Completely Positive Maps and Applications Part I
Groundwork for Operator Algebras Lecture Series 2021

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Let A, B be two C^*-algebras. A linear map $u: A \to B$ induces a linear map on the complexified C^*-algebras: for each $n \in \mathbb{N}$,

$$u_n : M_n(A) \to M_n(B)$$

$$\sum e_i e_j^* \otimes a_{ij} \mapsto \sum e_i e_j^* \otimes u(a_{ij})$$

We call u_n the n^{th} amplification of the map u.

Recall. $M_n(A)$ is a C^*-algebra being a closed $*$-subalgebra of $M_n(B(H)) \simeq B(\ell_2^n(H))$.

In particular, we know since A has a closed positive cone then $M_n(A)$ has a closed positive cone $\forall n \in \mathbb{N}$.

Similar case for B, $M_n(B)$.

$u: A \to B$ is positive if $u(A^+) \subset B^+$.

We say u is n-positive if u_n is positive and we say u is completely positive if u_n is positive $\forall n \in \mathbb{N}$.

Note if u is a $*$-homomorphism then it is automatically completely positive.

Given a C^*-algebra A, then $M_n(A)$ has a C^*-norm

$$\alpha_n : M_n(A) \to [0, \infty)$$

and we can talk about boundedness of the amplifications

It's a map $u: A \to B$. We say $u: A \to B$ is n-bounded if $\alpha_n(u_n) < \infty$. We say u is completely bounded if $\|u\|_{c.b.} := \sup_{n \in \mathbb{N}} \alpha_n(u_n) < \infty$.

Once again, if $u: A \to B$ is a $*$-homomorphism, then since $u_n: M_n(A) \to M_n(B)$ is also a $*$-homomorphism for each $n \in \mathbb{N}$ implies $\|u_n\|_{c.b.} \leq 1$ i.e. u_n is completely contractive.

Why do we care? What is the motivation to ever consider such objects?

For me, the tensor theory.
Given linear maps between vector spaces
$$u_1: V_1 \to W_1 \quad , \quad u_2: V_2 \to W_2$$
then we know there exists a unique linear map
$$u_1 \otimes u_2: V_1 \otimes V_2 \to W_1 \otimes W_2.$$
We expect to be able to take simple tensors of morphisms and induce a morphism on the tensor product (in our category)

In particular, there exist continuous maps such that their induced map on, say the minimal C^*-algebra tensor product is not continuous.
Thus, given two continuous maps on C^*-algebras, we cannot always guarantee that the induced tensor map is continuous.

Proposition: Let $u: A \to A$ be a unital positive isometry. It may happen that $u \otimes Id_A : A \otimes A \to A \otimes A$ is unbounded.

Not good, since we should be able to take continuous maps on our objects and induce a continuous map on the C^*-algebra tensor product.

For me, this is precisely the reason to consider the matricial structure of our objects.

If we require our maps to be completely positive, then the simple tensor is completely positive & hence continuous on both the minimal and maximal C^*-algebra tensor product.

Theorem: Let $u_1 : A_1 \to B_1$, $u_2 : A_2 \to B_2$ be completely positive maps. Then

$$u_1 \otimes u_2 : A_1 \otimes_\alpha A_2 \to B_1 \otimes_\alpha B_2$$

is completely positive (and hence continuous), $\alpha \in \{\min, \max\}$.

Pf Sketch: The minimal case:

$u_i : A_i \to B(H_i)$ w/ minimal Stinespring (V_i, π_i, K_i), $u_i(\cdot) = V_i^* \pi_i(\cdot) V_i$,

$\pi_i : A \to B(K_i)$ $*$-representation

$V_i : H_i \to K_i$ bounded linear operator.

$\|u_i\| = \|V\|^2$ and $\overline{\pi_i(A) V_i H_i}^{\|\cdot\|} = K_i$

$\Rightarrow$ we consider $V_1^* \pi_1(\cdot) V_1 \otimes V_2^* \pi_2(\cdot) V_2$

We use fact that $\pi_1 : A_1 \to B(H_1)$ and $\pi_2 : A_2 \to B(H_2)$ induce a $*$-homomorphism (unique) on $\otimes_{min}$.

Proposition: Let $\pi_A : A \to B(H_A)$ $\pi_B : B \to B(H_B)$ be representation $\Rightarrow \pi_A \otimes \pi_B$ extends to a unique representation on $A \otimes_{min} B \to B(H)$, $H := H_A \otimes_2 H_B$ such that $\pi_A \otimes \pi_B (a \otimes b) = \pi_A(a) \otimes \pi_B(b)$.

$$\pi_1 \otimes \pi_2 : A_1 \otimes_{min} A_2 \to B(H_1 \otimes_2 H_2).$$

We consider the map

$$u_1 \otimes u_2 := (V_1 \otimes V_2)^* \pi_1 \otimes \pi_2 (\cdot) (V_1 \otimes V_2)$$

$\rightsquigarrow$ On simple tensors,

$$u_1 \otimes u_2 (a_1 \otimes a_2) = (V_1 \otimes V_2)^* \pi_1 \otimes \pi_2 (a_1 \otimes a_2) V_1 \otimes V_2$$

$$= V_1^* \pi_1(a_1) V_1 \otimes V_2^* \pi_2(a_2) V_2$$

$$= u_1(a_1) \otimes u_2(a_2)$$

$\Rightarrow u_1 \otimes u_2$ is a genuine c.p. extension to $\otimes_{min}$. w/ range in $B_1 \otimes_{min} B_2$.

Furthermore,

$$\|u_1 \otimes u_2\|_{cb} = \|(V_1 \otimes V_2)^* \pi_1 \otimes \pi_2 (V_1 \otimes V_2)\|$$

$$= \|V_1 \otimes V_2\|^2$$

$$\leq \|V_1\|^2 \|V_2\|^2$$

$$= \|u_1\| \, \|u_2\|$$

Q.E.D.

Our objects:

The concrete scenario: Given a Hilbert space H then a concrete operator system is a self-adjoint unital subspace of $B(H)$.

Let $\mathcal{C} := \{\mathcal{C}_n\}_n$, $\mathcal{C}_n := M_n(V) \cap M_n(B(H))^+$

then we have $\mathbb{R}^+ \mathcal{C}_n \subset \mathcal{C}_n \quad \forall n \in \mathbb{N}$

$$\mathcal{C}_n + \mathcal{C}_n \subset \mathcal{C}_n \quad \forall n \in \mathbb{N}.$$

Let $x \in M_n(X) \cap M_n(B(H))^+$, $x : \ell_2^n(H) \to \ell_2^n(H)$.

positive operator.

$M_n(B(H)) \cong B(\ell_2^n(H))$ is a C^*-algebra

and in particular, if $a \in M_{n,k}$ we know $a^* x a$ is a positive operator on $M_k(B(H))$.

$\Rightarrow a^* \mathcal{C}_n a \subset \mathcal{C}_k$ for $a \in M_{n,k}$, $n, k \in \mathbb{N}$.

$\Rightarrow$ these three properties determine the properties of the positive cone.

Towards abstractness: Let X be a $*$-vector space and let $\mathcal{C}$ be a metrix ordering.

i.e., $\mathcal{C}+\mathcal{C} \subset \mathcal{C}$

$\mathbb{R}^+ \mathcal{C} \subset \mathcal{C}$

$a^* \mathcal{C}_n a \subset \mathcal{C}_k \quad \forall a \in M_{n,k}, \ n,k \in \mathbb{N}$

If $\mathcal{C} \cap -\mathcal{C} = \{0\}$ then we say $\mathcal{C}$ is a proper metrix ordering.

$\mathcal{C}_1$ induces partial ordering on X_h w/ $x_1 \leq x_2$ $\Leftrightarrow$ $x_2 - x_1 \in \mathcal{C}_1$.

Let $e \in \mathcal{C}_1$, such that for any $x \in X_h$ $\exists r>0$ s.t.

$$-re \leq x \leq er.$$

We call e an order unit. If $\forall \varepsilon>0$ $\varepsilon e + x \in \mathcal{C}_1$ $\Rightarrow x \in \mathcal{C}_1$ then we say $\mathcal{C}_1$ is Archimedean closed.

If $I_n \otimes e$ is an order unit in $M_n(X)$ $\forall n \in \mathbb{N}$ then we call e a metrix order unit. If $\overline{\mathcal{C}_n}^{e} = \mathcal{C}_n$ $\forall n$ then we say e is an Archimedean metrix order unit.

Definition: Let $(X, \mathcal{C})$ be a proper matrix ordered $*$-vector space and let e be an Archimedean matrix order unit for the pair $(X, \mathcal{C})$. Then the triple is called an *abstract operator system.*

Let $u: X \to Y$ be linear map between operator systems. Then if u is a linear isomorphism such that both u and u^{-1} are completely positive, we call u a *complete order isomorphism.*

If $u: X \to u(X)$ is a complete order isomorphism then we sometimes say u is a *complete order embedding.*

Theorem: Choi-Effros '77: Given any abstract operator system $(X, \mathcal{C}, e)$ then there exists a Hilbert space H and a concrete operator system $\tilde{X} \subset B(H)$ such that

$$X \simeq \tilde{X}$$

" X is completely order isomorphic to $\tilde{X}$.

Thus, just as is the case for C^*-algebras, there is a 1-1 correspondence between concrete and abstract operator systems.

Examples: • M_n $\forall n \in \mathbb{N}$; $B(H)$
• Comm unital C^*-algebra.

• span $\{u_i : -n \leq i \leq n, \; u_0 = I, \; u_{-i} = u_i^*\}$
$u_i = \pi_U(g_i)$ where $\mathbb{F}_n := \ast_1^n g_i$
$\pi_F : \mathbb{F}_n \to B(H_U)$

Here we have let $\pi_F : \mathbb{F}_n \to B(H_U)$ be the universal unitary representation of $\mathbb{F}_n$, i.e. our system above is the operator system corresponding to the universal unitary generators of $C^*(\mathbb{F}_n)$.

We have already established any C^*-algebra comes equipped w/ a family of norms α.

In particular, let $\mathcal{E} \subset B(H)$ be a subspace (closed). We know

• $x \in M_n(\mathcal{E})$, $y \in M_m(\mathcal{E})$, $\alpha_{n+m}(x \oplus y) = \max\{\alpha_n(x), \alpha_m(y)\}$

• $a, b \in M_n$, $\alpha_n(a x b) \leq \|a\| \|b\| \alpha_n(x)$.

Definition: An operator space is a pair $(\mathcal{E}, \alpha)$ where $\mathcal{E}$ is a vector space and $\alpha = (\alpha_n)$ is a sequence of norms satisfying Ruan's axioms:

R1: $x \in M_n(\mathcal{E})$, $y \in M_m(\mathcal{E})$, $\alpha_{n+m}(x \oplus y) = \max\{\alpha_n(x), \alpha_m(y)\}$

R2: $a, b \in M_n$. $\alpha_n(a x b) \leq \|a\| \|b\| \alpha_n(x)$.

$M_n \hookrightarrow M_{n+1}$ compatible s.t. $\alpha_{n+1}|_{M_n} = \alpha_n$.
$\Downarrow$

Equivalently, if we let $\mathcal{K}_0 := \bigcup M_n$ equipped w/ the norm coming from $\mathcal{B}(\ell_2)$ so the completion may be identified w/ $\mathcal{K}$. Then if $\mathcal{K}_0[\mathcal{E}] := \bigcup_n M_n(\mathcal{E}) \cong \mathcal{K}_0 \otimes \mathcal{E}$

the compact operators

define the norm $\alpha : \mathcal{K}_0[\mathcal{E}] \to [0, \infty)$, $\alpha(x) = \lim_{n \to \infty} \alpha_n(x)$

whose limit is stationary since if $x \in M_n(\mathcal{E})$ then $\alpha_m(x) = \alpha_n(x)$ $\forall m \geq n$.

Let $\mathcal{K} \otimes_\alpha \mathcal{E}$ be the completion of $\mathcal{K}_0 \otimes \mathcal{E}$ w/ α.

Then $(\mathcal{E}, \alpha)$ is an operator space if given any finite sequences $(a_i), (b_i) \subset \mathcal{K}_0$ and a finite sequence $(x_i) \subset \mathcal{K}_0[\mathcal{E}]$ then

$$\alpha\left(\sum a_i x b_i\right) \leq \left\|\sum a_i a_i^*\right\|^{1/2} \sup_i \alpha(x_i) \left\|\sum b_i^* b_i\right\|^{1/2}.$$

Theorem: Ruan '87: Given any abstract operator space $(\mathcal{E}, \alpha)$ then there exists a Hilbert space H and a concrete operator space $\tilde{\mathcal{E}} \subset \mathcal{B}(H)$ such that

$$\mathcal{E} \cong \tilde{\mathcal{E}}$$

Here, we identify operator spaces if $\exists$ a linear isomorphism $u : E \to \hat{E}$ such that u is a complete isometry.

Thus, as we saw for operator systems, there is a 1-1 correspondence between abstract and concrete operator spaces.

The Trick: Let $A \subset \tilde{A}$ be an inclusion of C^*-algebras w/ B a C^*-algebra. Let α be a C^*-algebra tensor product on $\tilde{A} \otimes B$

i.e. $\|x^* x\|_\alpha^2 = \|x\|_\alpha$, $\|xy\|_\alpha \leq \|x\|_\alpha \|y\|_\alpha$

$\|x^*\|_\alpha = \|x\|_\alpha \quad \forall x, y \in A \otimes B$

$A \otimes_\alpha B$ = completion of $(A \otimes B, \alpha)$

and let α_{rel} be the C^*-norm on $A \otimes B$ gotten by restricting

$$\alpha|_{A \otimes B} : A \otimes B \to [0, \infty).$$

If $\pi_A : A \to B(H)$, $\pi_B : B \to B(H)$ are $*$-homomorphisms w/ commuting ranges and if the product $*$-homomorphism is continuous wrt α_{rel} then $\exists$ a c.c.p. extension $\kappa_A : \hat{A} \to \pi_B(B)'$ of π_A.

Universality: Given a $*$-homomorphism $\pi : A_1 \odot A_2 \to C$ then there exists a unique extension to a $*$-homomorphism

$$\tilde{\pi} : A_1 \otimes_{max} A_2 \to C.$$

In particular, any pair of $*$-homomorphisms $\pi_1 : A_1 \to C$ $\pi_2 : A_2 \to C$ w/ commuting ranges induces a unique $*$-homomorphism

$$\pi_1 \times \pi_2 : A_1 \otimes_{max} A_2 \to C.$$

Restrictions: Given a nondegenerate $*$-homomorphism

$$\pi : A_1 \odot A_2 \to B(H)$$

then there exists nondegenerate $*$-homomorphisms

$$\pi_1 : A_1 \to B(H), \quad \pi_2 : A_2 \to B(H)$$

w/ commuting ranges s.t. $\pi = \pi_1 \times \pi_2$.

"For each nondegenerate representation $\pi : A \to B(H)$ there exists a unique normal extension $\tilde{\pi} : A^{**} \to B(H)$ such that

$$\tilde{\pi}\big|_A = \pi \quad \text{and} \quad \tilde{\pi}(A^{**}) = \pi(A)''."$$

$$\overline{\pi(\tilde{\pi})}^{w(A^{**}, A^{*})} = \pi(\tilde{\pi}) \Rightarrow \text{it is a vNa.}$$