$. Let

$$L_\alpha = \{x \in L : [h, x] = \alpha(h)x, \forall h \in H\}$$

Fix $x \in L_\alpha$. Then $\text{ad } x$ is nilpotent.

Proof. We can decompose $L$ as

$$L = H \oplus \bigoplus_{\beta \in \Phi} L_\beta$$

where $\Phi$ is finite. Then

$$(\text{ad } x)^2(H) = \text{ad } x(\text{ad } x(H)) = \text{ad } x(\{[x, h] : h \in H\}) \subset \text{ad } x(\text{span}\{x\}) = \{0\}$$

Thus $(\text{ad } x)^2(H) = \{0\}$. We will also show that $\text{ad } x$ acting on any $L_\beta$ is nilpotent. Let $\beta \in \Phi$.

$$\text{ad } x(L_\beta) = \{[x, y] : y \in L_\beta\} \subset [L_\alpha, L_\beta] \subset L_{\alpha + \beta}$$

by Lemma 10.1(i). Then

$$(\text{ad } x)^2(L_\beta) \subset \text{ad } x(L_{\alpha + \beta}) \subset [L_\alpha, L_{\alpha + \beta}] \subset L_{2\alpha + \beta}$$

and by a simple induction

$$(\text{ad } x)^n(L_\beta) \subset L_{n\alpha + \beta}$$

There are infinitely many $L_{n\alpha + \beta}$, but only finitely many nonzero root spaces of $L$. Thus for some $n$, we have $L_{n\alpha + \beta} = \{0\}$, so

$$(\text{ad } x)^n(L_\beta) \subset \{0\}$$

thus $\text{ad } x$ restricted to $L_\beta$ is nilpotent. We have shown that $\text{ad } x$ is nilpotent on each summand of $L$, so $\text{ad } x$ acts nilpotently on all of $L$. \qed

Proposition 15.2 (Exercise 10.2). Let $L = \text{sl}(n, \mathbb{C})$ with $n \geq 2$, and let $H = \text{span}\{h\}$ where $h = e_{11} - e_{22}$. Then $L$ decomposes into weight spaces as

$$L = L_0 \oplus L_\alpha \oplus L_\beta$$

where the weights $\alpha, \beta : H \to \mathbb{C}$ are defined by $\alpha(h) = 1$ and $\beta(h) = -1$ and the corresponding weight spaces are

$$L_0 = \{x \in L : [h, x] = 0\}$$
$$L_\alpha = \{x \in L : [h, x] = x\}$$
$$L_\beta = \{x \in L : [h, x] = -x\}$$

110
Proof. We compute \([h, e_{ij}]\).

\[
[h, e_{ij}] = [e_{11}, e_{ij}] - [e_{22}, e_{ij}] = \delta_{1j}e_{1j} - \delta_{1j}e_{11} + \delta_{2j}e_{2j} - \delta_{2j}e_{12}
\]

When \(i, j \geq 3\), all of the Kronecker deltas become zero, so \([h, e_{ij}] = 0\) for \(i, j \geq 3\). Somewhat surprisingly, when \(i, j \leq 2\), terms cancel and we again get zero. Thus \([h, e_{ij}] = 0\) for \(i, j \leq 2\). When \(i \leq 2\) but \(j \geq 3\), then \([h, e_{ij}] = \delta_{1j}e_{1j} + \delta_{2j}e_{2j} = e_{ij}\). Thus for \(i \leq 2, j \geq 3\), span\{\(e_{ij}\}\} \subset L_\alpha. When \(i \geq 3\) and \(j \leq 2\), then \([h, e_{ij}] = -\delta_{1j}e_{1j} - \delta_{2j}e_{22} = -e_{ij}\). Thus in this case, span\{\(e_{ij}\}\} \subset L_\beta. Thus we have allocated the entirety of the standard basis each to one of \(L_0, L_\alpha, L_\beta\). By definition, these weight spaces have trivial intersection, thus \(L = L_0 \oplus L_\alpha \oplus L_\beta\).

Proposition 15.3 (Exercise 10.3i). Let \(L\) be a semisimple complex finite-dimensional Lie algebra with Cartan subalgebra \(H\). Let \(\alpha: H \to \mathbb{C}\) be a root of \(L\) with \(L_\alpha \neq \{0\}\). Define \(\text{sl}(\alpha) = \text{span}\{x, y, [x, y]\}\) where \(x \in L_\alpha, y \in L_{-\alpha}\), and \([x, y] \in H\). We know that \(\text{sl}(\alpha) \cong \text{sl}(2, \mathbb{C})\). Then there is a basis \(\{e_\alpha, f_\alpha, h_\alpha\}\) of \(\text{sl}(\alpha)\) such that \(e_\alpha \in L_\alpha, f_\alpha \in L_{-\alpha}, h_\alpha \in H\), and \(\alpha(h_\alpha) = 2\).

Proof. We have fixed \(x, y, h\). We know that \(\alpha(h) \neq 0\) as shown in the Lemma. Let \(\lambda = 2/\alpha(h)\), then set \(e_\alpha = x, f_\alpha = \lambda y, h_\alpha = \lambda h\). Then \(e_\alpha \in L_\alpha, f_\alpha \in L_{-\alpha}\), and \(h_\alpha \in H\), and

\[
\alpha(h_\alpha) = \alpha(\lambda h) = \lambda \alpha(h) = \frac{2}{\alpha(h)} \alpha(h) = 2
\]

Proposition 15.4 (Exercise 10.3ii). Let \(\alpha, e_\alpha, f_\alpha, h_\alpha\) be as above. Then \(\theta: \text{sl}(\alpha) \to \text{sl}(2, \mathbb{C})\) defined by

\[
\theta(e_\alpha) = e \quad \theta(f_\alpha) = f \quad \theta(h_\alpha) = h
\]

is a Lie algebra isomorphism.

Proof. \(\theta\) is a linear isomorphism of vector spaces because it maps a basis to a basis. We need to show it preserves the bracket.

\[
\theta([e_\alpha, f_\alpha]) = \theta(h_\alpha) = h = [e, f] = [\theta(e_\alpha), \theta(f_\alpha)]
\]

\[
\theta([h_\alpha, e_\alpha]) = \theta(\alpha(h_\alpha)) e_\alpha = 2\theta(e_\alpha) = 2e = [h, e] = [\theta(h_\alpha), \theta(e_\alpha)]
\]

\[
\theta([h_\alpha, f_\alpha]) = \theta(-2f_\alpha) = -2f_\alpha = [h, f] = [\theta(h_\alpha), \theta(f_\alpha)]
\]

Proposition 15.5 (Exercise 10.5). Let \(L\) be a complex, finite-dimensional, semisimple Lie algebra with Cartan subalgebra \(H\). Let \(|\Phi|\) be the root system corresponding to \(H\). Then \(\dim L = \dim H + |\Phi|\).

Proof. We can decompose \(L\) as a direct sum of root spaces,

\[
L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha
\]

111
By Proposition 10.9, each \( L_\alpha \) is one dimensional. Thus

\[
\dim L = \dim H + \dim L_{\alpha_1} + \dim L_{\alpha_2} + \ldots = \dim H + |\Phi|
\]

\[\square\]

**Proposition 15.6** (Exercise 10.6). Let \( L = \mathfrak{sl}(3, \mathbb{C}) \), and let \( H = \text{span}\{e_{11} - e_{22}, e_{22} - e_{33}\} \) be the Cartan subalgebra of diagonal matrices. Then the set of roots for \( H \) is

\[
\Phi = \{\alpha, -\alpha, \beta, -\beta, \alpha + \beta, -\alpha - \beta\}
\]

where \( \alpha = \epsilon_1 - \epsilon_2, \beta = \epsilon_2 - \epsilon_3 \).

**Proof.** (For the definition of \( \epsilon_i \) see page 92.) As shown on page 92, if \( i \neq j \) and

\[
L_{ij} = \{x \in \mathfrak{sl}(3, \mathbb{C}) : \text{ad} h(x) = (\epsilon_i - \epsilon_j)(h)x, \forall h \in H\}
\]

then \( L_{ij} = \text{span}\{e_{ij}\} \) and \( L_{ij} \) is the root space for \( \epsilon_i - \epsilon_j \). Thus

\[
\begin{align*}
L_{12} &= L_\alpha \\
L_{21} &= L_{-\alpha} \\
L_{23} &= L_\beta \\
L_{32} &= L_{-\beta} \\
L_{13} &= L_{\alpha + \beta} \\
L_{31} &= L_{-\alpha - \beta}
\end{align*}
\]

And thus

\[
\mathfrak{sl}(3, \mathbb{C}) = H \oplus \bigoplus_{i \neq j} L_{ij} = H \oplus \bigoplus_{\gamma \in \Phi} L_\gamma
\]

\[\square\]
16 Chapter 11 Exercises

Proposition 16.1 (Exercise 11.1). In \( \mathbb{R}^n \) with the usual dot product, let \( e_i \) be the vector with a 1 in the \( i \)th position and zeroes elsewhere. Define

\[
R = \{e_i - e_j : 1 \leq i \neq j \leq n\}
\]

\[
E = \text{span } R
\]

Then \( R \) is a root system for \( E \).

Proof. (R1) Clearly \( R \) is finite, by definition of \( E \), \( R \) spans \( E \), and \( 0 \notin R \).

(R2) It is obvious from the definition of \( R \) that it doesn’t contain any multiples of \( x \) other than \( \pm x \) for \( x \in R \).

(R4) Let \( x = e_i - e_j, y = e_k - e_m \), so \( x, y \in R \).

\[
\langle x, y \rangle = \frac{2(x, y)}{\langle y, y \rangle} = \frac{2(x, y)}{2} = (x, y) = (e_i - e_j) \cdot (e_k - e_m) = \delta_{ik} - \delta_{jk} - \delta_{im} + \delta_{jm} \in \mathbb{Z}
\]

(R4) This one is much harder than the others. We need to show that for \( x \in R \), \( s_x \) is a permutation of \( R \). Since \( s_x \) is a reflection through a hyperplane, it is a bijection from \( \mathbb{R}^n \) to \( \mathbb{R}^n \). Thus all we need to show is that for \( x, y \in R \), we have \( s_x(y) \in R \). Let \( x = e_i - e_j \) and \( y = e_k - e_m \) (\( i \neq j \) and \( k \neq m \)). Then

\[
s_x(y) = x - \langle x, y \rangle = x - \frac{2x \cdot y}{y \cdot y}y = x - (x \cdot y)y
\]

Now we have a bunch of cases.

\[
x \cdot y = \delta_{ik} - \delta_{im} + \delta_{jm} - \delta_{jk}
\]

\[
= \begin{cases} 
2 & j = m, i = k \\
1 & i = k, j \neq m, j \neq k \text{ OR } i \neq k, i \neq m, j = m \\
0 & i \neq k, i \neq m, j \neq k, j \neq m \\
-1 & i \neq k, i \neq m, j = k \text{ OR } i = m, j \neq m, j \neq k \\
2 & i = m, j = k 
\end{cases}
\]

(Note that in the case where \( x \cdot y = 0 \), there might seem to be more possibilities, but those possibilities are ruled out since \( i \neq j \) and \( k \neq m \).) Thus in these same cases,

\[
s_x(y) = \begin{cases} 
x - 2y & j = m, i = k \\
x - y & i = k, j \neq m, j \neq k \text{ OR } i \neq k, i \neq m, j = m \\
x & i \neq k, i \neq m, j \neq k, j \neq m \\
x + y & i \neq k, i \neq m, j = k \text{ OR } i = m, j \neq m, j \neq k \\
x + 2y & i = m, j = k 
\end{cases}
\]

\[
= \begin{cases} 
(e_i - e_j) - 2(e_i - e_j) = e_j - e_i \\
(e_i - e_j) - (e_i - e_m) = e_m - e_i \\
(e_i - e_j) - (e_k - e_j) = e_i - e_k \\
(e_i - e_j) + (e - j - e_m) = e_i - e_m \\
(e_i - e_j) + (e_k - e_i) = e_k - e_j \\
(e_i - e_j) + 2(e_j - e_i) = e_j - e_i 
\end{cases}
\]

In every case, \( s_x(y) \in R \), so (R4)
Proposition 16.2 (Exercise 11.2). Let $R$ be a root system and let $\alpha, \beta \in R$ such that $(\alpha, \beta) \neq 0$. Then $(\alpha, s_\alpha(\beta)) \neq 0$.

Proof. 

\[
(\alpha, s_\alpha(\beta)) = \left( \alpha, \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)} \alpha \right) \\
= (\alpha, \beta) - \frac{2(\alpha, \beta)}{(\alpha, \alpha)} (\alpha, \alpha) \\
= (\alpha, \beta) - 2(\alpha, \beta) \\
= -(\alpha, \beta)
\]

If $(\alpha, \beta) \neq 0$, then certainly $-(\alpha, \beta) \neq 0$. \(\square\)

Proposition 16.3 (Exercise 11.2). Let $R$ be a root system in inner product space $E$. Let $R_i$ be the equivalence classes defined in Lemma 11.8. Then $R_i$ satisfies (R3).

Proof. We need to show that for $\alpha, \beta \in R_i$, we have $s_\alpha(\beta) \in R_i$. Let $\alpha, \beta \in R_i$. Then there exist $\gamma_1, \gamma_2, \ldots, \gamma_n$ such that $\alpha = \gamma_1, \beta = \gamma_2$, and $(\gamma_k, \gamma_{k+1}) \neq 0$ for $k = 1, 2, \ldots, (n-1)$. Since $s_\alpha$ preserves the inner product,

\[
0 \neq (\gamma_k, \gamma_{k+1}) = (s_\alpha(\gamma_k), s_\alpha(\gamma_{k+1}))
\]

Note that $s_\alpha(\gamma_1) = s_\alpha(\alpha) = -\alpha$. (Note that since $(-\alpha, s_\alpha(\gamma_2)) = -(\alpha, s_\alpha(\gamma_2))$, we get $(\alpha, s_\alpha(\gamma_2) \neq 0$.) Thus we have $\alpha, s_\alpha(\gamma_2), s_\alpha(\gamma_3), \ldots, s_\alpha(\gamma_n) = s_\alpha(\beta)$ such that

\[
(\alpha, s_\alpha(\gamma_2)) \neq 0 \\
(s_\alpha(\gamma_2), s_\alpha(\gamma_3)) \neq 0 \\
\vdots \\
(s_\alpha(\gamma_{n-1}), s_\alpha(\beta)) \neq 0
\]

Thus $s_\alpha(\beta) \sim \alpha$, so $s_\alpha(\beta) \in R_i$. \(\square\)

Proposition 16.4 (Exercise 11.4). Let $R = \{\pm(e_i - e_j) : 1 \leq i < j \leq l + 1\}$ and let $E = \text{span } R$. ($E$ is a subspace of the inner product space $\mathbb{R}^{l+1}$.) Let $\alpha_i = e_i - e_{i+1}$ for $1 \leq i \leq l$. Let $B = \{\alpha_1, \ldots, \alpha_l\}$. Then $B$ is a base for $R$.

Proof. First we need to show that $B$ is a basis for $E$. $B$ spans $E$ since the $-(e_i - e_j)$ contribute nothing to the span of $R$. We need to show that $B$ is linearly independent. Suppose that

\[
\sum_{i=1}^{l} c_i \alpha_i = 0
\]
Then

\[ 0 = \sum_{i=1}^{l} c_i(e_i - e_{i+1}) \]

\[ = \sum_{i=1}^{l} c_i e_i - \sum_{i=1}^{l} c_i e_{i+1} \]

\[ = \sum_{i=1}^{l} c_i e_i - \sum_{i=2}^{l+1} c_{i-1} e_i \]

\[ = c_1 e_1 + (c_1 - c_2) e_2 + \ldots + (c_{l-1} - c_l) e_l - c_l e_{l+1} \]

Then since \( \{e_1, \ldots, e_{l+1}\} \) is linearly independent, it follows that \( c_1 = 0, c_1 = c_2, c_3 = c_2, \ldots \) so we have \( c_1 = c_2 = \ldots = c_{l+1} = 0 \). Thus \( B \) is linearly independent, so it is a basis for \( E \).

Now we need to show that every \( \beta \in R \) can be written as

\[ \sum_{\alpha \in B} k_\alpha \alpha \]

where the nonzero \( k_\alpha \) have the same sign. Let \( \beta \in R \). Then \( \beta = \pm(e_i - e_j) \), where \( 1 \leq i < j \leq l + 1 \). Then

\[ \beta = \pm(e_i + (-e_{i+1} + e_{i+1}) + (-e_{i+2} + e_{i+2}) + \ldots + (-e_{j-1} + e_{j-1} - e_j)) \]

\[ = \pm((e_i - e_{i+1}) + (e_{i+1} - e_{i+2}) + \ldots + (e_{j-1} - e_j)) \]

\[ = \pm(\alpha_i + \alpha_{i+1} + \ldots + \alpha_{j-1}) \]

\[ = \pm \sum_{\alpha \in B} k_\alpha \alpha \]

where

\[ k = \begin{cases} 
1 & \alpha \in \{\alpha_i, \ldots, \alpha_{j-1}\} \\
0 & \text{otherwise}
\end{cases} \]

The \( \pm \) distributes to all the \( k_\alpha \), so all the nonzero \( k_\alpha \) have the same sign. Thus \( B \) is a base for \( R \).

**Proposition 16.5** (Exercise 11.4). Let \( R, B \) be as in the above proposition. Then the positive roots of \( R \) are

\[ \{e_i - e_j : i < j\} \]

(Every other root is negative.)

**Proof.** As shown above, if \( \beta = e_i - e_j \) where \( i < j \), then \( \beta = \sum_{\alpha} k_\alpha \alpha \) where \( k_\alpha \in \{0,1\} \). Thus \( \beta \) is a positive root. If \( \beta = -(e_i - e_j) \) then \( \beta = -\sum_{\alpha} k_\alpha \alpha \), so \( \beta \) is a negative root. \( \square \)
Proposition 16.6 (Exercise 11.5). Let $R$ be a root system with base $B$. Fixe $\gamma \in R$. Then
\[ s_{\gamma}(B) = \{ s_{\gamma}(\alpha) : \alpha \in B \} \]
is a base for $R$.

Proof. We know that $s_{\gamma} : E \to E$ is a linear bijection, and $B$ is a basis for $E$, so $s_{\gamma}(B)$ is a basis for $E$. We also need to show that for $\beta \in R$, $\beta$ can be written as
\[ \beta = \sum_{\alpha \in s_{\gamma}(B)} c_{\alpha} \alpha \]
where the nonzero $c_{\alpha}$ all have the same sign. Since $s_{\gamma}$ permutes $R$, there exists $\beta_0 \in R$ such that $s_{\gamma}(\beta_0) = \beta$. Since $B$ is a base, we can write $\beta_0$ as
\[ \beta_0 = \sum_{\alpha \in B} k_{\alpha} \alpha \]
where the nonzero $k_{\alpha}$ have the same sign. By linearity of $s_{\gamma}$,
\[ \beta = s_{\gamma}(\beta_0) = \sum_{\alpha \in B} k_{\alpha} s_{\gamma}(\alpha) = \sum_{\alpha \in s_{\gamma}(\beta)} k_{\alpha} \alpha \]
Thus we have written $\beta$ in the necessary form. \qed
17 Appendix A - Linear Algebra

17.1 Quotient Spaces

Definition 17.1. Let $V$ be a vector space over $F$ and let $W$ be a subset of $V$. Then $W$ is a **subspace** of $V$ if $W$ is also a vector space over $F$ (with the same addition as in $V$).

Proposition 17.2 (Condition for being a subspace). Let $V$ be a vector space over $F$ and let $W \subset V$ such that for $v, u \in W$ and $t \in F$, $v + u \in W$ and $tv \in W$. Then $W$ is a subspace of $V$.

Proof. Let $V, W, F$ be as described. Since $0v \in W$, $W$ is nonempty. By hypothesis, $W$ is closed under addition. Let $v \in W$. Then $(-1)v \in W$, so $-v \in W$, so $W$ is also closed under taking inverses, so $W$ is a subgroup of $V$ as an Abelian group, so $W$ is an Abelian group. Thus $W$ satisfies properties 1-5 in the definition. Properties 6-10 follow for $W$ since they hold for all elements of $V$ and $F$. Thus $W$ is a vector space over $F$. \qed

Definition 17.3. Let $V$ be a vector space over $F$ and let $W$ be a subspace of $V$. A **coset** of $W$ is a set of the form

$$v + W = \{v + w : w \in W\}$$

Proposition 17.4. Let $V$ be a vector space over $F$ and let $W$ be a subspace of $V$. Let $v, v' \in V$. Then the cosets $v + W$ and $v' + W$ are equal if and only if $v - v' \in W$.

Proof. Suppose $v - v' \in W$. Let $w_0 = v - v'$, rearranging we have $v = w_0 + v'$. Let $x \in v + W$. Then $x = v + w_1$ for $w_1 \in W$, so $x = w_0 + w_1 + v'$. Since $W$ is closed under addition, $w_0 + w_1 \in W$ so $x = v' + (w_0 + w_1) \in v' + W$. Likewise, suppose $y \in v' + W$. Then $y = v' + w_2 = w_0 + v + w_1 = v + (w_0 + w_2) \in v + W$. Thus if $v - v' \in W$, then $v + W = v' + W$.

Now suppose $v + W = v' + W$. Since $0 \in W$, $v + 0 = v \in v + W$, and since $v + W = v' + W$, $v \in v' + W$. Then there exists $w_0$ such that $v = v' + w_0$. Then $w_0 = v - v'$, so $v - v' \in W$. \qed

Definition 17.5. Let $V$ be a vector space over $F$ and let $W$ be a subspace of $V$. The **quotient space** $V/W$ is the set of all cosets of $W$, that is, the set

$$V/W = \{v + W : v \in V\}$$

We then define addition in this space by

$$(v + W) + (v' + W) = (v + v') + W$$

where $v, v' \in V$. We define scalar multiplication from $F$ by

$$\lambda(v + W) = \lambda v + W$$

where $\lambda \in F$.

Proposition 17.6. Addition and scalar multiplication for quotient spaces are well-defined.
Proof. First we show that scalar multiplication is well-defined. Let \( v, v' \in V \) such that \( v + W = v' + W \). We need to show that for \( \lambda \in F \), \( \lambda(v+W) = \lambda(v'+W) \). Since \( v + W = v' + W \), by Proposition 17.4, \( v - v' \in W \). Since \( W \) is closed under scalar multiplication, \( \lambda(v - v') = \lambda v - \lambda v' \in W \). Then using the other direction of Proposition 17.4, \( \lambda v + W = \lambda v' + W \), so we have shown what was needed to show, since \( \lambda v + W = \lambda(v + W) \) by definition.

Now we will show that addition is well-defined. Let \( v_1, v_2 \in V \). We will show now that the addition \( (v_1 + W) + (v_2 + W) \) does not depend on coset representative. Let \( v'_1, v'_2 \in V \) such that \( v'_1 + W = v_1 + W \) and \( v'_2 + W = v_2 + W \). Now we need to show that \( (v_1 + w) + (v_2 + W) = (v'_1 + W) + (v'_2 + W) \). By Proposition 17.4, \( v'_1 = v_1 + w_1 \) and \( v'_2 = v_2 + w_2 \) for some \( w_1, w_2 \in W \). Note that since \( W \) is closed under addition, \( w_1 + w_2 \in W \), so

\[
(v'_1 + W) + (v'_2 + W) = (v'_1 + v'_2) + W
= ((v_1 + w_1) + (v_2 + w_2)) + W
= (v_1 + v_2 + (w_1 + w_2)) + W
= (v_1 + v_2) + W
= (v_1 + W) + (v_2 + W)
\]

\[
\square
\]

Proposition 17.7. Let \( V \) be a vector space over \( F \), and let \( W \) be a subspace. Then the quotient space \( V/W \) is a vector space over \( F \).

Proof. Let \( v, u, w \in V \) and \( a, b \in F \), so \( v + W, u + W \in V/W \).

(Closure) By closure of addition in \( V \), \( v + u \in V \) so \( (v+W) + (u+W) = (v+u) + W \in V/W \).

(Associativity of addition) By associativity of addition in \( V \),

\[
(v + W) + ((u+W) + (w+W)) = (v + W) + ((u+w) + W)
= (v + (u+w)) + W
= ((v+u) + w) + W
= ((v+W) + (u+W)) + (w+W)
\]

(Commutativity of addition) By commutativity of addition in \( V \),

\[
(v + W) + (u + W) = (v + u) + W = (u + v) + W = (u + W) + (v + W)
\]

(Identity for Addition) \( 0 + W = W \) is the identity for addition because

\[
(0 + W) + (v + W) = (0 + v) + W = v + W
\]

(Inverses for Addition) \( -v + W \) is an additive inverse for \( (v + W) \) because

\[
(-v + W) + (v + W) = (-v + v) + W = 0 + W
\]

(Closure of scalar multiplication) For \( v \in V, a \in F \), \( av \in V \) by closure of scalar multiplication in \( V \), so \( a(v + W) = av + W \in V/W \).

(Associativity of scalar multiplication) \( ab(v + W) + ab(v + W) = (ab)v + W = (ab)(v + W) \). 

118
(Identity for scalar multiplication) 1 (the multiplicative identity for $F$) is the identity for $V/W$ also, since $1(v + W) = 1v + W = v + W$.

(Distributivity over vector sums)

$$a((v + W) + (u + W)) = a((v + u) + W) = a(v + u) + W = (av + au) + W = (av + W) + (au + W)$$

(Distributivity over scalar sums)

$$(a + b)(v + W) = (a + b)v + W = (av + bv) + W = (av + W) + (bv + W)$$

Proposition 17.8. Let $V$ be a vector space with subspace $W$. If $v_1, v_2, \ldots, v_k$ are vectors in $V$ such that the cosets $v_1 + W, v_2 + W, \ldots, v_3 + W$ form a basis for the quotient space $V/W$, then $v_1, \ldots, v_k$ together with any basis for $W$ forms a basis for $V$.

Proof. Asserted on page 190 of Erdmann and Wildon.

17.2 Linear Maps

Lemma 17.9. Let $V, W$ be vector spaces over a field $F$ and let $\phi : V \rightarrow W$ be an onto linear map. Let $\beta$ be a basis for $V$. Then $\phi(\beta)$ is a spanning set for $W$.

Proof. Let $\beta = \{v_1, \ldots, v_n\}$. We need to show that any $w \in W$ can be written as a linear combination of $\{\phi(v_1), \ldots, \phi(v_n)\}$. Let $w \in W$. Since $\phi$ is onto, there exists $v \in V$ such that $\phi(v) = w$. We can write $v$ as a unique linear combination of the basis vectors, $v = \sum a^i v_i$ where $a^i \in F$. Then

$$w = \phi(w) = \phi \left( \sum a^i v_i \right) = a^i \sum \phi(v_i)$$

Thus $\phi(\beta)$ is a spanning set for $W$.

Lemma 17.10. Let $V, W$ be vector spaces over a field $F$ and let $\phi : V \rightarrow W$ be a one-to-one linear map. Let $\beta$ be a basis for $V$. Then $\phi(\beta)$ is linearly independent.

Proof. Let $\beta = \{v_1 \ldots v_n\}$. Suppose that

$$\sum a^i \phi(v_i) = 0$$

for $a^i \in F$. Then by linearity of $\phi$, $\phi \left( \sum a^i v_i \right) = 0$.

Since $\phi$ is one-to-one, the kernel is only zero, so $\sum a^i v_i = 0$. Since $\beta$ is a basis, this implies that $a^i = 0$ for all $i$. Thus $\phi(\beta)$ is linearly independent.
Corollary 17.11. Let $V, W$ be vector spaces over a field $F$ and let $\phi : V \to W$ be a linear bijection. Let $\beta$ be a basis for $V$. Then $\phi(\beta)$ is a basis for $W$.

Proof. By the previous lemmas, since $\phi$ is one-to-one and onto, $\phi(\beta)$ is a linearly independent spanning set, so it is a basis. \hfill \Box

Proposition 17.12. Let $\phi : V \to W$ be a linear map, where $V$ and $W$ are vector spaces over $F$. Then $\ker \phi$ is a subspace of $V$.

Proof. Let $v_1, v_2 \in \ker \phi$. Then since $\phi$ is a linear map, $\phi(v_1 + v_2) = \phi(v_1) + \phi(v_2) = 0 + 0 = 0$, so $v_1 + v_2 \in \ker \phi$. Let $a \in F$. Then $\phi(av_1) = a\phi(v_1) = (a)0 = 0$ so $av_1 \in \ker \phi$. Thus, $\ker \phi$ is closed under vector addition and $a$.

Proposition 17.13 (Rank-Nullity Theorem). Let $V, W$ be finite-dimensional vector spaces over $F$ and let $\phi : V \to W$ be a linear map. Then

$$\dim V = \dim \text{im} \phi + \dim \ker \phi \quad (17.1)$$

Proof. (Proof by Lawrence Valby) Let $\phi : V \to W$ be a linear map and let $n = \dim V$. By Proposition 17.12, $\ker \phi$ is a subspace of $V$. Since $\ker \phi$ is a vector space, it has a basis $B = \{b_1, b_2, \ldots, b_k\}$ where $k = \dim \ker \phi$, and where $k \leq n$. If $k = n$, then $\ker \phi = V$ and $\dim \phi = 0$ so the result is true, so assume that $k \neq n$. Since $B$ is a linearly independent subset of $V$, we can extend it to a basis for $V$, finding $C = \{c_1, c_2, \ldots, c_{n-k}\}$ where $B \cup C$ is a basis for $V$. We claim that $\phi(C) = \{\phi(c_1), \phi(c_2), \ldots, \phi(c_{n-k})\}$ is a basis for $\text{im} \phi$. If it is, then $\dim \phi = n - k$ and then $\dim \phi + \dim \ker \phi = (n - k) + k = n = \dim V$.

First we show that $\phi(C)$ spans $\text{im} \phi$. Let $w \in \text{im} \phi \subseteq W$. Then there exists $v \in V$ such that $\phi(v) = w$. Since $B \cup V$ spans $V$, there exist scalars $r_1, r_2, \ldots, r_k \in F$ and $s_1, s_2, \ldots, s_{n-k} \in F$ such that $v = \sum_{i=1}^{k} r_i b_i + \sum_{i=1}^{n-k} s_i c_i$. Then by the linear property of $\phi$,

$$\phi(v) = \phi \left( \sum_{i=1}^{k} r_i b_i + \sum_{i=1}^{n-k} s_i c_i \right) = \sum_{i=1}^{k} r_i \phi(b_i) + \sum_{i=1}^{n-k} s_i \phi(c_i) = \sum_{i=1}^{n-k} s_i \phi(c_i)$$

since $b_i \in \ker \phi$ for all $i$. Thus, $\phi(v) = w$ can be written as a linear combination of elements in $\phi(C)$, so $\phi(C)$ spans $\text{im} \phi$.

Now we show that $\phi(C)$ is linearly independent. Let $s_1, s_2, \ldots, s_{n-k} \in F$ be scalars such that $\sum_{i=1}^{n-k} s_i \phi(c_i) = 0$. Then if we show that $s_1 = s_2 = \ldots = s_{n-k} = 0$ we have show that $\phi(C)$ is linearly independent. By the linear property of $\phi$, $\phi(\sum_{i=1}^{n-k} s_i c_i) = 0$ so $\sum_{i=1}^{n-k} s_i c_i \in \ker \phi$. Since $B$ spans $\ker \phi$, there exist scalars $r_1, r_2, \ldots, r_k \in F$ such that

$$\sum_{i=1}^{k} r_i b_i = \sum_{i=1}^{n-k} s_i c_i$$

We can rearrange this to give

$$\sum_{i=1}^{k} (-r_i) b_i + \sum_{i=1}^{n-k} s_i c_i = 0$$

120
Since \( B \cup C \) is a basis, it is linearly independent, so all the scalars \( r_i, s_i \) are equal to zero. Thus \( \phi(C) \) is linearly independent.

We have show that \( \phi(C) \) spans \( \text{im} \phi \) and is linearly independent, so it is a basis for \( \text{im} \phi \). Thus \( \dim \text{im} \phi = n - k \), so \( \dim \text{im} \phi + \dim \ker \phi = (n - k) + k = n = \dim V \).

**Proposition 17.14.** Let \( V \) be a vector space over a field \( F \) and let \( U, W \) be subspaces of \( V \). Then \( U + W \) is a subspace of \( V \).

**Proof.** Using Propositions 17.2, we must show that \( U + W \) is closed under vector addition and scalar multiplication, then it will be shown that it is a subspace.

Let \( v_1, v_2 \in U + W \). Then by definition of \( U + W \) there exist \( u_1, u_2, w_1, w_2 \) such that \( v_1 = u_1 + w_1 \) and \( v_2 = u_2 + w_2 \). Then \( v_1 + v_2 = (u_1 + w_1) + (u_2 + w_2) = (u_1 + u_2) + (w_1 + w_2) \). Since \( U, W \) are subspaces, \( u_1 + u_2 \in U \) and \( w_1 + w_2 \in W \), thus \( v_1 + v_2 \in U + W \).

Let \( v_1 \in U + W \) and \( t \in F \). Then \( tv_1 = tu_1 + tw_1 \) for some \( u_1 \in U, w_1 \in W \). Then \( tv_1 \in U + W \).

17.3 Matrices and Diagonalisation

**Definition 17.15.** Let \( V \) be an \( n \)-dimensional vector space over \( F \) with \( \beta = \{v_1, v_2, \ldots, v_n\} \) a basis for \( V \). For \( v \in V \), we can write \( v \) uniquely as a linear combination of basis elements, \( v = \sum_{i=1}^{n} a^i v_i \) where \( a^i \in F \). We define a map \( [\ ]_\beta : V \to F^n \) by \( [v]_\beta = (a^1, a^2, \ldots, a^n) \).

**Definition 17.16.** Let \( V \) be an \( n \)-dimensional vector space over \( F \) with basis \( \beta \). Let \( x : V \to V \) be a linear map. Then the matrix of \( x \) is \( [x]_\beta \), the unique matrix in \( \text{gl}(n, F) \) satisfying

\[
[x]_\beta [v]_\beta = [x(v)]_\beta
\]

for all \( v \in V \). We usually suppress the subset \( \beta \) since the choice of basis is clear from context, writing

\[
[x][v] = [x(v)]
\]

**Lemma 17.17.** Let \( V \) be an \( n \)-dimensional vector space over \( F \) with basis \( \beta = \{v_1, v_2, \ldots, v_n\} \). Let \( x : V \to V \) be a linear map, and \( [x] = (a_{ij}) \) be the matrix of \( x \) with respect to \( \beta \). Then

\[
x(v_j) = \sum_{i=1}^{n} a_{ij} v_i
\]

for \( v_j \in \beta \).

**Proof.** Let \( v_j \in \beta \). Then \( [v_j] = (\delta_{ij})_{i=1}^{n} = (0, 0, \ldots, 1, \ldots, 0) \) where the 1 is in the \( j \)th position. By the definition of \( [x] \),

\[
[x(v_j)] = [x][v_j] = (a_{ij})(\delta_{ij}) = (a_{1j}, a_{2j}, \ldots, a_{nj})
\]

So we have \( [x(v_j)] = (a_{1j}, a_{2j}, \ldots, a_{nj}) \). Thus by the definition of the map \( [\ ]_\beta \),

\[
x(v_j) = \sum_{i=1}^{n} a_{ij} v_i
\]
Lemma 17.18. Let $V$ be an $n$-dimensional vector space over $F$ with basis $\beta = \{v_1, \ldots, v_n\}$. Let $v, w \in V$ and let $\lambda \in F$. Then

$$[v]_\beta + [w]_\beta = [v + w]_\beta$$

$$\lambda [v]_\beta = [\lambda v]_\beta$$

Proof. Let $v = \sum_i a^i v_i$ and let $w = \sum_i b^i v_i$. Then by definition,

$$[v]_\beta = (a^1, \ldots, a^n)$$

$$[w]_\beta = (b^1, \ldots, b^n)$$

Then

$$[v]_\beta + [w]_\beta = (a^1, \ldots, a^n) + (b^1, \ldots, b^n) = (a^1 + b^1, \ldots, a^n + b^n)$$

$$v + w = \sum_i a^i v_i + \sum_i b^i v_i = \sum_i (a^i + b^i) v_i$$

$$[v + w]_\beta = (a^1 + b^1, \ldots, a^n + b^n)$$

Thus $[v]_\beta + [w]_\beta = [v + w]_\beta$. Also,

$$\lambda [v]_\beta = \lambda (a^1, \ldots, a^n) = (\lambda a^1, \ldots, \lambda a^n) = [\lambda v]_\beta$$

Lemma 17.19. Let $V$ be an $n$-dimensional vector space over $F$ with basis $\beta$ and let $x, y : V \to V$ be linear maps and let $\lambda \in F$. Then

$$[x + y]_\beta = [x]_\beta + [y]_\beta$$

$$[\lambda x]_\beta = \lambda [x]_\beta$$

Proof. Let $v \in V$. Then

$$[x + y]_\beta = [(x + y)(v)]_\beta = [x(v) + y(v)]_\beta = [x(v)]_\beta + [y(v)]_\beta$$

$$= [x]_\beta [v]_\beta + [y]_\beta [v]_\beta = ([x]_\beta + [y]_\beta) [v]_\beta$$

Thus $[x + y]_\beta = [x]_\beta + [y]_\beta$. To show the scalar multiplication property, observe that

$$[\lambda x][v] = [(\lambda x)(v)] = \lambda [x(v)] = \lambda [x]_\beta [v]$$

Proposition 17.20 (Exercise 16.1i). Let $V$ be a $n$-dimensional vector space with basis $\beta$ and let $x, y : V \to V$ be linear maps. Then

$$[y \circ x]_\beta = [y]_\beta [x]_\beta$$
Proof. We omit the subscript $\beta$s for clarity. By definition of $[x]$ and $[y]$ we have

$$[x][v] = [x(v)]$$
$$[y][v] = [y(v)]$$

for all $v \in V$. Then since $x(v) \in V$,

$$[y][x][v]) = [y][x(v)] = [y(x(v)) = [y \circ x](v)]$$

By definition of $[y \circ x]$,

$$[y \circ x][v] = [y \circ x(v)]$$

Thus $[y \circ x] = [y][x]$. \qed

Lemma 17.21. Let $A, B$ be similar matrices. Then the set of eigenvalues for $A$ is equal to the set of eigenvalues for $B$.

Proof. Since $A, B$ are similar, there exists an invertible matrix $P$ such that $A = PB^P_{-1}$. Let $\lambda$ be an eigenvalue of $A$. Then $Av = \lambda v$ for some vector $v$. Then $PB^P_{-1}v = \lambda v$, so $B(P^{-1}v) = P^1\lambda v = \lambda(P^{-1}v)$. Thus $\lambda$ is an eigenvalue of $B$, with corresponding eigenvector $P^{-1}v$. \qed

Proposition 17.22 (Exercise 16.2). Let $x \in \text{gl}(V)$, and let $f(t)$ be a polynomial with $f(x) = 0$. Then $m_x(t)$ (the minimal polynomial of $x$) divides $f(t)$.

Proof. The Euclidean division algorithm for polynomials says that there exist polynomials $q(t), r(t)$ with $\deg r(t) < \deg m_x(t)$ such that

$$f(t) = q(t)m_x(t) + r(t)$$

We can rearrange this to get $r(t) = f(t) - q(t)m_x(t)$. From this, we also get $r(x) = f(x) - q(x)m_x(x)$. By the definition of $m_x$, $m_x(x) = 0$, and by hypothesis $f(x) = 0$, so $r(x) = 0$. Since $m_x(t)$ is the lowest degree polynomial that kills $x$, and $r(x) = 0$ and $\deg r(t) < \deg m_x(t)$, it must be that $r(t) = 0$. Thus $f(t) = q(t)m_x(t)$, so $m_x(t)$ divides $f(t)$. \qed

Proposition 17.23 (Lemma 16.7). Let $V$ be a vector space and let $x_1, x_2, \ldots, x_k : V \to V$ be diagonalizable linear maps. Then there exists a basis $\beta$ of $V$ that simultaneously diagonalizes each $x_i$ if and only if for each $i, j$, $x_i \circ x_j = x_j \circ x_i$.

Proof. From now on when writing a composition of maps, we omit the $\circ$ and simply write $x_ix_j$. Suppose that there is a basis $\beta$ that simultaneously diagonalizes each $x_i$. Then

$$[x_i x_j]_{\beta} = [x_i]_{\beta}[x_j]_{\beta} = [x_j]_{\beta}[x_i]_{\beta} = [x_j x_i]_{\beta}$$

The matrix representations $[x_i]_{\beta}, [x_j]_{\beta}$ commute because they are diagonal matrices, and since the matrix representation of $x_i x_j$ and $x_j x_i$ are equal, they are equal as linear maps. This completes the easier direction of the proof.
Now suppose that \( x_1, x_2, \ldots, x_k \) all pairwise commute. We will show that they are all simultaneously diagonalizable. We proceed by induction on \( k \). The base case is \( k = 2 \). Suppose \( x_1, x_2 \) are diagonalizable, commuting linear maps. Since \( x_1 \) is diagonalizable, we can write \( V \) as a direct sum of eigenspaces for \( x_1 \).

\[
V = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_r}
\]

If \( v \in V_{\lambda_i} \), then \( x_1(v) = \lambda_i v \). We also claim that \( w = x_2(v) \in V_{\lambda_i} \).

\[
x_1(w) = x_1(x_2(v)) = x_2(x_1(v)) = x_2(\lambda_i v) = \lambda_i x_2(v) = \lambda_i w
\]

so we have \( x_1(w) = \lambda_i w \), so \( w \) is an eigenvector of \( x_1 \) with eigenvalue \( \lambda_i \), so \( w = x_2(v) \in V_{\lambda_i} \) by definition of \( V_{\lambda_i} \). Thus \( V_{\lambda_i} \) is an \( x_2 \) invariant space. By Corollary 16.5(a), \( x_2 \) restricted to \( V_{\lambda_i} \) is diagonalizable. Thus there is a basis \( \gamma_i \) of \( V_{\lambda_i} \) consisting of eigenvectors for \( x_2 \). If \( v \in \gamma_i \), then \( v \) is a linear combination of eigenvectors of \( x_i \) with eigenvalue \( \lambda_i \), so \( v \) is an eigenvector of \( x_1 \) with eigenvalue \( \lambda_i \). Hence

\[
\gamma = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_r
\]

is a basis of eigenvectors for both \( x_1 \) and \( x_2 \). Thus \( x_1, x_2 \) are simultaneously diagonalized by the basis \( \gamma \) of \( V \). The completes the base case.

For the inductive step, suppose that if \( x_1, x_2, \ldots, x_k \) commute then they are simultaneously diagonalizable. Suppose \( x_1, x_2, \ldots, x_{k+1} \) commute. Then there is a basis \( \beta \) that simultaneously diagonalizes \( x_1, \ldots, x_k \). Consider the composition \( x_1 x_2 \ldots x_k \). It has matrix representation

\[
\prod_{i=1}^k x_i = \prod_{i=1}^k [x_i]_{\beta}^\beta
\]

But each \( [x_i]_{\beta} \) is diagonal, so the matrix of the product is also diagonal. Since \( x_{k+1} \) commutes with each \( x_i \), it commutes with the product, so by the base case, there is a basis \( \Omega \) that simultaneously diagonalizes this product and \( x_{k+1} \).

\[\square\]

### 17.4 Interlude: The Diagonal Fallacy

**Proposition 17.24** (Exercise 16.3). Let \( V \) be a 2-dimensional vector space with basis \( \beta = \{v_1, v_2\} \). Let \( x : V \to V \) be a linear map with matrix

\[
[x]_{\beta} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]

If \( U \) is a subspace of \( V \) such that \( x(U) \subseteq U \), then \( U = \{0\}, U = V \), or \( U = \text{span}\{v_1\} \).

**Proof.** One can check that \( x(\{0\}) = \{0\} \) and \( x(V) = \text{span}\{v_1\} \subseteq V \) and \( x(\text{span}\{v_1\}) = \{0\} \subseteq \text{span}\{v_1\} \). This deals with all zero- and two-dimensional subspaces of \( V \). We must show that no other 1-dimensional subspace of \( V \) is \( x \)-invariant. Let \( w = a^1 v_1 + a^2 v_2 \). Then any 1-dimensional subspace of \( V \) will be of the form \( \text{span}\{w\} \). Then

\[
[x(w)]_{\beta} = [x]_{\beta} [w]_{\beta} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a^1 \\ a^2 \end{pmatrix} = \begin{pmatrix} a^2 \\ 0 \end{pmatrix}
\]
Thus $x(\text{span}\{w\}) = \text{span}\{v_1\}$. So if $\text{span}\{w\}$ is $x$-invariant, it must be that $w \in \text{span}\{v_1\}$, so it must be that $\text{span}\{w\} = \text{span}\{v_1\}$. Thus the only 1-dimensional subspace of $V$ that is $x$-invariant is $\text{span}\{v_1\}$.

**Proposition 17.25** (Exercise 16.6i). Let $V$ be a vector space over a field $F$. Let $\text{Hom}(V, V) = \{x : V \to V\}$ be the set of linear maps from $V$ to itself, and for $x, y \in \text{Hom}(V, V)$ and $\lambda \in F$ define

$$(x + y)v = x(v) + y(v)$$
$$(\lambda x)v = \lambda(x(v))$$

Then $\text{Hom}(V, V)$ is a vector space over $F$ under these operations. It has dimension $(\dim V)^2$.

**Proof.** Closure of addition - $x + y$ is clearly linear. Inverse - $(x + (-x))v = x(v) - x(v) = 0$. Associativity - from associativity in $V$. Commutativity - from commutativity in $V$. Identity - zero map, $0(v) = 0$. To see that the dimension of $\text{Hom}(V, V)$ is $(\dim V)^2$, note that there is a simple bijection between $\text{Hom}(V, V)$ and the set of $\dim V \times \dim V$ matrices with entries in $F$. 

**Proposition 17.26** (Exercise 16.10). Let $V$ be a finite-dimensional vector space over $F$ and let $\text{Hom}(V, V)$ be the vector space of linear transformations from $V$ to itself. Define

$$\beta : \text{Hom}(V, V) \times \text{Hom}(V, V) \to F$$

by $\beta(x, y) = \text{tr}(xy)$. Then $\beta$ is a symmetric, non-degenerate, bilinear form.

**Proof.** $\beta$ is symmetric because $\text{tr}(xy) = \text{tr}(yx)$. $\beta$ is linear because the trace function is linear:

$$\beta(ax + y, z) = \text{tr}((ax + y)z)) = a \text{tr}(xz) + \text{tr}(yz) = a\beta(x, y) + \beta(y, z)$$

To show that $\beta$ is non-degenerate, we need to show that $\text{Hom}(V, V)^\perp = \{0\}$.

$$\text{Hom}(V, V)^\perp = \{x \in \text{Hom}(V, V) : \text{tr}(xy) = 0 \forall y \in \text{Hom}(V, V)\} = \{0\}$$

(According to the Wikipedia page on trace, “This follows from the fact that $\text{tr}(A^*A) = 0$ if and only if $A = 0$.”) Thus $\beta$ is non-degenerate.
18 Connection to Lie Groups

**Definition 18.1.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable and let $v \in \mathbb{R}^n$ with $v = (v^1, v^2, \ldots, v^n)$. Then $df : \mathbb{R}^n \to T\mathbb{R}^n$ is defined by
\[
df(v) = \sum_{i=1}^n \frac{\partial f}{\partial x^i} v^i
\]

**Definition 18.2.** Let $U \subset \mathbb{R}^m$ and let $F : U \to \mathbb{R}^n$ be differentiable. Let $\alpha : I \to U$ be a smooth curve with $\alpha(0) = x$ and $\alpha'(0) = v$. Then $dF_x : U \to T_{F(x)}\mathbb{R}^n$ is defined by
\[
dF_x(v) = (F \circ \alpha)'(0)
\]

**Proposition 18.3 (Exercise 1.3.1a).** Let $U \subset \mathbb{R}^m$ and let $F : U \to \mathbb{R}^n$ be a differentiable map. Let $x \in U$ and $v \in T_x\mathbb{R}^m$ be a tangent vector. Let $I \subset \mathbb{R}$ be an interval, and let $\alpha, \beta : I \to U$ be smooth curves with
\[
\alpha(0) = \beta(0) = x \quad \alpha'(0) = \beta'(0) = v
\]
Then $(F \circ \alpha)'(0) = (F \circ \beta)'(0)$.

**Proof.** We know that $F(x) = (f^1(x), f^2(x), \ldots, f^n(x))$ for differentiable functions $f^i : U \to \mathbb{R}$. Since $f^i \circ \alpha, f^i \circ \beta : I \to \mathbb{R}$ are differentiable function from $\mathbb{R}$ to $\mathbb{R}$, we can use the one-dimensional chain rule to get
\[
(f^i \circ \alpha)'(t) = \alpha'(t)((f^i) \circ \alpha)(t)
\]
\[
(f^i \circ \beta)'(t) = \beta'(t)((f^i) \circ \beta)(t)
\]
So $v(f^i)'(x) = (f^i \circ \alpha)'(0) = (f^i \circ \beta)'(0)$. Now we compute
\[
(F \circ \alpha)(t) = ((f^1 \circ \alpha)(t), \ldots, (f^n \circ \alpha)(t))
\]
\[
(F \circ \alpha)'(t) = ((f^1 \circ \alpha)'(t), \ldots, (f^n \circ \alpha)'(t))
\]
\[
(F \circ \alpha)'(0) = ((f^1 \circ \alpha)'(0), \ldots, (f^n \circ \alpha)'(0))
\]
\[
= ((\alpha'(0)(f^1 \circ \alpha)(0), \ldots, (f^n \circ \alpha)(0))
\]
\[
= ((\beta'(0)(f^1 \circ \beta)(0), \ldots, (f^n \circ \beta)(0))
\]
\[
= ((f^1 \circ \beta)'(0), \ldots, (f^n \circ \beta)'(0))
\]
\[
= (F \circ \beta)'(0)
\]

**Lemma 18.4 (for Exercise 1.3.1b).** Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable. Then $df(v + w) = df(v) + df(w)$.

**Proof.** Let $v = (v^1, v^2, \ldots, v^n), w = (w^1, w^2, \ldots, w^n)$. \[
df(v + w) = df((v^1 + w^1, v^2 + w^2, \ldots, v^n + w^n))
\]
\[
= \frac{\partial f}{\partial x^i}(v^i + w^i)
\]
\[
= \frac{\partial f}{\partial x^i}v^i + \frac{\partial f}{\partial x^i}w^i
\]
\[
= df(v) + df(w)
\]

126
Proof. Let $F(x) = (f^1(x), f^2(x), \ldots f^n(x))$ for some differentiable functions $f^i : U \to \mathbb{R}$. Let $\alpha, \beta : I \to U$ be smooth curves such that $\alpha(0) = \beta(0) = x$ and $\alpha'(0) = v$ and $\beta'(0) = w$. Then

\[
d F_x(v + w) = d F_x(v) + d F_x(w)
\]

And likewise for $\beta$,

\[
d F_x(w) = (F \circ \beta)'(0) = (df^1(\beta'(0)), df^2(\beta'(0)), \ldots df^n(\beta'(0)))
\]

Then we add them together and get

\[
d F_x(v) + d F_x(w) = (F \circ \alpha)'(0) + (F \circ \beta)'(0)
\]

\[
= (df^1(\alpha'(0)), df^2(\alpha'(0)), \ldots df^n(\alpha'(0)))
\]

\[
+ (df^1(\beta'(0)), df^2(\beta'(0)), \ldots df^n(\beta'(0)))
\]

\[
= (df^1(\alpha'(0)) + df^1(\beta'(0)), \ldots df^n(\alpha'(0) + \beta'(0)))
\]

\[
= (df^1(v + w), \ldots df^n(v + w))
\]

\[
d F_x(v + w)
\]

Proposition 18.6 (Exercise 1.3.1b). Let $U \subset \mathbb{R}^m$ and let $F : U \to \mathbb{R}^n$ be differentiable. Let $x \in U$ and $v, w \in T_x \mathbb{R}^m$. Then

\[
d F_x(cv) = c \, d F_x(v)
\]

Proof. Let $f^1, f^2, \ldots f^n$ be the component functions of $F$, that is, $F(x) = (f^1(x), f^2(x), \ldots f^n(x))$. Let $\alpha : I \to \mathbb{R}^m$ be a smooth curve such that $\alpha(0) = x$ and $\alpha'(0) = v$. Defined $\beta : I \to \mathbb{R}^m$ by $\beta(t) = \alpha(ct)$. Then $\beta(0) = x$ and $\beta'(0) = cv$, so $d F_x(cv) = (F \circ \beta)'(0)$. First we do some preliminary calculations.

\[
(f^i \circ \beta)(t) = f^i(\beta(t)) = f^i(\alpha(ct)) = (f^i \circ \alpha)(ct)
\]

\[
(f^i \circ \beta)'(t) = (f^i \circ \alpha)'(ct) = c(f^i \circ \alpha)'(t)
\]

\[
(F \circ \beta)(0) = ((f^1 \circ \beta)(0), \ldots (f^n \circ \beta)(0))
\]
Now we can use these to evaluate $dF_x(cv)$.

$$dF_x(cv) = (F \circ \beta)'(0)$$

$$= ((f^1 \circ \beta)'(0), \ldots, (f^n \circ \beta)'(0))$$

$$= (c(f^1 \circ \alpha)'(0), \ldots, c(f^n \circ \alpha)'(0))$$

$$= c(F \circ \alpha)'(0)$$

$$= c\ dF_x(v)$$

Let $G$ be a Lie group. Then for $g \in G$ we can define maps $L_g : G \to G$ by $L_g(h) = gh$. For $h \in G$, the differential $(dL_g)_h$ of $L_g$ at $h$ is a linear map $T_hG \to T_{gh}G$.

**Proposition 18.7** (Exercise 1.4.3a). Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. Let $v \in \mathfrak{g}$ and let $\tilde{v}$ be the vector field defined by

$$\tilde{v}(g) = (dL_g)_e(v)$$

Then for $g, h \in G$,

$$\tilde{v}(gh) = dL_g(\tilde{v}(h))$$

**Proof.** We know that $L_{gh} = L_g \circ L_h$. Then since $L_g, L_h : G \to G$ are differentiable, we can use the Chain Rule,

$$d(L_{gh})_e = d(L_g \circ L_h)_e = (dL_g)_h \circ (dL_h)_e$$

Now we can compute

$$\tilde{v}(gh) = (dL_{gh})_e(v)$$

$$= ((dL_g)_h \circ (dL_h)_e)(v)$$

$$= (dL_g)_h((dL_h)_e(v))$$

$$= (dL_g)_h(\tilde{v}(h))$$

$$= dL_g(\tilde{v}(h))$$

**Proposition 18.8** (Exercise 1.4.4). Let $V = a^i \frac{\partial}{\partial x^i}, W = b^j \frac{\partial}{\partial x^j}$ be vector fields on a Lie group $G$. Then $[V, W]$ is a vector field on $G$, and in particular,

$$[V, W] = c^j \left( a^i \frac{\partial b^j}{\partial x^i} - b^j \frac{\partial a^i}{\partial x^j} \right) \frac{\partial}{\partial x^j}$$

**Proof.** Let $f : G \to \mathbb{R}$ be a smooth function. Then

$$[V, W](f) = V(W(f)) - W(V(f))$$

$$= V \left( a^i \frac{\partial}{\partial x^i} \right) - W \left( b^j \frac{\partial}{\partial x^j} \right)$$

$$= a^i \frac{\partial}{\partial x^i} \left( a^i \frac{\partial}{\partial x^i} \right) - b^j \frac{\partial}{\partial x^j} \left( b^j \frac{\partial}{\partial x^j} \right)$$

128
Now we use the product rule.

\[ a^i \left( \frac{\partial b^j}{\partial x^i} \frac{\partial f}{\partial x^j} + b^j \frac{\partial^2 f}{\partial x^i \partial x^j} \right) - b^j \left( \frac{\partial a^i}{\partial x^i} \frac{\partial f}{\partial x^j} + a^i \frac{\partial^2 f}{\partial x^i \partial x^j} \right) \]

By Clairaut’s Theorem,

\[ \frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i} \]

so we get nice cancellation of the second term of each sum. Thus

\[ [V, W](f) = a^i \frac{\partial b^j}{\partial x^i} \frac{\partial f}{\partial x^j} - b^j \frac{\partial a^i}{\partial x^i} \frac{\partial f}{\partial x^j} \]

\[ = a^i \frac{\partial b^j}{\partial x^i} \frac{\partial f}{\partial x^j} - b^j \frac{\partial a^i}{\partial x^i} \frac{\partial f}{\partial x^j} \]

\[ = \left( a^i \frac{\partial b^j}{\partial x^i} - b^j \frac{\partial a^i}{\partial x^i} \right) \frac{\partial f}{\partial x^j} \]

Thus we have written \([V, W](f)\) as a linear combination of \(\frac{\partial}{\partial x^j} f\), and we know that \(\{\frac{\partial}{\partial x^j}\}\) is a basis for the tangent space, so

\[ [V, W] = \left( a^i \frac{\partial b^j}{\partial x^i} - b^j \frac{\partial a^i}{\partial x^i} \right) \frac{\partial}{\partial x^j} \]

Proposition 18.9. Let \(G\) be a Lie group with Lie algebra \(\mathfrak{g}\) and identity \(e\). Let \(V\) be a left-invariant vector field on \(G\), that is, \(V : G \to TG\) with \(V(gh) = (dL_g)_h(V(h))\). (Note that \((dL_g)_h : T_hG \to T_{gh}G\).) Let \(v = V(e)\). Then \(V = \tilde{v}\), where \(\tilde{v} : G \to TG\) is defined by \(\tilde{v}(g) = (dL_g)_e(v)\).

Proof. To show: For \(g \in G\), \(V(g) = \tilde{v}(g)\).

\[ \tilde{v}(g) = (dL_g)_e(v) = (dL_g)_e(V(e)) \]

\[ V(g) = V(ge) = (dL_g)_e(V(e)) \]

What the previous proposition shows is that every left-invariant vector field on \(G\) is equal to \(\tilde{v}\) for some \(v \in \mathfrak{g}\).

In the next proposition, \(M(n, \mathbb{R})\) refers to the set of all \(n \times n\) matrices with real entries and \(GL(n, \mathbb{R})\) refers to the set of invertible \(n \times n\) matrices.

Proposition 18.10 (Exercise 1.4.8a). Let \(G\) be a subgroup of \(GL(n, \mathbb{R})\) and denote the identity (matrix) by \(I\). Let \(L_g : G \to G\) be the map \(L_g(h) = gh\). Then \((dL_g)_1 : T_1G \to T_gG\) is given by \((dL_g)_1(A) = gA\).
Proof. Let $h : (-\epsilon, \epsilon) \to G$ be a smooth curve with $h(0) = I$ and $h'(0) = A \in M(n, \mathbb{R})$. Then
\[
(dL_g)_I(A) = \frac{d}{dt}(L_g \circ h)(t)|_{t=0} = \frac{d}{dt}(gh(t))|_{t=0} = gh'(t)|_{t=0} = gh'(0) = gA
\]

Corollary 18.11 (to Exercise 1.4.8a). Define the vector field $\tilde{A}$ on $G$ by $\tilde{A}(g) = gA$. Then $\tilde{A}$ is left-invariant.

Proof.
\[
\tilde{A}(gh) = ghA = g(hA) = g(\tilde{A}(h)) = (dL_g)_h(\tilde{A}(h))
\]

19 Connection of $\text{SL}(n, \mathbb{C})$ to $\text{sl}(n, \mathbb{C})$

Definition 19.1. Let $A \in \text{gl}(n, \mathbb{C})$. The matrix exponential is defined by
\[
\exp(A) = \sum_{n=1}^{\infty} \frac{A^n}{n!} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots
\]

Lemma 19.2. For a matrix $A \in \text{gl}(n, \mathbb{C})$, $\det(\exp(A)) = e^{\text{tr}A}$.

Proposition 19.3. $\text{sl}(n, \mathbb{C})$ is the tangent space at the identity of $\text{SL}(n, \mathbb{C})$.

Proof. Note that
\[
\text{Sl}(n, \mathbb{C}) = \text{det}^{-1}(1) \subset \text{GL}(n, \mathbb{C})
\]
We know that $\dim \text{SL}(n, \mathbb{C}) = n^2 - 1$, so the tangent space at the identity must also have $\dim n^2 - 1$. Let $A \in \text{sl}(n, \mathbb{C})$, so $\text{tr}A = 0$. Let $\alpha : (-\epsilon, \epsilon) \to \text{gl}(n, \mathbb{C})$ be a curve with $\alpha(t) = \exp(tA)$. Then
\[
\alpha(0) = I \\
\alpha'(t) = A \exp(tA) \implies \alpha'(0) = A \\
\det(\alpha(t)) = \det(\exp(tA)) = e^{\text{tr}A} = e^0 = 1
\]
Thus $\alpha$ is a curve in $\text{SL}(n, \mathbb{C})$ with $A = \alpha'(0)$ in the tangent space, thus $\text{sl}(n, \mathbb{C})$ is contained in the tangent space. Since $\dim \text{sl}(n, \mathbb{C}) = n^2 - 1$, it must be the whole tangent space.
References


