Lecture II: Completely Positive Maps and Applications

Lance's Weak Expectation Property and Kirchberg's Conjecture:

Proposition: Let $A \subset \hat{A}$ be an inclusion of C^*-algebras. TFAE.

(i) there exists a ccp map $u: \hat{A} \to A^{**}$ such that $\hat{u}(a) = a \ \forall a \in A$

(ii) For every $*$-homomorphism $\pi: A \to B(H)$ there exists a c.c.p. map $u: \hat{A} \to \pi(A)''$ such that $u(a) = \pi(a)$ $\forall a \in A$.

(iii) For every C^*-algebra C we have an inclusion of C^*-algebras

$$A \otimes_{max} C \subset \hat{A} \otimes_{max} B$$

Pf: (i) $\Leftrightarrow$ (ii)

Let $u: \hat{A} \to A^{**}$ be a ccp map such that $u(a) = a \ \forall a \in A$. and let $\pi: A \to B(H)$ be a representation of A.
Consider the unique $w(A^{**}, A^*)$-continuous extension of π,

$$\tilde{\pi}: A^{**} \to B(H)$$

which satisfies $\tilde{\pi}(a) = \pi(a) \ \forall a \in A$, and such that

$$\tilde{\pi}(A^{**}) = \pi(A)''.$$

Thus, we consider $\tilde{\pi} u : \hat{A} \to \pi(A)''$ which satisfies $\tilde{\pi} u(a) = \pi(a)$ $\forall a \in A$ and $\tilde{\pi} u$ c.c.p.

Conversely if $\forall$ representations $\pi : A \to B(H)$ $\exists$ c.c.p. $u : \tilde{A} \to \pi(A)''$ s.t. $u(a) = \pi(a)$ $\forall a \in A$, then we consider the universal representation,

$$\pi_U : A \to B(H_U) \Rightarrow \exists \; u : \tilde{A} \to \pi_U(A)'' = A^{**}$$

c.c.p. w/ $u(a) = \pi_U(a) = \iota(a) = a \quad \forall a \in A.$

Assume (iii)

(iii) $\Leftrightarrow$ (ii) : Let $\pi : A \to B(H)$ be a representation, and let $C := \pi(A)'$

$$\left.\begin{array}{l} \pi : A \to B(H) \\ \sigma : \pi(A)' \hookrightarrow B(H) \end{array}\right\} \text{representations}$$

Then π, σ have commuting ranges.

$\pi \times \sigma : A \odot C \to B(H)$ is a $*$-homomorphism which extends uniquely to $A \otimes_{max} C$. By the Trick $\exists$ a c.c.p. map

Universal property of $\otimes_{max}$

$u : \tilde{A} \to (\pi(A)')' = \pi(A)''$ extending π i.e. $u(a) = \pi(a)$ for all $a \in A$.

Assume (ii): We claim $A \otimes_{max} B \subset \tilde{A} \otimes_{max} B$.

By universality of $\otimes_{max}$ we have a canonical $*$-homomorphism

$$\pi : A \otimes_{max} B \to \tilde{A} \otimes_{max} B.$$

We claim π is injective. Let $\sigma : A \otimes_{max} B \to B(H)$ be a faithful representation, w/ restriction $*$-homomorphisms

$$\sigma_A : A \to B(H)$$
$$\sigma_B : B \to B(H)$$

existence of restrictions.

s.t. $\sigma = \sigma_A \times \sigma_B$.

Then σ_A and σ_B have commuting ranges. In particular $\sigma_B(B) \subset \sigma_A(A)'$. Consider the inclusion $*$-homomorphisms

$$\sigma_A(A)'' \hookrightarrow B(H)$$
$$\sigma_B(B) \hookrightarrow B(H)$$

By universality the product $*$-homomorphism extends uniquely

$$\sigma_A(A)'' \otimes_{max} \sigma_B(B) \to B(H)$$

By (ii) let $u : \hat{A} \to \sigma_A(A)''$ be a c.c.p. map s.t.
$u(a) = \sigma_A(a) \quad \forall a \in A$.

Since both u and σ_B are completely positive implies

$$u \otimes \sigma_2 : \hat{A} \otimes_{max} B \to \sigma_1(A)'' \otimes_{max} \sigma_2(B) \text{ is continuous.}$$

Recall, this follows from lecture I.

$$\begin{tikzcd}
\hat{A} \otimes_{max} B \arrow[r, "u \otimes \sigma_2"] \arrow[dr, phantom, "\#"] & \sigma_1(A)'' \otimes_{max} \sigma_2(B) \arrow[d, "\pi_1 \times \pi_2"] \\
A \otimes_{max} B \arrow[u] \arrow[r, "\sigma"] & B(H)
\end{tikzcd}$$

Since σ is faithful implies the vertical map on the left is injective.

Thus (ii) $\Leftrightarrow$ (iii).

Q.E.D.

Definition: Let A be a C^*-algebra, and let A^{**} be its bidual. Then A has the weak expectation property if there exists unital completely positive extension $\hat{\iota}: B(H) \to A^{**}$ extending the canonical embedding of $A \hookrightarrow A^{**}$. i.e. $\hat{\iota}(a) = a \; \forall a \in A$, $A \subset A^{**}$

Equivalently, A has the WEP if and only if for every inclusion $A \subset \hat{A}$ and C^*-algebra B,

$$A \otimes_{max} B \subset \hat{A} \otimes_{max} B.$$

Kirchberg's Conjecture: $C^*(F_\infty)$ has Lance's WEP.

It follows that $C^*(F_\infty)$ is a test object for a C^*-algebra having WEP. In particular, a C^*-algebra A has the WEP if and only if

$$A \otimes_{min} C^*(F_\infty) = A \otimes_{max} C^*(F_\infty).$$

Kirchberg's Conjecture false?

$MIP^* = RE$: Zhengfeng Ji, Anand Natarajan, Thomas Vidick, John Wright, Henry Yuen

We make some remarks regarding the road from Connes '76 to Kirchberg '93 to Junge et al. '11 to $MIP^* = RE$ '20.

In his 1976 Annals paper Alain Connes commented that every II_1 factor should embed in an ultrapower of the hyperfinite II_1 factor R.

Recall, M is type II if it is semifinite and there exists no nonzero abelian projection.

II_1 = finite w/ no nonzero abelian projection.

In 1993 Eberhard Kirchberg discovered an equivalence to Connes' remark. In particular, Connes' remark was indeed true if and only if $C^*(F_\infty)$ has Lance's WEP.

Fix $n, k \in \mathbb{N}$. We say $p := \{p(a,b|x,y) : x,y \in [k], a,b \in [n]\}$ is a correlation if $p(a,b|x,y) \geq 0$ $\forall a,b,x,y$ and $\sum_{a,b} p(a,b|x,y) = 1$ for each $x, y \in [k]$.

Tsirelson '80: Does $\underbrace{\overline{C_q(n,k)}}_{\text{closure of the quantum correlations}} = \underbrace{C_{qc}(n,k)}_{\text{quantum-commuting correlations}}$ for all $n,k \in \mathbb{N}$.

$p \in C_{qc}(n,k) \iff$ there exists a Hilbert space H, $\eta \in S_H$, PVMs $\{E_{xa}\}_{a=1}^n$, $\{F_{yb}\}_{b=1}^n$ for each $x, y \in [k]$ such that $E_{xa}F_{yb} = F_{yb}E_{xa}$ for each x,y,a,b and

$$p(a,b|x,y) = \langle \eta | E_{xa} F_{yb} \eta \rangle$$

for each $x,y \in [k]$, $a,b \in [n]$.

If H is finite-dimensional then $p \in C_q(n,k)$.

What does this have to do w/ operator systems?

Theorem: Araiza-Russell '20: Let $n, k \in \mathbb{N}$. Then $p \in C_{qc}(n,k)$ if and only if there exists an operator system $\mathcal{X}$ w/ generators $\{ Q(ab|xy) : a,b \in [n], x,y \in [k] \} \subset \mathcal{X}$, such that $\forall x,y$, $\sum_{a,b} Q(ab|xy) = e$, $\sum_{b} Q(ab|xy) =: E(a|x)$,

$\sum_{a} Q(ab|xy) =: F(b|y)$ are well-defined, and each

$Q(ab|xy)$ is a projection in the C^*-envelope of $\mathcal{X}$, and there exists a state $\varphi : \mathcal{X} \to \mathbb{C}$ such that

$$p(ab|xy) = \varphi(Q(ab|xy)) \quad \forall a,b,x,y.$$

Junge et al '11: Tsirelson's Problem is true iff Kirchberg's Conjecture is true.

Used operator system/ operator space theory to prove equivalence.

Ji et al. '20: $\overline{C_q(n,k)} \neq C_{qc}(n,k)$ for some very large values of n,k.

In lecture I mentioned that if F is any free group that $C^*(F)$ has the LLP. We discussed how $C^*(F_\infty)$ is a test object for WEP. It follows $B(H)$ is a test object for the LLP.

Theorem: (Kirchberg): Let F be any free group and let $C^*(F)$ be the universal group C^*-algebra. Then

$$C^*(F) \otimes_{min} B(H) = C^*(F) \otimes_{max} B(H).$$

Theorem: Pisier: Let A_1, A_2 be unital C^*-algebras and let $\{u_i\}$ (resp. $\{v_j\}$) be the unitary generators of A_1 (resp. A_2). Let

$$\mathcal{E}_1 := \text{span}\{u_i : i \in I\}$$
$$\mathcal{E}_2 := \text{span}\{v_j : j \in J\}$$

Then the following are equivalent:

(i) The inclusion $i: \mathcal{E}_1 \otimes_{min} \mathcal{E}_2 \hookrightarrow A_1 \otimes_{max} A_2$ is completely isometric.

(ii) $A_1 \otimes_{min} A_2 = A_1 \otimes_{max} A_2$.

Good books for more information on operator spaces and operator systems:

(1) Introduction to Operator Space Theory
- Gilles Pisier

(2) Completely Bounded Maps and Operator Algebras
- Vern Paulsen

(3) Operator Spaces
- Edward Effros & Zhong-Jin Ruan