Coaiza Expository Talk Exercises.

Exercise 2: Suppose $A = B(H)$ and let $a, b \in B(H)$, $\xi \in H$.

If $\begin{pmatrix} I_H & a \\ a^* & b \end{pmatrix} \in M_2(B(H))^+$ then

$$\left\langle \begin{pmatrix} I_H & a \\ a^* & b \end{pmatrix} \begin{pmatrix} -a\xi \\ \xi \end{pmatrix} \middle| \begin{pmatrix} -a\xi \\ \xi \end{pmatrix} \right\rangle \geq 0$$

$$\Rightarrow \left\langle \begin{pmatrix} 0 \\ (-a^*a + b)\xi \end{pmatrix} \middle| \begin{pmatrix} -a\xi \\ \xi \end{pmatrix} \right\rangle \geq 0$$

$$\Rightarrow \langle (-a^*a + b)\xi \mid \xi \rangle \geq 0 \quad \forall \xi \in H$$

$$\Rightarrow b - a^*a \in B(H)^+.$$

Conversely, if $b - a^*a \in B(H)^+ \Rightarrow \begin{pmatrix} 0 & 0 \\ 0 & b-a^*a \end{pmatrix} \in M_2(B(H))^+$.

Consider $\begin{pmatrix} I_H & a \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} I_H & a \\ 0 & 0 \end{pmatrix} \in M_2(B(H))^+$

and $\begin{pmatrix} I_H & a \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} I_H & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_H & 0 \\ a^* & 0 \end{pmatrix} \begin{pmatrix} I_H & a \\ 0 & 0 \end{pmatrix}$

$$= \begin{pmatrix} I_H & a \\ a^* & a^*a \end{pmatrix} \in M_2(B(H))^+$$

Thus since $B(H)^+ + B(H)^+ \subset B(H)^+$, we conclude

$$\begin{pmatrix} I_H & a \\ a^* & b \end{pmatrix} \in M_2(B(H))^+ .$$

Q.E.D.

Exercise 3: $u: X \to Y$,

$u_n: M_n(X) \to M_n(Y)$, $\sum_{ij} e_i e_j^* \otimes x_{ij} \mapsto \sum_{ij} e_i e_j^* \otimes u(x_{ij})$

Let $a, b \in M_n$. Then we have for $x \in M_n(X)$

$$a x b = \sum_{ij} e_i e_j^* \otimes \sum_{kl} a_{ik} x_{kl} b_{lj}$$

In particular, the ij^{th} entry of axb is precisely

$$\sum_{kl} a_{ik} x_{kl} b_{lj} .$$

We then have,

$$u_n(axb) = u_n\Big(\sum_{ij} e_i e_j^* \otimes \sum_{kl} a_{ik} x_{kl} b_{lj} \Big)$$

$$= \sum_{ij} e_i e_j^* \otimes u\Big(\sum_{kl} a_{ik} x_{kl} b_{lj} \Big)$$

$$= \sum_{ij} e_i e_j^* \otimes \sum_{kl} a_{ik} u(x_{kl}) b_{lj}$$

$$= a\, u_n(x)\, b .$$

Q.E.D.

Exercise 4) Let $w \in M_n(\mathcal{V}_X)$,

$$w = \sum e_i e_j^{\circ} \otimes w_{ij}, \quad w_{ij} = \begin{pmatrix} \lambda_{ij11} & x_{ij12} \\ y^*_{ij21} & \mu_{ij22} \end{pmatrix}$$

Since $w \in M_n(\mathcal{V}_X) \subset M_n(M_2(A))$ we use the canonical shuffle to identify $M_n(M_2(A)) \simeq M_2(M_n(A))$.

Under the canonical shuffle $w \rightsquigarrow \begin{pmatrix} \Lambda & X \\ Y^* & M \end{pmatrix}$

w/ $\Lambda = \sum e_i e_j^{\circ} \otimes \lambda_{ij}$, $M = \sum e_i e_j^{\circ} \otimes \mu_{ij}$

$X = \sum e_i e_j^{\circ} \otimes x_{ij}$, $Y = \sum e_i e_j^{\circ} \otimes y_{ij}$.

In particular, $u_n(w)$ under the canonical shuffle becomes

$$\begin{pmatrix} \Lambda & u_n(X) \\ u_n(Y)^* & M \end{pmatrix}.$$

Thus, we claim $\begin{pmatrix} \Lambda & X \\ Y^* & M \end{pmatrix}$ is positive implies

$\begin{pmatrix} \Lambda & u_n(X) \\ u_n(Y)^* & M \end{pmatrix}$ is positive. This will prove complete positivity.

If $\begin{pmatrix} \Lambda & X \\ Y^* & M \end{pmatrix}$ positive it must follow Λ, M are positive, self-adjoint and $X = Y$. Thus, we consider $\begin{pmatrix} \Lambda & X \\ X^* & M \end{pmatrix}$

and let $\varepsilon > 0$. Let $\Lambda_\varepsilon := \Lambda + \varepsilon I_n$, $M_\varepsilon := M + \varepsilon I_n$ so that both Λ_ε and M_ε are positive and invertible.

positive and invertible.

Notice,

$$\begin{pmatrix}\Lambda_\varepsilon^{-1/2} & \\ & M_\varepsilon^{-1/2}\end{pmatrix}\begin{pmatrix}\Lambda_\varepsilon & X \\ X^* & M_\varepsilon\end{pmatrix}\begin{pmatrix}\Lambda_\varepsilon^{-1/2} & \\ & M_\varepsilon^{-1/2}\end{pmatrix}$$

$$= \begin{pmatrix}\Lambda_\varepsilon^{1/2} & \Lambda_\varepsilon^{-1/2}X \\ M_\varepsilon^{-1/2}X^* & M_\varepsilon^{1/2}\end{pmatrix}\begin{pmatrix}\Lambda_\varepsilon^{-1/2} & \\ & M_\varepsilon^{-1/2}\end{pmatrix} = \begin{pmatrix} I & \Lambda_\varepsilon^{-1/2}XM_\varepsilon^{-1/2} \\ M_\varepsilon^{-1/2}X^*\Lambda_\varepsilon^{-1/2} & I\end{pmatrix}$$

is positive (being conjugation of a positive element).

Thus, $\|\Lambda_\varepsilon^{-1/2} X M_\varepsilon^{-1/2}\| \leq 1$ (by Exercise 2)

By Exercise 3 we know $\Lambda_\varepsilon^{-1/2} u_n(X) M_\varepsilon^{-1/2} = u_n(\Lambda_\varepsilon^{-1/2} X M_\varepsilon^{-1/2})$
and since u is a complete contraction
$\|u_n(\Lambda_\varepsilon^{-1/2} X M_\varepsilon^{-1/2})\| \leq 1$.

By Exercise 2, this implies $\begin{pmatrix} I & u_n(\Lambda_\varepsilon^{-1/2}XM_\varepsilon^{-1/2}) \\ u_n(\Lambda_\varepsilon^{-1/2}XM_\varepsilon^{-1/2})^* & I\end{pmatrix}$

is in $M_2(M_n(B))^+$. Therefore, we know

$$\begin{pmatrix}\Lambda_\varepsilon^{1/2} & \\ & M_\varepsilon^{1/2}\end{pmatrix}\begin{pmatrix} I & u_n(\Lambda_\varepsilon^{-1/2}XM_\varepsilon^{-1/2}) \\ u_n(\Lambda_\varepsilon^{-1/2}XM_\varepsilon^{-1/2})^* & I\end{pmatrix}\begin{pmatrix}\Lambda_\varepsilon^{1/2} & \\ & M_\varepsilon^{1/2}\end{pmatrix}$$

must also be positive in $M_2(M_n(B))^+$. But the above becomes

$$\begin{pmatrix} \Lambda_\varepsilon^{1/2} & \Lambda_\varepsilon^{1/2} u_n(\Lambda_\varepsilon^{-1/2} X M_\varepsilon^{-1/2}) \\ M_\varepsilon^{1/2} u_n(\Lambda_\varepsilon^{-1/2} X M_\varepsilon^{-1/2})^* & M_\varepsilon^{1/2} \end{pmatrix} \begin{pmatrix} \Lambda_\varepsilon^{1/2} & \\ & M_\varepsilon^{1/2} \end{pmatrix}$$

$$= \begin{pmatrix} \Lambda_\varepsilon & u_n(X) \\ u_n(X)^* & M_\varepsilon \end{pmatrix} \in M_2(M_n(B))^+ .$$

This holds $\forall \varepsilon > 0 \Rightarrow \begin{pmatrix} \Lambda & u_n(X) \\ u_n(X)^* & M \end{pmatrix} \in M_2(M_n(B))^+$.

Thus, $u : \mathcal{V}_X \to M_2(B)$ is completely positive. Q.E.D.