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Applications of Random Matrices to the structure of vNa's: 1-Bounded entropy

Review of Hartglass' talks:

A tracial vNa is a pair (M,τ) where M is a vNa & $\tau: M \to \mathbb{C}$ is linear &

- a state, $\tau(x^* x) \geq 0$ & $\tau(1)=1$ $\forall x \in M$
- faithful, $\tau(x^* x) = 0 \iff x=0$.
- normal, $\tau|_{\{\|x\|_\infty \leq 1\}}$ is SOT-cont.
- tracial, $\tau(xy) = \tau(yx)$ $\forall x,y \in M$.

Ex: $\cdot (L^\infty(X,\mu), \int \cdot \, d\mu)$,

- $(M_n(\mathbb{C}), tr)$

$$tr(A) = \frac{1}{n}\sum_{j=1}^{n} A_{jj}$$

- $(L(G), \tau)$

$L(G) = \overline{\mathrm{Span}}^{SOT}\{\lambda(g) : g \in G\}$

$\lambda : G \to \mathcal{U}(\ell^2(G))$ left regular rep'n given by $(\lambda(x)\xi)(y) = \xi(x^{-1}y)$ $\xi \in \ell^2(G)$, $y \in G$).

$\tau(x) = \langle x\delta_e, \delta_e \rangle$.

Tracial vNa's are noncommutative probability spaces.

For $r \in \mathbb{N}$, let $\mathbb{C}\langle t_1, \ldots, t_r \rangle$ be all NC polys in r variables.

For (M,τ) tracial vNa & $x \in M_{s.a.}^r$, let

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& $x \in M_{s.a.}^r$, let

$ev_x : \mathbb{C}\langle t_1, \ldots, t_r\rangle \to \mathbb{C}$

be unique homom s.t.

$ev_x(t_j) = x_j$, set $P(x) = ev_x(P)$.

Give $\mathbb{C}\langle t_1, \ldots, t_N\rangle$ unique

$*$-structure s.t. $t_j^* = t_j$.

Then ev_x is a $*$-homom.

Define $\ell_x = ev_x \circ \tau$, call

ℓ_x the law of x.

Thm (Voiculescu Asy. Freeness Thm)

Thm). Let $X \in$ ^{N}

e r independent $\cdot$ GUE

matrices. Then, with high

probability, $\ell_X \to \ell_S$

$S = (s_1, \ldots, s_r)$ are freely

independent semicirculars.

For

$\Sigma_{d,R} = \{ \ell_x : x \in M_{s.a.}^d \}$

where $x \in M_{s.a.}^d$ ranging

over all (M,τ) tracial

vNa & $x \in M_{s.a.}^d$;

Prop: Given $\ell : \mathbb{C}\langle t_1, \ldots, t_d\rangle \to \mathbb{C}$

linear have $\ell \in \Sigma_{d,R} \iff$

- $\ell(P^*P) \geq 0$, $\ell(1) = 1$,
- $\ell(PQ) = \ell(QP)$ $\forall Q, P$

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- $\ell(PQ) = \ell(QP)$ $\forall Q, P$
- $\ell(t_{i_1} \cdots t_{i_s}) \leq R^s$ $\forall i_1, \ldots, i_s \in \{1, \ldots, d\}$

Pf of $\Leftarrow$ is GNS

Free Entropy Dimension Theory:

Measure "how many" approximations (M, τ) has and if deduce many structural results if there are many (e.g. if $M = L(\mathbb{F}_r)$)

For $x \in M^d_{s.a.}$ & $\mathcal{O}$ a wk* -nbhd of ℓ_x define

$\Gamma^{(N)}(\mathcal{O}) = \{A \in M_N(\mathbb{C})^d_{s.a.} :$ $\ell_A \in \mathcal{O}$ & $\|A_j\| \leq \|x_j\|$ $\forall j = 1, \ldots, d\}$.

For $x \in M^d_{s.a.}$ Voiculescu defined $\delta_0(x)$.

Say M is diffuse if $\forall p \in M$ proj $p \neq 0$ $\Longrightarrow \exists q \in M$ proj $q \neq 0$, $q \leq p$ $q \neq p$.

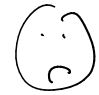
Exercise: $L^\infty(X, \mu)$ is diffuse $\Longleftrightarrow$ (X, μ) atomless.

Exercise*: $L(G)$ is diffuse if G is infinite.

Voiculescu showed $\delta_0(x) \geq 1$ if $W^*(x)$ is diffuse.

- $\delta_0(x) = d$ if $x = (x_1, \ldots, x_d)$

- $\delta_0(X) = d$ if $X=(x_1,\dots,x_d)$ are freely ind.
- $\delta_0(X)>1 \Rightarrow$ interesting structural constraints.

: Unknown if $W^*(X)=W^*(Y) \Rightarrow \delta_0(X)=\delta_0(Y)$.

Implicit in work of Jung & explicit in H. is the 1-bounded entropy $h(X)$ a modification of $\delta_0(X)$. $W^*(X)=W^*(Y) \Rightarrow h(X)=h(Y)$. So can define $h(M)=h(X)$ if $W^*(X)=M$.

Properties:
- $h(M)\in\{-\infty\}\cup[0,+\infty]$.

$h(M)\geq 0 \iff \forall X\in M_{s.a.}^d$ $\forall \mathcal{O}$ wk*-nbhd of ℓ_X, $\Gamma^{(N)}(\mathcal{O})\neq\emptyset$ $\forall N\gg 0$.

- $h(M)=0$ if M is diffuse & amenable (e.g. M is abelian or $M=L(G)$ G amenable).
- $h(M)=\infty$ if $M=W^*(X)$ & $\delta_0(X)>1$. E.g. $M=L(\mathbb{F}_r)$, $r>1$ or $M=M_1 * M_2$, M_j diffuse & $h(M_j)\geq 0$.

• $h(W^*(N_M(N))) \leq h(N)$

$N_M(N) = \{u \in \mathcal{U}(M) : uNu^* = N\}$

$\forall N \leq M$ diffuse.

• $h(M_1 \vee M_2) \leq h(M_1) + h(M_2)$, if $M_1 \cap M_2$ is diffuse

Say $N \leq M$ is regular if $W^*(N_M(N)) = M$.

Cor (Voiculescu) $L(\mathbb{F}_r)$, $r > 1$ have any diffuse, abelian subalgebras.

Cor. (Ge) For $r > 1$, $L(\mathbb{F}_r) \neq W^*(M_1, M_2)$ if $M_j \leq L(\mathbb{F}_r)$ are diffuse & commute.

Pf:

Exercise: M_j diffuse $\Rightarrow \exists A_j \leq M_j$ abelian & diffuse.

Then
$M_1 \vee A_2 \leq W^*(N_M(A_1))$
& $A_1 \vee M_2 \leq W^*(N_M(A_2))$

$\Rightarrow h(M_1 \vee A_2) \leq 0$
$h(A_1 \vee M_2) \leq 0$

& $M_1 \vee A_2 \cap (A_1 \vee M_2) \supseteq A_1 \vee A_2$,

which is diffuse

& $M_1 \cup A_2 \cap (A_1 \cup M_2) \supseteq A_1 \cup A_2$,

which is diffuse

$\Rightarrow h(M) \le h(M_1 \cup A_2) + h(A_1 \cup M_2) \le 0.$

□

Digression on the defn.

For $A \in M_n(\mathbb{C})^d_{s.a.}$, define

$$\|A\|_2^2 = \sum_{j=1}^{d} tr(A_j^* A_j)$$

Given $\Xi \subseteq \Omega \subseteq M_n(\mathbb{C})^d_{s.a.}$ & $\varepsilon > 0$,

Say Ξ is orbitally ε-dense in Ω if $\forall A \in \Omega$ $\exists B \in \Xi$ & $U \in \mathcal{U}(n)$ s.t.

$$\|A - U^* B U\|_2 < \varepsilon.$$

Let $K_\varepsilon^{orb}(\Omega)$ = minimal cardinality of an orbitally ε-dense subset of Ω. Define

$$h_\varepsilon(\mathcal{O}) = \overline{\lim_{n\to\infty}} \frac{1}{n^2} \log K_\varepsilon^{orb}(\Omega)$$

$$h(X) = \sup_{\varepsilon > 0} \inf_{\mathcal{O}} h_\varepsilon(\mathcal{O})$$

where the inf is over all wk*-neighborhoods of ℓ_X.

Exercise*: Use Voiculescu's Asymptotic Freeness Theorem to prove $h(S) = +\infty$ if prove $h(S) = +\infty$ if $S = (S_1, \ldots, S_l)$ are free semicirculars.

It will be helpful to know that $\exists\, C > 0$ st $\mathcal{U}(n)$ has an ε-dense set with respect to operator norm of cardinality $\leq \left(\frac{C}{\varepsilon}\right)^{n^2}$

Thm [H.-Jekel-Kunnawalkam Elayavalli]

If M is a Property (T) factor or if $M = L(G)$ G Property (T) $\Rightarrow$ $h(L(G)) < +\infty$.

In particular if $M_j, j = 1, 2$ are diffuse & $h(M_j) \geq 0$ $\Rightarrow M_1 * M_2 \neq N_1 \vee N_2$ where N_j are either (T) or amenable & $N_1 \cap N_2$ is diffuse.

Generalizes previous work of Voiculescu, Ge, Shen, Ge-Shen, ...

... ucintescu, Ge, Shen,
Ge-Shen, Jung-Shlyakhtenko,
Jung, Shlyakhtenko.