5.3 Roots of Polynomial Equations

In this section you will learn to:

• find zeros of polynomial equations
• solve polynomial equations with real and imaginary zeros
• find possible rational roots of polynomial equations
• understand properties of polynomial equations
• use the Linear Factorization Theorem

**Zeros of Polynomial Functions** are the values of $x$ for which $f(x) = 0$.

(Zero = Root = Solution = $x$-intercept (if the zero is a real number))

**Example 1:** Consider the polynomial that only has 3 and $\frac{1}{2}$ as zeros.

(a) How many polynomials have such zeros?

(b) Find a polynomial that has a leading coefficient of 1 that has such zeros.

(c) Find a polynomial, with integer coefficients, that has such zeros.

If the same factor $(x - r)$ occurs $k$ times, then the zero $r$ is called a zero with **multiplicity** $k$.

**Even Multiplicity** → Graph touches $x$-axis and turns around.

**Odd Multiplicity** → Graph crosses $x$-axis.

**Example 2:** Find the equation of a polynomial that has the following characteristics:

(a) graph falls left as $x \to -\infty$ and falls right as $x \to \infty$
(b) crosses the $x$-axis at 2 and -5
(c) touches the $x$-axis at 0 and 13
(d) has degree = 8
Example 3: Find all of the (real) zeros for each of the polynomial functions below. Give the multiplicity of each zero and state whether the graph crosses the x-axis or touches (and turns at) the x-axis at each zero. Use this information and the Leading Coefficient Test to sketch a graph of each function

(a) \( f(x) = x^3 + 2x^2 - 4x - 8 \) \hspace{1cm} (b) \( f(x) = -x^4 + 4x^2 \) \hspace{1cm} (c) \( g(x) = x^4 - 4x^3 + 4x^2 \)

The Rational Zero Theorem: If \( f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0 \) has integer coefficients and \( \frac{p}{q} \) (reduced to lowest terms) is a rational zero of \( f \), then \( p \) is a factor of the constant term, \( a_0 \), and \( q \) is a factor of the leading coefficient, \( a_n \).

Example 4: List all possible rational zeros of the polynomials below.

(a) \( f(x) = 2x^3 - 5x^2 + x + 2 \) Possible Rational Zeros: _________________________

(b) \( f(x) = -x^5 + 7x^2 - 12 \) Possible Rational Zeros: _________________________

(c) \( p(x) = 6x^3 - 8x^2 - 8x + 8 \) Possible Rational Zeros: _________________________
Example 5: Find all zeros of \( f(x) = 2x^3 - 5x^2 + x + 2 \).

Example 6: Solve \( x^4 - 8x^3 + 64x - 105 = 0 \).
Linear Factorization Theorem:

If \( f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 \), where \( n \geq 1 \) and \( a_n \neq 0 \), then

\[ f(x) = a_n (x - c_1)(x - c_2) \ldots (x - c_n) \]

where \( c_1, c_2, c_3, \ldots, c_n \) are complex numbers.

Example 7: Find all complex zeros of \( f(x) = 2x^3 + 3x^2 + 3x - 2 \), and then write the polynomial \( f(x) \) as a product of linear factors.

\[ f(x) = \ldots \]

Properties of Polynomial Equations:

Given the polynomial \( f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 \).

1. If a polynomial equation is of degree \( n \), then counting multiple roots (multiplicities) separately, the equation has \( n \) roots.

2. If \( a + bi \) is a root of a polynomial equation \( (b \neq 0) \), then the imaginary number \( a - bi \) is also a root. In other words, imaginary roots, if they exist, occur in conjugate pairs.

Example 8: Find all zeros of \( f(x) = x^4 - 4x^2 - 5 \). (Hint: Use factoring techniques from Chapter 1.) Write \( f(x) \) as a product of linear factors.

\[ f(x) = \ldots \]
**Example 9:** Find a third-degree polynomial function, \( f(x) \), with real coefficients that has 4 and \( 2i \) as zeros and such that  \( f(-1) = 50 \).

Step 1: Use the zeros to find the factors of \( f(x) \).

Step 2: Write as a linear factorization, then expand/multiply.

Step 3: Use \( f(-1) = 50 \) to substitute values for \( x \) and \( f(x) \).

Step 4: Solve for \( a_n \).

Step 5: Substitute \( a_n \) into the equation for \( f(x) \) and simplify.

Step 6: Use your calculator to check.
5.3 Homework Problems:

For Problems 1 – 4, use the Rational Zero Theorem to list all possible rational zeros for each function.

1. \( f(x) = x^3 + 3x^2 - 6x - 8 \)  
2. \( f(x) = 2x^4 + 3x^3 - 11x^2 - 9x + 15 \)  
3. \( f(x) = 3x^4 - 11x^3 - 3x^2 - 6x + 8 \)  
4. \( f(x) = 4x^5 - 8x^3 - x + 2 \)

For Problems 5 – 8, find the zeros for the given functions.

5. \( f(x) = x^3 - 2x^2 - 11x + 12 \)  
6. \( f(x) = 2x^3 - 5x^2 + x + 2 \)  
7. \( f(x) = 2x^3 + x^2 - 3x + 1 \)  
8. \( f(x) = x^3 - 4x^2 + 8x - 5 \)

For Problems 9 – 12, solve each of the given equations.

9. \( x^3 - 2x^2 - 7x - 4 = 0 \)  
10. \( x^3 - 5x^2 + 17x - 13 = 0 \)  
11. \( 2x^3 - 5x^2 - 6x + 4 = 0 \)  
12. \( x^4 - 2x^2 - 16x - 15 = 0 \)

For Problems 13-16, find an \( nth \) degree polynomial function, \( f(x) \), with real coefficients that satisfies the given conditions.

13. \( n = 3; \) 1 and 5\( i \) are zeros; \( f(-1) = -104 \)  
14. \( n = 4; 2, -2, \) and \( i \) are zeros; \( f(3) = -150 \)  
15. \( n = 3; \) 6 and -5 + 2\( i \) are zeros; \( f(2) = -636 \)  
16. \( n = 4; \) \( i \) and 3\( i \) are zeros; \( f(-1) = 20 \)

5.3 Homework Answers:  
1. \( \pm 1, \pm 2, \pm 4, \pm 8 \)  
2. \( \pm 1, \pm 3, \pm 5, \pm 15, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \pm \frac{15}{2} \)  
3. \( \pm 1, \pm 2, \pm 4, \pm 8, \pm \frac{1}{2}, \pm \frac{2}{3}, \pm \frac{4}{3}, \pm \frac{8}{3} \)  
4. \( \pm 1, \pm 2, \pm \frac{1}{2}, \pm \frac{1}{4} \)  
5. -3, 1, 4  
6. -\( \frac{1}{2} \), 1, 2

7. \( \frac{1}{2}, \frac{-1 \pm \sqrt{5}}{2} \)  
8. \( \frac{3 \pm i \sqrt{11}}{2} \)  
9. \( \{-1, 4\} \)  
10. \( \{1, 2 \pm 3i\} \)  
11. \( \left\{ \frac{1}{2}, 1 \pm \sqrt{5} \right\} \)

12. \( \{-1, 3, -1 \pm 2i\} \)  
13. \( f(x) = 2x^3 - 2x^2 + 50x - 50 \)  
14. \( f(x) = -3x^4 + 9x^2 + 12 \)  
15. \( f(x) = 3x^3 + 12x^2 - 93x - 522 \)  
16. \( f(x) = x^4 + 10x^2 + 9 \)