GOALS Free Prob Lecture II

Connections between free & classical probability.

<u>Thm</u>: Wigner ('55): For each $n\in\mathbb{N}$, let A_N be a self-adjoint $N\times N$ random matrix with entries A_{ij} satisfying

- $\overline{A_{ji}} = A_{ij}$
- If $i\neq j$, $Re(A_{ij}), Im(A_{ij})$ are independent Gaussians of variance $\frac{1}{2n}$
- A_{ii} is a real Gaussian of variance $\frac{1}{N}$
- for $i\leq j$, entries A_{ij} are classically independent

Then A_N converges to the semicircular law of variance 1 i.e.

$\frac{1}{N}\mathbb{E}\circ Tr(A_n^k) \longrightarrow \varphi(x^k)$ where $x\in(A,\varphi)$ is semicircular of variance 1.

Voiculescu upgraded this result to "freeness"

<u>Thm</u> Voiculescu ('85): Let $\{A_N(s)\mid s\in S\}$ be a family of independent Gaussian random matrices i.e. each $A_N(s)$ has entries distributed as A_N above $\{(A_N(s))_{ij}\mid i\leq j, s\in S\}$ are independent.

<u>Then</u>: $\{A_N(s)\}_{s\in S}$ converges in distribution to a free semicircular family $\{x_s\}_{s\in S}$ i.e. $x_s\in(A,\varphi)$ free semicirc (var 1) &

$$\frac{1}{N}\mathbb{E}\circ tr(A_N(s_1)\cdots A_N(s_k)) \xrightarrow{N\to\infty} \varphi(x_{s_1}\cdots x_{s_k})$$

Furthermore $\{A_N(s_i)\}$ are "asymptotically free" from the algebra of diagonal matrices.

Comment: This shows that the free semicircular law has matrical microstates i.e. the law of $\{X_s\}_{s\in S}$ is approximated by that of matrices.
Connes Embedding (recently proven false) asks whether any generating set $\{y_i\}_{i\in I}$ in any II_1 factor has matrical microstates. No explicit counterexample has been found.

Def: A circular element in (A,φ), y, is one where $\mathrm{re}(y)$ & $\mathrm{im}(y)$ are free-semicircular of the same variance.

Cor: Suppose $\{B_N(s)\}_{s\in S}$ is a family of (not) self-adjoint complex matrices s.t.

- $\mathrm{Re}(B_N(s)_{ij})$, $\mathrm{Im}(B_N(s)_{ij})$ are independent Gaussians of variance $\frac{1}{2N}$
- $\{B_N(s)_{ij} \mid i,j,s\}$ is a classically independent family.

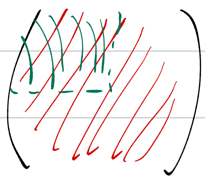
Then $\{B_N(s)\}_{s\in S}$ converges to a free circular family $\{y_s\}_{s\in S}$ of variance 1

Pf: Note $\frac{B_N(s)+B_N(s)^*}{2}$ & $\frac{B_N(s)-B_N(s)^*}{2i}$ satisfy conditions of Voiculescu's theorem.

Applications to the structure of free group factors:

Lemma: If $x\in(A,\varphi)$ is semicircular & $p\in(A,\varphi)$ is a projection that is free from x, then $\frac{pxp}{\sqrt{\varphi(p)}}$ is semicircular in $\left(pAp, \frac{1}{\varphi(p)}\varphi\right)$

Pf: Model X as a limit of A_N above & p by the matrix
$$\begin{pmatrix} \overbrace{1 \ \ddots \ 1}^{M(N)} & 0 \\ 0 & 0 \ \ddots \ 0 \end{pmatrix}$$
where $\frac{M(N)}{N}\to\varphi(p)$.

$X =$ 

Then $\frac{1}{\sqrt{\varphi(p)}}pxp$ is modeled by $M(N)\times M(N)$ self adjoint Gaussian matrix of the appropriate variance.

Lemma: Let $\{X(i,s) \mid i\in\{1,\dots,n\}, s\in S\}$ be a free semicirc family & $\{y(i,j,s) \mid 1\le i<j\le n, s\in S\}$ be a free circular family free from X's in (A,φ). In $(A\otimes M_n, \varphi\otimes tr)$ set

$$X_s = \frac{1}{\sqrt{N}}\begin{pmatrix} X(1,s) & y(1,2,s)^* & \cdots & y(1,n,s)^* \\ y(1,2,s) & \ddots & & \vdots \\ \vdots & & \ddots & \\ y(1,n,s) & \cdots & & X(n,s) \end{pmatrix}.$$

Then $\{X_s\}_{s\in S}$ is a free semicircular family.

If a is a normal element free from $\{X(i,s)\}$ & $\{y(i,j,s)\}$ then
$$\begin{pmatrix} f_1(a) & & 0 \\ & \ddots & \\ 0 & & f_n(a) \end{pmatrix}$$
is free from $\{X_s\}$

Pf: use matrix models & approx $x(i,j)$ by self-adjoint Gaussian matrices & $y(i,j,j,s)$ by Gaussian matrices.

X_S is approximated by a self-adjoint Gaussian matrix.

Model a by a diagonal matrix. Then the above matrix is also a diagonal matrix

Prop: $L(A_S)_{\frac{1}{N}} \cong L(A(N^2(|S|-1)+1))$

Sketch of pf for $N=2$ & $S=\{1,2\}$:

Model $L(A_2)$ by: $\begin{bmatrix} X_1 & Y^* \\ Y & X_2 \end{bmatrix}$ X_1, X_2, Y free, X_1 & X_2 semicircular Y circular &

$\begin{bmatrix} u & 0 \\ 0 & 2u \end{bmatrix}$ for u a Haar Unitary. free from X_1, X_2, Y. By examining spectral projections,

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ & $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are in this vNA, hence so are: $\begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ Y & 0 \end{bmatrix}$ & $\begin{bmatrix} 0 & 0 \\ 0 & X_2 \end{bmatrix}$

Polar decomp of y is $y = vb$ v Haar unitary & b diffuse. & both free

So get: $\begin{bmatrix} u & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ v & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$, $\begin{bmatrix} x_1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & x_2 \end{bmatrix}$. The generators

under e_{11} get: $\begin{bmatrix} u & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} v^* u v & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} x_1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} v^* x_2 v & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} v^* b v & 0 \\ 0 & 0 \end{bmatrix}$

which generates $L(\mathbb{F}_5)$ $(5 = 4(2-1)+1)$

Can use idea to develop $L(\mathbb{F}_t)$ $t > 1$: $\boxed{L(\mathbb{F}_t)_s = L(\mathbb{F}(\tfrac{1}{s^2}(t-1)+1))}$