3.3 Polynomial and Other Functions

In this section you will learn to:

- understand characteristics of polynomial functions
- find intervals on which a function is increasing, decreasing, or constant
- find the relative maximum or minimum of a function
- determine whether a function is even, odd, or neither
- graph and evaluate piecewise functions

A polynomial function of degree \( n \) is defined by

\[ f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0, \]

where \( n \) is a nonnegative integer, and \( a_n, a_{n-1}, a_{n-2}, \ldots, a_2, a_1, a_0 \) are real numbers and \( a_n \neq 0 \).

\( a_n \) is called the leading coefficient.

\( a_0 \) is called the constant term.

The degree of the polynomial is \( n \)

Graphs of polynomials are smooth (rounded curves) and continuous (no breaks).

A polynomial of degree \( n \) has at most \( n-1 \) turning points (graph changes direction).

Recall:

\[ y = f(x) = c \hspace{2cm} \text{degree} = _____ \]
\[ y = f(x) = mx + b \hspace{2cm} \text{degree} = _____ \]
\[ y = f(x) = ax^2 + bx + c \hspace{2cm} \text{degree} = _____ \]
\[ y = f(x) = ax^3 + bx^2 + cx + d \hspace{2cm} \text{degree} = _____ \]

Example 1: Determine which functions are polynomial functions. For those that are, identify the degree. For those that are not, explain why they are not polynomial functions.

(a) \( f(x) = 5x^2 + 3x^2 - 7 \) Yes No ______________________________________________________________________

(b) \( g(x) = 10 \) Yes No ______________________________________________________________________

(c) \( h(x) = x\sqrt{7} + \pi x^3 \) Yes No ______________________________________________________________________

(d) \( f(x) = \frac{3x^2 + 5}{x} \) Yes No ______________________________________________________________________

(e) \( g(x) = |x| \) Yes No ______________________________________________________________________

(f) \( h(x) = \frac{3x^2 + 5}{2} \) Yes No ______________________________________________________________________
End Behavior of a Polynomial (what happens to the graph of the function to the far left \((x \to -\infty)\) and far right \((x \to \infty)\)) and Leading Coefficient \((a_n)\) Test

<table>
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<tr>
<th>Degree (n) is Even</th>
<th>Degree (n) is Even</th>
<th>Degree (n) is Odd</th>
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<td>(a_n &gt; 0)</td>
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Think: \(y = x^2\)  
Think: \(y = -x^2\)  
Think: \(y = x^3\)  
Think: \(y = -x^3\)

Example 2: Without using a calculator, determine the end behavior of the following.

(a) \(f(x) = x^3 - x^2 - 1\)  
(b) \(f(x) = -4x^4 - 3x^2 + 7\)  
(c) \(f(x) = -5(x-3)^2(x+2)^3\)  
(d) \(f(x) = -x^3 + 8x^4 + 4x^2 + 2\)

Relative Maximum/Minimum: The point(s) at which a function changes its increasing or decreasing behavior. These points are also called turning points.  
A function is increasing if the \(y\) values increase on the graph of \(f\) from left to right.  
A function is decreasing if the \(y\) values decrease on the graph of \(f\) from left to right.  
A function is constant if the \(y\) values remain unchanged on the graph of \(f\) from left to right.

Example 3: 

\[y = f(x)\]

\(f\) has a relative minimum(s) at _____________.
The relative minimum(s) of \(f\) are _____________.
\(f\) has a relative maximum(s) at _____________.
The relative maximum(s) \(f\) are _____________.

On which intervals is \(f\) increasing? ____________ decreasing? ____________ constant? ____________
**Example 4:** Use the following steps to graph $f(x) = x^4 - 4x^2$.

Steps for Graphing a Polynomial Function:

1. Use **Leading Coefficient Test** to determine End Behavior.
2. Find the $x$-intercept(s). Let $f(x) = 0$.
3. Find the $y$-intercept. Let $x = 0$.
4. Determine where the graph is above or below $x$-axis.
5. Plot a few points and draw a smooth, continuous graph.
6. Use # of **turning points** to check graph accuracy.

<table>
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<th>Even Functions</th>
<th>Odd Functions</th>
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**Note:** If $f(x) \neq f(-x)$ and $f(x) \neq -f(-x)$, then $f(x)$ is “**neither**” odd nor even.

**Example 5:** Determine whether each of the functions below is even, odd, or neither.

\[
\begin{align*}
  f(x) &= x^3 - 3x \quad \underline{\text{even}} \quad f(x) = x^4 - 2x^2 + 1 \quad \underline{\text{even}} \quad f(x) = x^3 - x^2 - 6x \quad \underline{\text{odd}}
\end{align*}
\]
Example 6: Refer to the graph below to answer/find the following:

(a) Is the graph a function? _______
(b) Is this a graph of a polynomial function? ____
(c) domain __________
(d) range __________
(e) $f(0) = __________$
(f) $x$-intercept(s) __________
(g) $y$-intercept(s) __________
(h) increasing (interval) __________
(i) decreasing (interval) __________
(j) constant (interval) __________
(k) For what value(s) of $x$ does $f(x) = -5$? ______

**Piecewise Function:** A function that is defined by two (or more) equations over a specified domain.

Example 7: Graph $f(x) = \begin{cases} 
-x, & x < 0 \\
x^2 - 3, & x \geq 0 
\end{cases}$

Example 8: Graph $f(x) = \begin{cases} 
1, & \text{if } x > 0 \\
0, & \text{if } x = 0 \\
-1, & \text{if } x < 0 
\end{cases}$

Example 9: Given $f(x) = \begin{cases} 
2x, & \text{if } x \geq 5 \\
x^2 - 3, & \text{if } x < 5 
\end{cases}$, evaluate each of the following:

(a) $f(0) - f(5)$
(b) $5f(-3) - \left[f(6)\right]^2$
3.3 Homework Problems:

1. Determine which functions are polynomial functions.
   (a) \( f(x) = x^5 - \sqrt[3]{x^2} \)  
   (b) \( g(x) = x^{-2} + 8x^{-1} - 9 \)  
   (c) \( h(x) = 2.5x^3 - \pi x^2 + 2 \)  
   (d) \( g(x) = 6x^7 + \frac{1}{x} \)  
   (e) \( f(x) = x^{\frac{1}{2}} - 5 \)  
   (f) \( h(x) = \frac{3x^3 + 2x^2}{x^3} \)

2. Use the Leading Coefficient Test to determine the end behavior of the graph of \( f \).
   (a) \( f(x) = -x^4 - x^2 \)  
   (b) \( f(x) = 7x^3 - 4x^2 \)  
   (c) \( f(x) = x^8 \)  
   (d) \( f(x) = 9 - x^3 \)  
   (e) \( f(x) = (x - 2)^2(x + 3)^3 \)  
   (f) \( f(x) = -2x(x + 3)^2(x - 5) \)

3. Consider the graph of the function \( f(x) = x^4 - 9x^2 \).
   (a) Use Leading Coefficient Test to determine the end behavior of the function.
   (b) Find the \( x \)-intercept(s).
   (c) Find the \( y \)-intercept.
   (d) For what intervals is the graph above the \( x \)-axis?

4. Consider the graph of the function \( f(x) = 6x^2 + x^3 - x^4 \).
   (a) Use Leading Coefficient Test to determine the end behavior of the function.
   (b) Find the \( x \)-intercept(s).
   (c) Find the \( y \)-intercept.
   (d) For what intervals is the graph above the \( x \)-axis?

5. Determine whether each function is even, odd, or neither.
   (a) \( f(x) = x^4 + 5x^2 \)  
   (b) \( g(x) = -5x^3 - 3x^2 + 7 \)  
   (c) \( h(x) = x^5 + 2x^3 - x \)  
   (d) \( g(x) = 5x^2 + 6 \)  
   (e) \( f(x) = -3 \)  
   (f) \( f(x) = x^3 - 1 \)
6. Refer to the graph of \( f \) below to determine each of the following:
   (Use interval notation whenever possible.)

   (a) the domain of \( f \)
   (b) the range of \( f \)
   (c) \( x \)-intercept(s)
   (d) \( y \)-intercept(s)
   (e) interval(s) on which \( f \) is increasing
   (f) interval(s) on which \( f \) is decreasing
   (g) values of \( x \) for which \( f(x) < 0 \)
   (h) number(s) at which \( f \) has a relative maximum
   (i) relative maximum of \( f \)
   (j) \( f(-2) \)
   (k) value(s) for which \( f(x) = 0 \)
   (l) values for which \( f(x) = 2 \)
   (m) Is \( f \) even, odd, or neither?

7. Given the piecewise function \( f(x) = \begin{cases} 
  x & \text{if } x < 0 \\
  x^2 & \text{if } x \geq 0 
\end{cases} \), evaluate the following:

   (a) \( f(0) \)  (b) \( f(10) \)  (c) \( f(-3) + f(5) \)  (d) \(-3f(-1) \cdot f(2) \)  (e) \(7 - f(-5) \)

8. Given the piecewise function \( f(x) = \begin{cases} 
  0 & \text{if } x < -2 \\
  x + 2 & \text{if } -2 \leq x \leq 2 \\
  \sqrt{x} & \text{if } x > 2 
\end{cases} \), evaluate the following:

   (a) \( f(-4) + f(0) + f(4) \)  (b) \( 3f(-2) - 5f(2) \)  (c) \( 2[f(9)]^2 \)  (d) \( \frac{f(100) + f(1)}{13 + f(-10)} \)

3.3 Homework Answers: 1. (a) polynomial; (b) not a polynomial; (c) polynomial; (d) not a polynomial; (e) not a polynomial; (f) not a polynomial
   2. (a) falls right and left; (b) falls left and rises right; (c) rises right and left; (d) rises left and falls right; (e) left and rises right; (f) falls left and right
   3. (a) rises left and right; (b) -3, 0, 3; (c) 0; (d) \((-\infty, -3) \) and \((3, \infty) \)
   4. (a) falls left and right; (b) 0, 3, -2; (c) 0; (d) (-2, 0) and (0, 3)
   5. (a) even; (b) neither; (c) odd; (d) even; (e) even; (f) neither
   6. (a) [-7, 6]; (b) [-2, 5]; (c) -6; (d) 2; (e) (-7, -4) and (0, 6); (f) (-4, 0); (g) [-7, -6]; (h) \(x = -4 \); (i) 4; (j) 3; (k) \( x = -6 \); (l) \( x = -5, 0 \); (m) neither
   7. (a) 0; (b) 100; (c) 22; (d) 12; (e) 12
   8. (a) 4; (b) -20; (c) 18; (d) 1