The following exercise can be found on p. 283 of the textbook *Applied Functional Analysis and Partial Differential Equations*, by Milan Miklavčič, and it demonstrates nonuniqueness for semilinear parabolic equations under many definitions of a solution that are currently being used.

10. Show that

\[ u(x,t) = \left( c + \frac{\pi^2}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{e^{-2(n^2+m^2)t}}{n^2 + m^2} \right)^{-1/4} \sum_{k=1}^{\infty} \frac{\sin kx}{k} e^{-k^2t} \]

satisfies, for every \( c \geq 0 \),

\[ u_t(x,t) = u_{xx}(x,t) + \left( \int_0^{\pi} |u_x(s,t)|^2 ds \right)^2 u(x,t) \quad \text{for} \quad t > 0, \quad 0 \leq x \leq \pi \]

\[ u(0,t) = u(\pi,t) = 0 \quad \text{for} \quad t \geq 0 \]

\[ \lim_{t \to 0^+} \sup_{0 \leq x \leq \pi} |u(x,t)| = 0. \]

(*Hint:* evaluate the Fourier sine series of \( \pi - x \) and see Exercise 16 in Chapter 4.) Show that the PDE can be set as a semilinear parabolic equation in \( L^2(0,\pi) \), with \( \alpha = 1/2 \) (see also Exercise 7) and that the uniqueness fails in this case because

\[ \int_0^1 \left( \int_0^{\pi} |u_x(x,t)|^2 dx \right)^2 dt = \infty. \]

This example, in more abstract form, was first published by the author in *Pacific J. Math.* **118**(1985), pp. 199-214. In response to the article, Dan Henry sent to the author an example similar to the above one.