4.1 Exponential Functions and Their Graphs

In this section you will learn to:
- evaluate exponential functions
- graph exponential functions
- use transformations to graph exponential functions
- use compound interest formulas

An exponential function \( f \) with base \( b \) is defined by
\[
 f(x) = b^x \quad \text{or} \quad y = b^x , \quad \text{where} \quad b > 0, \ b \neq 1, \ \text{and} \ x \ \text{is any real number.}
\]

Note: Any transformation of \( y = b^x \) is also an exponential function.

Example 1: Determine which functions are exponential functions. For those that are not, explain why they are not exponential functions.

(a) \( f(x) = 2^x + 7 \) \hspace{1cm} Yes No ______________________________
(b) \( g(x) = x^2 \) \hspace{1cm} Yes No ______________________________
(c) \( h(x) = 1^x \) \hspace{1cm} Yes No ______________________________
(d) \( f(x) = x^x \) \hspace{1cm} Yes No ______________________________
(e) \( h(x) = 3 \cdot 10^{-x} \) \hspace{1cm} Yes No ______________________________
(f) \( f(x) = -3^{x+1} + 5 \) \hspace{1cm} Yes No ______________________________
(g) \( g(x) = (-3)^{x+1} + 5 \) \hspace{1cm} Yes No ______________________________
(h) \( h(x) = 2x - 1 \) \hspace{1cm} Yes No ______________________________

Example 2: Graph each of the following and find the domain and range for each function.

(a) \( f(x) = 2^x \) \hspace{1cm} domain: __________
    \hspace{1cm} range: __________
(b) \( g(x) = \left(\frac{1}{2}\right)^x \) \hspace{1cm} domain: __________
    \hspace{1cm} range: __________
### Characteristics of Exponential Functions \( f(x) = b^x \)

<table>
<thead>
<tr>
<th>( b &gt; 1 )</th>
<th>( 0 &lt; b &lt; 1 )</th>
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<tbody>
<tr>
<td>Domain:</td>
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<td>Range:</td>
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**Transformations of** \( g(x) = b^x \) (\( c > 0 \)):  
(Order of transformations is H S R V.)

- **Horizontal:**
  - \( g(x) = b^{x+c} \) (graph moves \( c \) units left)
  - \( g(x) = b^{x-c} \) (graph moves \( c \) units right)

- **Stretch/Shrink:**
  - (Vertical) \( g(x) = cb^x \) (graph stretches if \( c > 1 \))
  - (graph shrinks if \( 0 < c < 1 \))

- **Stretch/Shrink:**
  - (Horizontal) \( g(x) = b^{-x} \) (graph shrinks if \( c > 1 \))
  - (graph stretches if \( 0 < c < 1 \))

- **Reflection:**
  - \( g(x) = -b^x \) (graph reflects over the \( x\)-axis)
  - \( g(x) = b^{-x} \) (graph reflects over the \( y\)-axis)

- **Vertical:**
  - \( g(x) = b^x + c \) (graph moves up \( c \) units)
  - \( g(x) = b^x - c \) (graph moves down \( c \) units)
Example 3: Use \( f(x) = 2^x \) to obtain the graph \( g(x) = -2^{x+3} - 1 \).

Domain of \( g \): __________

Range of \( g \): __________

Equation of any asymptote(s) of \( g \): __________

\[ f(x) = e^x \] is called the **natural exponential function**, where the irrational number \( e \) (approximately 2.718282) is called the **natural base**.

(The number \( e \) is defined as the value that \( \left(1 + \frac{1}{n}\right)^n \) approaches as \( n \) gets larger and larger.)

Example 4: Graph \( f(x) = e^x \), \( g(x) = e^{x-3} \), and \( h(x) = -e^x \) on the same set of axes.
<table>
<thead>
<tr>
<th>Periodic Interest Formula</th>
<th>Continuous Interest Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A = P\left(1+\frac{r}{n}\right)^{nt}$</td>
<td>$A = Pe^{rt}$</td>
</tr>
</tbody>
</table>

$A =$ balance in the account (Amount after $t$ years)  

$P =$ principal (beginning amount in the account)  

$r =$ annual interest rate (as a decimal)  

$n =$ number of times interest is compounded per year  

$t =$ time (in years)

**Example 5:** Find the accumulated value of a $5000 investment which is invested for 8 years at an interest rate of 12% compounded:

(a) annually  

(b) semi-annually  

(c) quarterly  

(d) monthly  

(e) continuously
4.1 Homework Problems

1. Use a calculator to find each value to four decimal places.
   (a) $5\sqrt[3]{3}$  (b) $7^\pi$  (c) $2^{-5.3}$  (d) $e^2$  (e) $e^{-2}$  (f) $-e^{0.25}$  (g) $\pi^{-1}$

2. Simplify each expression without using a calculator. (Recall: $b^n \cdot b^m = b^{n+m}$ and $(b^m)^n = b^{mn}$)
   (a) $6\sqrt[3]{6\sqrt[3]{6}}$  (b) $(3\sqrt[3]{3})\sqrt[3]{3}$  (c) $b^{\sqrt[3]{3}}$  (d) $(5\sqrt[3]{3})\sqrt[3]{3}$  (e) $\frac{1}{4^{\frac{1}{4}}} \cdot \frac{1}{4^{\frac{1}{4}}}$  (f) $b^{\frac{1}{\sqrt[3]{b}}} \cdot b^{\frac{1}{\sqrt[3]{b}}}$

For Problems 3 – 14, graph each exponential function. State the domain and range for each along with the equation of any asymptotes. Check your graph using a graphing calculator.

3. $f(x) = 3^x$  4. $f(x) = -(3^x)$  5. $f(x) = 3^{-x}$  6. $f(x) = \left(\frac{1}{3}\right)^x$

7. $f(x) = 2^x - 3$  8. $f(x) = 2^{x-3}$  9. $f(x) = 2^{x+5} - 5$  10. $f(x) = -2^{-x}$

11. $f(x) = -2^{x+3} + 1$  12. $f(x) = \left(\frac{1}{2}\right)^{x-3} - 4$  13. $f(x) = e^{-x} + 2$  14. $f(x) = -e^{x+2}$

15. $10,000$ is invested for 5 years at an interest rate of 5.5%. Find the accumulated value if the money is
   (a) compounded semiannually; (b) compounded quarterly; (c) compounded monthly; (d) compounded
   continuously.

16. Sam won $150,000$ in the Michigan lottery and decides to invest the money for retirement in 20
   years. Find the accumulated value for Sam’s retirement for each of his options:
   (a) a certificate of deposit paying 5.4% compounded yearly
   (b) a money market certificate paying 5.35% compounded semiannually
   (c) a bank account paying 5.25% compounded quarterly
   (d) a bond issue paying 5.2% compounded daily
   (e) a saving account paying 5.19% compounded continuously

4.1 Homework Answers:  1. (a) 16.2425; (b) 451.8079; (c) .0254; (d) 7.3891; (e) .1353; (f) -1.2840; (g) .3183
   2. (a) 36\sqrt[2]{2}; (b) 9; (c) b^4; (d) 125; (e) 4; (f) b^{\frac{1}{3}} \cdot \sqrt[3]{b}  3. Domain: $(-\infty, \infty)$; Range: $(0, \infty)$; $y = 0$
   4. Domain: $(-\infty, \infty)$; Range: $(-\infty, 0)$; $y = 0$  5. Domain: $(-\infty, \infty)$; Range: $(0, \infty)$; $y = 0$
   6. Domain: $(-\infty, \infty)$; Range: $(0, \infty)$; $y = 0$  7. Domain: $(-\infty, \infty)$; Range: $(-3, \infty)$; $y = -3$
   8. Domain: $(-\infty, \infty)$; Range: $(0, \infty)$; $y = 0$  9. Domain: $(-\infty, \infty)$; Range: $(-5, \infty)$; $y = -5$
   10. Domain: $(-\infty, \infty)$; Range: $(-\infty, 0)$; $y = 0$  11. Domain: $(-\infty, \infty)$; Range: $(-\infty, 1)$; $y = 1$
   12. Domain: $(-\infty, \infty)$; Range: $(-4, \infty)$; $y = -4$  13. Domain: $(-\infty, \infty)$; Range: $(2, \infty)$; $y = 2$
   14. Domain: $(-\infty, \infty)$; Range: $(-\infty, 0)$; $y = 0$  15. (a) $13,116.51$; (b) $13,140.67$; (c) $13,157.04$; (d) $13,165.31$  16. (a) $429,440.97$; (b) $431,200.96$; (c) $425,729.59$; (d) $424,351.12$; (e) $423,534.64$

Page 5 (Section 4.1)