The Baum-Connes Conjecture.

Let G be a countable (discrete) group. The assembly map

$$\mu_G : \underbrace{K_i^G(\underline{E}G)}_{\text{G-equivariant K-homology of the space } \underline{E}G} \longrightarrow \underbrace{K_i(C_r^*(G))}_{\text{K-theory of the reduced group } C^*\text{-algebra of } G} \quad (\text{for } i=0,1)$$

is an isomorphism.

G-equivariant K-homology of the space $\underline{E}G$ - space classifying G-proper actions of G

K-theory of the reduced group C^*-algebra of G.

Known to be true for large classes of groups including: (not exhaustive)

- all semisimple Lie groups (Wassermann, Lafforgue)
- Connected Lie groups (Lafforgue)
- all groups satisfying Haagerup property [Higson-Kasparov].
- amenable groups (Higson-Kasparov)
- hyperbolic groups [Lafforgue].

Open:

· Surjectivity of Baum-Connes unknown for $SL(n,\mathbb{Z})$, $n \geq 3$.

Implications - in topology & geometry.

- if G is torsion free, surjectivity of Baum-Connes conjecture implies $C_r^*(G)$ has no idempotents (other than 0 or 1) [Kaplansky-Kadison conjecture]

LHS of BCC: $K_i^G(\underline{E}G)$.

let us restrict to G being discrete and torsionfree, then

$K_i^G(\underline{E}G) \cong K_i(BG)$ where BG is roughly the "path connected ~~top.~~ space X such that $\underbrace{\pi_1(X)}_{\text{fundamental group of } X} = G$"

$B\mathbb{Z} = S^1$. ← unit circle.

RHS of BCC: $K_i(C^*_r(G))$. Recall $C^*_r(G) = \overline{\lambda(\mathbb{C}G)}^{\|\cdot\|}$.

K-theory is a sequence of abelian groups that forms an invariant of C^*-algebras. (in fact "generalised homology theory")

Some properties

- $K_i(A) \cong K_{i+2}(A)$ Bott periodicity. $K_1(A) \cong K_3(A) \cong \ldots$ $K_0(A) \cong K_2(A) \cong \ldots$
- half exact, homotopy invariant, stable.

Ex: $K_i^G(\underline{E}\mathbb{Z}) \cong K_i(B\mathbb{Z}) \cong K_i(S^1)$ ($\mathbb{Z}$ torsion free discrete)

$K_i(C^*_r(\mathbb{Z})) \cong K_i(C(S^1))$ ≈ $C^*_r(\mathbb{Z}) \cong C(S^1)$

↘ $K_i(C(X)) \cong K_i(X)$.

There are results where LHS & RHS are computed and the isomorphism explicitly shown:

Flores-Pooya-Valette 17 - lamplighter groups of finite groups.

Pooya-Valette 18 - solvable Baumslag-Solitar groups

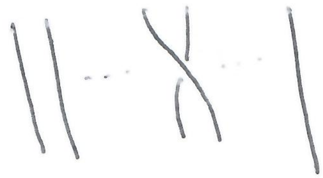
Pooya 19 - finite wreath products of free groups.

Baum-Connes known ~~true~~ before for all Braid groups + pure Braid groups — Chabert Echterhoff — Oyono-Oyono — Schick

(in a collaboration, we show this (for pure Braid groups) by computing both sides).

Braid groups (on n strands)

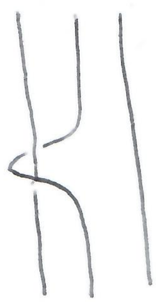
$B_n = \langle \sigma_1, \sigma_2, \dots, \sigma_{n-1} \mid \sigma_j \sigma_i = \sigma_i \sigma_j, \ |i-j|>1, \ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for all } i \rangle$

σ_i  

each determines a permutation.

$P_n = \mathrm{Ker}\,(\pi : B_n \to S_n)$ $\quad \pi(\sigma_i) = (i, i+1)$

Consider P_3, one generator is σ_1^2

σ_1^2 

Turns out that

$$P_n = F_{n-1} \rtimes P_{n-1} \simeq F_{n-1} \rtimes F_{n-2} \rtimes \cdots \rtimes F_2 \rtimes F_1$$

$P_3 = F_2 \rtimes \mathbb{Z} \cong F_2 \times \mathbb{Z}$ (due to center)

$$P_4 \cong F_3 \rtimes F_2 \rtimes \mathbb{Z}$$
$$\cong (F_3 \rtimes F_2) \times \mathbb{Z}$$

This iterated semidirect product allows the right hand side to be computed.

↓ free groups specifically.

We will need both Pimsner-Voiculescu six term exact sequences & the Künneth Formula.

<u>Pimsner-Voiculescu six term exact sequences</u>

Let D, be a C^*-algebra endowed with an action of the free group F_n by $\phi : F_n \to Aut(D)$. Then we have two six term exact sequences:

(a)
$$\begin{tikzcd}
K_0(D) \arrow[r] & K_0(D \rtimes_{\phi',r} F_{n-1}) \arrow[r] & K_0(D \rtimes_{\phi,r} F_n) \arrow[d] \\
K_1(D \rtimes_{\phi,r} F_n) \arrow[u] & K_1(D \rtimes_{\phi',r} F_{n-1}) \arrow[l] & K_1(D) \arrow[l]
\end{tikzcd}$$

and

(b)
$$\begin{tikzcd}
(K_0(D))^n \arrow[r] & K_0(D) \arrow[r] & K_0(D \rtimes_r F_n) \arrow[d] \\
K_1(D \rtimes_r F_n) \arrow[u] & K_1(D) \arrow[l] & (K_1(D))^n \arrow[l]
\end{tikzcd}$$

<u>Künneth Formula</u> Let A, B be C^*-algebras with A nuclear, and such that $K_*(A)$ or $K_*(B)$ is torsion-free, then

$$K_0(A \otimes B) \cong K_0(A) \otimes K_0(B) \oplus K_1(A) \otimes K_1(B)$$

$$K_1(A \otimes B) \cong K_0(A) \otimes K_1(B) \oplus K_1(A) \otimes K_0(B).$$

<u>Fact</u>: $C^*_r(P_4) \cong C^*_r(F_3 \rtimes F_2 \times \mathbb{Z})$

$$\cong C^*_r(F_3 \rtimes F_2) \otimes C^*_r(\mathbb{Z})$$

$$\cong (C^*_r(F_3) \rtimes_r F_2) \otimes C^*_r(\mathbb{Z}).$$

Then we need:

$$K_i(C^*_r(P_4)) \cong K_i(C^*_r(F_3 \rtimes_r F_2) \otimes C^*_r(\mathbb{Z})) \text{ and so}$$

need:

$$K_i(C^*_r(F_3) \rtimes_r F_2) \quad \text{and} \quad K_i(C^*_r(\mathbb{Z}))$$

<u>Note</u>: $K_i(C^*_r(\mathbb{Z})) \cong \mathbb{Z}$ for $i=0,1$. and $K_i(C^*_r(F_n)) \cong \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}^n & i=1 \end{cases}$.

<u>Aim</u>: Compute $K_i(C^*_r(F_3) \rtimes_r F_2)$, let $A = C^*_r(F_3) \rtimes_r F_2$.

Using Pimsner–Voiculescu (a), with $D := C^*_r(F_3)$, we have

$$\begin{tikzcd}
K_0(C^*_r(F_3)) \arrow[r] & K_0(C^*_r(F_3) \rtimes_r F_1) \arrow[r] & K_0(A) \arrow[d] \\
K_1(A) \arrow[u] & K_1(C^*_r(F_3) \rtimes_r F_1) \arrow[l] & K_1(C^*_r(F_3)) \arrow[l]
\end{tikzcd}$$

We need to find $K_i(C^*_r(F_3) \rtimes_r F_1)$, to do this we use Pimsner–Voiculescu (b).

Let $C^*_r(F_3) \rtimes_r F_1 = B$.

Pimsner–Voiculescu (b): $n=1$

$$\begin{tikzcd}
\overset{\cong \mathbb{Z}}{K_0(C^*_r(F_3))} \arrow[r, "\star"] & \overset{\cong \mathbb{Z}}{K_0(C^*_r(F_3))} \arrow[r] & K_0(B_1) \arrow[d] \\
K_1(B_1) \arrow[u] & \underset{\mathbb{Z}^3}{\underset{\cong}{K_1(C^*_r(F_3))}} \arrow[l] & \underset{\mathbb{Z}^3}{\underset{\cong}{K_1(C^*_r(F_3))}} \arrow[l, "\star"]
\end{tikzcd}$$

$\star$ maps are zero maps by relations/generators.

Due to zeromaps, we obtain two short exact sequences :-

$$0 \to \mathbb{Z} \to K_0(B_1) \to \mathbb{Z}^3 \to 0$$

$$0 \to \mathbb{Z}^3 \to K_1(B_1) \to \mathbb{Z} \to 0$$

So $K_i(B_1) = \mathbb{Z} \oplus \mathbb{Z}^3 \cong \mathbb{Z}^4$.

Now replacing in (a), we obtain

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\star"] & \mathbb{Z}^4 \arrow[r] & K_0(A) \arrow[d] \\
K_1(A) \arrow[u] & \mathbb{Z}^4 \arrow[l] & \mathbb{Z}^3 \arrow[l, "\star"]
\end{tikzcd}$$

$\star$ zero again.

and get

$$0 \to \mathbb{Z}^4 \to K_0(A) \to \mathbb{Z}^3 \to 0$$

$$0 \to \mathbb{Z}^4 \to K_1(A) \to \mathbb{Z} \to 0$$

and so

$$K_0(A) \cong \mathbb{Z}^7 \qquad K_1(A) \cong \mathbb{Z}^5.$$

that is

$$K_0(C^*_r(F_3) \rtimes_r F_2) \cong \mathbb{Z}^7 \qquad K_1(C^*_r(F_3) \rtimes_r F_2) \cong \mathbb{Z}^5.$$

Now Künneth, let $A = C_r^*(F_3) \rtimes_r F_2$ and $B = C^*(\mathbb{Z})$

$K_0(A) = \mathbb{Z}^7$ $\qquad K_0(B) = \mathbb{Z}$

$K_1(A) = \mathbb{Z}^5$ $\qquad K_1(B) = \mathbb{Z}$

By Künneth:

$K_0(A \otimes B) \cong \mathbb{Z}^7 \otimes \mathbb{Z} \oplus \mathbb{Z}^5 \otimes \mathbb{Z} \cong \mathbb{Z}^7 \oplus \mathbb{Z}^5 \cong \mathbb{Z}^{12}$

$K_1(A \otimes B) \cong \mathbb{Z}^7 \otimes \mathbb{Z} \oplus \mathbb{Z}^5 \otimes \mathbb{Z} \cong \mathbb{Z}^{12}$.

<u>Theorem</u>. (joint with Sara Azzali, Maria-Paula Gomez Aparicio, Lauren Ruth, Hang Wang)

$K_i(C_r^*(P_n)) = \mathbb{Z}^{n!/2}$.

<u>Remark</u>: Such computations for Braid groups are not so easy due to torsion possibilities.

For example, Li, Omland-Spielberg (21) show that

$K_0(C_r^*(B_4)) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

$K_1(C_r^*(B_4)) \cong \mathbb{Z}$.