Non-separable metric space of non-commutative laws

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Goals

- Connect non-commutative probability theory and group theory
- Distance between non-commutative probability measures
- Metric space of non-commutative probability measures is not separable!
- Use geometric group theory result of Olshanskii.

Non-commutative probability

A *tracial von Neumann algebra* is a von Neumann algebra M with a faithful normal tracial state $\tau : M \to \mathbb{C}$.

If M is commutative, (M, τ) is isomorphic to $L^\infty(\Omega, P)$ for some probability space P, and τ corresponds to the expectation $\tau(f) = \int f \, dP$.

Tracial von Neumann algebras are *non-commutative probability spaces*.

Non-commutative probability

classical	non-commutative
$L^\infty(\Omega, P)$	A
expectation $\mathbb{E}$	trace τ
bounded random variable Z	$Z \in M$
bdd real random variable	self-adjoint $Z \in M$
random variable in $\mathbb{T}$	unitary $U \in M$
random variable in $\mathbb{T}^d$	d-tuple $(U_1, \dots, U_d)$ of unitaries

This talk: unitary d-tuples.

Agreement in law

Definition

$U = (U_1, \ldots, U_d)$ unitary d-tuple of from (M, τ_M),
$V = (V_1, \ldots, V_d)$ unitary d-tuple from (N, τ_N).
U and V *agree in law*, or $U \sim V$, if

$$\tau_M(U_{i_1}^{a_1} \ldots U_{i_n}^{a_n}) = \tau_N(V_{i_1}^{a_1} \ldots V_{i_n}^{a_n})$$

for all $n \in \mathbb{N}$, $i_1, \ldots, i_n \in \mathbb{N}$, $a_1, \ldots, a_n \in \{1, -1\}$.

Example

If $d = 3$ and U and V agree in law, then

$$\tau_M(U_1 U_2^* U_3 U_1) = \tau_N(V_1 V_2^* V_3 V_1).$$

Agreement in law

Observation

Agreement in law is an equivalence relation.

Proposition

Let $U = (U_1, \dots, U_d)$ be a unitary tuple from (M, τ_M) and let $V = (V_1, \dots, V_d)$ be a unitary tuple from (N, τ_N). TFAE:

1. U and V agree in law.
2. There is $*$-isomorphism $\phi : \mathrm{W}^*(U) \to \mathrm{W}^*(V)$ such that $\phi(U_j) = V_j$ and $\tau_N(\phi(X)) = \tau_M(X)$ for all $X \in \mathrm{W}^*(U)$.

Agreement in law classifies unitary d-tuples in tracial von Neumann algebras up to isomorphism.

So wait. . . what is a *law*?

For $U_1, \ldots, U_d \in \mathcal{U}(M)$, $\exists!$ grp. hom. $\phi_U : \mathbb{F}_d \to \mathcal{U}(M)$ s.t. $\phi_U(g_j) = U_j$.

Definition

For $U = (U_1, \ldots, U_d)$ unitary d-tuple from (M, τ_M), the *non-commutative law of U* is the function $\ell : \mathbb{F}_d \to \mathbb{C}$,

$$\ell_U(w) = \tau_m(\phi_U(w)) \text{ for all } w \in \mathbb{F}_d.$$

Example

$$d = 3, \qquad \ell_U(g_1 g_2^{-1} g_3 g_1) = \tau_M(U_1 U_2^* U_3 U_1).$$

Non-commutative laws and characters

Definition (Representation theory)

A *character* on a group G is a unital, conjugation-invariant, positive-definite function $\mu : G \to \mathbb{C}$. Here *positive-definite* means that $[\mu(g_i^* g_j)]_{i,j=1}^n \geq 0$ for all $g_1, \dots, g_n \in G$.

Proposition

$\mu : \mathbb{F}_d \to \mathbb{C}$ is character iff $\mu = \ell_U$ for some $U \in \mathcal{U}(M)$, some (M, τ).

$(\Longleftarrow)$ Trace property of τ_M implies conjugation invariance of ℓ_U, positivity of τ_M implies positive-definiteness of ℓ_U.

$(\Longrightarrow)$ If μ is a character, then μ defines an inner product on $\mathbb{C}G$. Complete to Hilbert space. Left translation of G produces unitary representation of G. Vector δ_e defines tracial state.

Wasserstein distance

Definition

Υ_d is the space of characters on $\mathbb{F}_d$.

Definition (Biane-Voiculescu 2001)

For $\mu, \nu \in \Upsilon_d$,

$$d_W(\mu, \nu) := \inf\{\|U - V\|_2 : U, V \in \mathcal{U}(M)^d, \\ \ell_U = \mu, \ell_V = \nu, \\ (M, \tau_M) \text{ tracial von Neumann algebra}\}.$$

Here $U = (U_1, \dots, U_d)$ and $V = (V_1, \dots, V_d)$ and

$$\|U - V\|_2 = \left(\sum_{j=1}^{d} \tau_M((U_j - V_j)^*(U_j - V_j)) \right)^{1/2}.$$

Wasserstein distance

Proposition (Biane-Voiculescu 2001)

The infimum defining $d_W(\mu, \nu)$ is achieved. Also, d_W is a metric on Υ_d.

Existence of minimizer: The quantity $\|U - V\|_2$ depends continuously on the law $\ell_{(U,V)} \in \Upsilon_{2d}$. Also, Υ_{2d} is compact w.r.t. the topology of pointwise convergence on $\mathbb{F}_{2d}$.

Triangle inequality: Take U, V in (A, τ_A) and V', W in (B, τ_B) that achieve the minimum for (λ, μ) and (μ, ν). Let (M, τ_M) be the free product of (A, τ_A) and (B, τ_B) with amalgamation over $\mathrm{W}^*(V) \cong \mathrm{W}^*(V)$, so $d_W(\lambda, \nu) \leq \|U - W\|_2$.

The connection to classical probability

Let $U = (U_1, \ldots, U_d)$ in a commutative von Neumann algebra $(M, \tau_M) = (L^\infty(\Omega), P)$.

- U_j is measurable function $\Omega \to \mathbb{T}$.
- U is random variable in $\mathbb{T}^d$.
- Classical law of U is the unique Borel measure m_U on $\mathbb{T}^d$ such that

$$\int_{\mathbb{T}^d} z_1^{a_1} \ldots z_d^{a_d} \, dm_U(z_1, \ldots, z_d) = \tau_M(U_1^{a_1} \ldots U_d^{a_d})$$

  for all $a_1, \ldots, a_d \in \mathbb{Z}$.
- m_U completely determines ℓ_U, e.g. $d = 3$,

$$\ell_U(g_1 g_2^{-1} g_3 g_1) = \tau_M(U_1 U_2^* U_3 U_1) = \int_{\mathbb{T}^3} z_1^2 z_2^{-1} z_3 \, dm_U(z_1, z_2, z_3)$$

The connection to classical probability

For commuting unitaries $(U_1, \dots, U_d)$, how do you get from ℓ_U to μ_U.

- ℓ_U is defined by representation π of $\mathbb{F}_d$ sending g_j to U_j.
- The U_j's commute, so π factors through the abelianization $\mathbb{F}_d \to \mathbb{Z}^d$.
- So ℓ_U corresponds to a character on $\mathbb{Z}^d$.
- Characters on $\mathbb{Z}^d \iff$ states on $\mathrm{C}^*(\mathbb{Z}^d)$.
- $\mathrm{C}^*(\mathbb{Z}^d) \cong C(\mathbb{T}^d)$.
- Character on $\mathbb{Z}^d \iff$ probability measure on $\mathbb{T}^d$.

The classical Wasserstein distance

Proposition

1. The Wasserstein distance defines a metric on $\mathcal{P}(\mathbb{T}^d)$.
2. The Wasserstein distance metrizes the weak-$*$ topology on $\mathcal{P}(\mathbb{T}^d)$.
3. $(\mathcal{P}(\mathbb{T}^d), d_W)$ is separable.

Proof of separability: Measures of the form

$$\sum_{j=1}^{k} a_j \delta_{(z_1^{(j)},\dots,z_d^{(j)})}, \qquad a_j \in \mathbb{Q}$$

form a countable dense subset.

Non-separability

Today's main theorem (GJNS)

For $d > 1$, the metric space (Υ_d, d_W) is not separable.

Corollary

The topology on Υ_d generated by d_W is *not* the same as the weak-$*$ topology (the topology of pointwise convergence as functions on $\mathbb{F}_d$).

Proof of Corollary: Υ_d is compact with respect to the weak-$*$ topology. Every compact metric space is separable.

Kazhdan's Property (T)

- For $\pi : G \to \mathcal{U}(H)$ unitary representation, $\xi \in H$ is *invariant* if $\pi(g)\xi = \xi$ for all $g \in G$.
- P_{inv} is the projection of H onto the subspace of invariant vectors.

Definition (Kazhdan)

A group G has *Property (T)* if there exists $g_1, \dots, g_d \in G$ and $K > 0$ such that, for every representation $\pi : G \to \mathcal{U}(H)$ and every $\xi \in H$, we have

$$\|P_{\text{inv}}\xi - \xi\| \leq K \left(\sum_{j=1}^{d} \|\pi(g_j)\xi - \xi\|^2 \right)^{1/2}$$

We call $\{g_1, \dots, g_d\}$ a *Kazhdan set* and K the *Kazhdan constant*.

Kazhdan's Property (T)

1. Kazhdan set generates G.
2. Lattices in $SL_n(\mathbb{Z})$ for $n \geq 3$ have Property (T).
3. Property (T) is opposite to amenability.
4. Property (T) plus amenability $\implies$ finite group.
5. Property (T) means *rigidity* of representations of G.

Property (T) and Wasserstein distance

Theorem (Gromov, **Olshanskii**, Ozawa)

There exists a group G with Property (T) and an uncountable family $(H_\alpha)_{\alpha \in I}$ of distinct normal subgroups of G.

Proposition (GJNS)

Given Property (T) group G, Kazhdan tuple $g_1, \dots, g_d$, Kazhdan constant K: For every normal $H \subseteq G$, there is a NC law μ_H such that

$$d_W(\mu_H, \mu_{H'}) \geq 1/2K \text{ for } H \neq H'.$$

Proof of main theorem:

- Take Property (T) group from Olshanskii theorem.
- By Proposition, we get uncountable $1/2K$-separated set in Υ_d.
- So Υ_d is not separable, *for some d*.
- Get for all $d > 1$ by matrix amplification trick.

Proof of Proposition

- H normal subgroup of G.
- $q : G \to G/H$ quotient map.
- $\lambda_{G/H} : G/H \to \mathcal{U}(\ell^2(G/H))$ left regular representation.
- $L(G/H)$ von Neumann algebra generated by $\lambda_{G/H}$.
- $\tau_{G/H}$ trace on $L(G/H)$ given by δ_e.
- $U = (U_1, \dots, U_d)$ where $U_j = \lambda_{G/H}(q(g_j))$.
- Let $\mu_H = \ell_U$.

Alternative interpretation:

- $\psi : \mathbb{F}_d \to G$ sending generators to g_j.
- $1_H : G \to \mathbb{C}$ indicator function of H.
- $\ell_H = 1_H \circ \psi$.
- It's the same since $\tau_{G/H}(q(\psi(w))) = \delta_{\psi(w) \in H}$.

Proof of Proposition

- Consider $H_1 \neq H_2$, quotient map $q_j : G \to G/H_j$.
- Let $U_j = \lambda_{G/H_1}(q_1(g_j)))$, $V_j = \lambda_{G/H_2}(q_2(g_j))$.
- So $\mu_{H_1} = \ell_U$ and $\mu_{H_2} = \ell_V$.

- Take U', V' from (M, τ_M) such that

$$\ell_{U'} = \ell_U, \quad \ell_{V'} = \ell_V, \quad \|U' - V'\|_2 = d_W(\mu_{H_1}, \mu_{H_2}).$$

- $\mathrm{W}^*(U') \cong \mathrm{W}^*(U) = L(G/H_1)$.
- So $L(G/H_1) \overset{\alpha}{\hookrightarrow} M$ with $\alpha(U_j) = U_j'$
- Similarly, $L(G/H_2) \overset{\beta}{\hookrightarrow} M$ with $\beta(V_j) = V_j'$.

Proof of Proposition

To apply Property (T), define representation $\pi : G \to \mathcal{U}(L^2(M, \tau_M))$,

$$\pi(g)\xi = \alpha(\lambda_{G/H_1}(q_1(g)))\ \xi\ \beta(\lambda_{G/H_2}(q_2(g)))^{-1}.$$

Note:

$$\begin{aligned}\|\pi(g)\widehat{1} - \widehat{1}\| &= \|\alpha(\lambda_{G/H_1}(q_1(g)))\beta(\lambda_{G/H_2}(q_2(g)))^{-1} - 1\|_2 \\ &= \|\alpha(\lambda_{G/H_1}(q_1(g))) - \beta(\lambda_{G/H_2}(q_2(g)))\|_2.\end{aligned}$$

So

$$\|\pi(g_j)\widehat{1} - \widehat{1}\|_2 = \|\alpha(U_j) - \beta(V_j)\|_2$$

So

$$\left(\sum_{j=1}^{d} \|\pi(g_j)\widehat{1} - \widehat{1}\|^2\right)^{1/2} = d_W(\mu_{H_1}, \mu_{H_2}).$$

Estimate on the Wasserstein distance

- Apply Property (T): $\|P_{\mathrm{inv}}\widehat{1} - \widehat{1}\| \leq K\, d_W(\mu_{H_1}, \mu_{H_2})$.
- Invariance of $P_{\mathrm{inv}}\widehat{1}$ plus triangle inequality:

$$\|\pi(g)\widehat{1} - \widehat{1}\| \leq 2\|P_{\mathrm{inv}}\widehat{1} - \widehat{1}\| \text{ for all } g \in G.$$

- $\|\alpha(\lambda_{G/H_1}(q_1(g))) - \beta(\lambda_{G/H_2}(q_2(g)))\|_2 \leq 2K\, d_W(\mu_{H_1}, \mu_{H_2})$.
- $\tau_M(\alpha(\lambda_{G/H_1}(q_1(g)))) = \delta_{g \in H_1}$. Same for H_2.
- $|\delta_{g \in H_1} - \delta_{g \in H_2}| \leq 2K\, d_W(\mu_{H_1}, \mu_{H_2})$.
- $H_1 \neq H_2$, so $d_W(\mu_{H_1}, \mu_{H_2}) \geq 1/2K$.

References (in order of appearance) I

1. Wilfrid Gangbo, David Jekel, Kyeongsik Nam, and Dimitri Shlyakhtenko. Duality for optimal couplings in free probability. arXiv:2105.12351
2. Philippe Biane and Dan-Virgil Voiculescu. A free probability analogue of the Wasserstein metric on the trace-state space. Geometric and Functional Analysis, 11:1125–1138, 2001.
3. D. Kazhdan, "On the connection of the dual space of a group with the structure of its closed subgroups", Functional Analysis and its Applications, 1 (1967), no. 1: 63–65
4. Narutaka Ozawa. There is no separable universal II_1 factor. Proc. Amer. Math. Soc., 132 (2004), 487–490.
5. A. Yu. Olshanskii, On residualing homomorphisms and G-subgroups of hyperbolic groups. Internat. J. Algebra Comput. 3 (1993), no. 4, 365–409.

References (in order of appearance) II

6. Scott Atkinson and Srivatsav Kunnawalkam Ellayavalli. On ultraproduct embeddings and amenability for tracial von Neumann algebras. International Mathematics Research Notices, 2021(4):2882–2918, 2021.