The algebraic structure of group operator algebras

Matthew Kennedy

University of Waterloo, Waterloo, Canada

July 19, 2020

C*-simplicity and the unique trace property

Let G be a discrete group. The **reduced C*-algebra** $\mathrm{C}^*_r(G)$ is the C*-algebra generated by the left regular representation of G on $\ell^2(G)$,

$$\mathrm{C}^*_r(G) = \mathrm{C}^*(\{\lambda_g : g \in G\}),$$

where $\ell^2(G) = \mathrm{span}\{\delta_h : h \in G\}$ and

$$\lambda_g \delta_h = \delta_{gh}, \quad g, h \in G.$$

Let G be a discrete group. The **reduced C*-algebra** $\mathrm{C}^*_r(G)$ is the C*-algebra generated by the left regular representation of G on $\ell^2(G)$,

$$\mathrm{C}^*_r(G) = \mathrm{C}^*(\{\lambda_g : g \in G\}),$$

where $\ell^2(G) = \mathrm{span}\{\delta_h : h \in G\}$ and

$$\lambda_g \delta_h = \delta_{gh}, \quad g, h \in G.$$

There is a "canonical" tracial state τ on $C^*_r(G)$ determined by

$$\tau(\lambda_g) = \begin{cases} 1 & g = e \\ 0 & \text{otherwise} \end{cases}.$$

Let G be a discrete group. The **reduced C*-algebra** $\mathrm{C}^*_r(G)$ is the C*-algebra generated by the left regular representation of G on $\ell^2(G)$,

$$\mathrm{C}^*_r(G) = \mathrm{C}^*(\{\lambda_g : g \in G\}),$$

where $\ell^2(G) = \mathrm{span}\{\delta_h : h \in G\}$ and

$$\lambda_g \delta_h = \delta_{gh}, \quad g, h \in G.$$

There is a "canonical" tracial state τ on $C^*_r(G)$ determined by

$$\tau(\lambda_g) = \begin{cases} 1 & g = e \\ 0 & \text{otherwise} \end{cases}.$$

Basic questions: For which G is $C^*_r(G)$ simple? For which G does it have a unique tracial state?

Let G be a discrete group. The **reduced C*-algebra** $\mathrm{C}^*_r(G)$ is the C*-algebra generated by the left regular representation of G on $\ell^2(G)$,

$$\mathrm{C}^*_r(G) = \mathrm{C}^*(\{\lambda_g : g \in G\}),$$

where $\ell^2(G) = \mathrm{span}\{\delta_h : h \in G\}$ and

$$\lambda_g \delta_h = \delta_{gh}, \quad g, h \in G.$$

There is a "canonical" tracial state τ on $C^*_r(G)$ determined by

$$\tau(\lambda_g) = \begin{cases} 1 & g = e \\ 0 & \text{otherwise} \end{cases}.$$

Basic questions: For which G is $C^*_r(G)$ simple? For which G does it have a unique tracial state?

Theorem (Murray-von Neumann 1936)

The von Neumann algebra $L(G)$ is a factor if and only if it has a unique trace if and only if G is ICC (i.e. every non-trivial conjugacy class is infinite).

Let $\mathbb{F}_2$ denote the free group on two generators.

Theorem (Powers 1975)

The reduced C-algebra* $\mathrm{C}^*_r(\mathbb{F}_2)$ *is simple and has a unique trace.*

Let $\mathbb{F}_2$ denote the free group on two generators.

Theorem (Powers 1975)

The reduced C-algebra $\mathrm{C}^*_r(\mathbb{F}_2)$ is simple and has a unique trace.*

We say that $\mathbb{F}_2$ is **C*-simple** and has the **unique trace property**.

Let $\mathbb{F}_2$ denote the free group on two generators.

Theorem (Powers 1975)

The reduced C-algebra $\mathrm{C}^*_r(\mathbb{F}_2)$ is simple and has a unique trace.*

We say that $\mathbb{F}_2$ is **C*-simple** and has the **unique trace property**.

Variants of Powers' proof became the main method for establishing these properties.

Definition

A group G has *Powers' averaging property* if for every $a \in \mathrm{C}^*_r(G)$ and $\epsilon > 0$ there are $g_1, \ldots, g_n \in G$ such that

$$\left\| \frac{1}{n} \sum \lambda_{g_i} a \lambda_{g_i^{-1}} - \tau(a) 1 \right\| < \epsilon.$$

Definition

A group G has *Powers' averaging property* if for every $a \in \mathrm{C}^*_r(G)$ and $\epsilon > 0$ there are $g_1, \ldots, g_n \in G$ such that

$$\left\| \frac{1}{n} \sum \lambda_{g_i} a \lambda_{g_i^{-1}} - \tau(a) 1 \right\| < \epsilon.$$

Theorem (Powers 1975)

A group with Powers' averaging property is C-simple and has the unique trace property.*

Definition

A group G has *Powers' averaging property* if for every $a \in \mathrm{C}^*_r(G)$ and $\epsilon > 0$ there are $g_1, \ldots, g_n \in G$ such that

$$\left\| \frac{1}{n} \sum \lambda_{g_i} a \lambda_{g_i^{-1}} - \tau(a) 1 \right\| < \epsilon.$$

Theorem (Powers 1975)

A group with Powers' averaging property is C-simple and has the unique trace property.*

Proof.

For C*-simplicity, let I be a non-trivial closed two-sided ideal of $\mathrm{C}^*_r(G)$. By faithfulness there is $a \in \mathrm{C}^*_r(G)$ with $\tau(a) = 1$. Applying Powers' averaging property implies $1 \in I$. The unique trace property is similarly straightforward.

□

Theorem (Powers 1975)

The free group $\mathbb{F}_2$ has Powers' averaging property. Hence it is C-simple and has the unique trace property.*

Question

Is there an (intrinsic) group-theoretic characterization of C*-simplicity and the unique trace property?

A group G is C*-simple iff whenever ρ is a unitary representation of G,

$$\rho \prec \lambda \Longrightarrow \rho \sim \lambda,$$

i.e. weak containment implies weak equivalence.

A group G is C*-simple iff whenever ρ is a unitary representation of G,

$$\rho \prec \lambda \Longrightarrow \rho \sim \lambda,$$

i.e. weak containment implies weak equivalence.

In other words, if

$$\lambda_g \to \rho_g, \quad g \in G$$

extends to a bounded *-homomorphism, then it is necessarily an isomorphism.

A group G is C*-simple iff whenever ρ is a unitary representation of G,

$$\rho \prec \lambda \Longrightarrow \rho \sim \lambda,$$

i.e. weak containment implies weak equivalence.

In other words, if

$$\lambda_g \to \rho_g, \quad g \in G$$

extends to a bounded *-homomorphism, then it is necessarily an isomorphism.

Proposition

C*-simple groups have no non-trivial normal amenable subgroups.

A group G is C*-simple iff whenever ρ is a unitary representation of G,

$$\rho \prec \lambda \Longrightarrow \rho \sim \lambda,$$

i.e. weak containment implies weak equivalence.

In other words, if

$$\lambda_g \to \rho_g, \quad g \in G$$

extends to a bounded *-homomorphism, then it is necessarily an isomorphism.

Proposition

C*-simple groups have no non-trivial normal amenable subgroups.

Proof.

If $N < G$ is amenable and normal then $\lambda_{G/N} \precnsim \lambda_G$. □

Many (40+) years of work shows that the converse almost always holds.

Many (40+) years of work shows that the converse almost always holds.

C*-simple and unique trace equivalent to no non-trivial normal subgroups	**Authors**
Free groups $\mathbb{F}_n$ for $n \geq 2$	Powers (1975)
$\vdots$	$\vdots$
Linear groups	T. Poznansky (2008)
Groups with non-zero first ℓ^2-Betti number	J. Peterson and A. Thom (2010)
Acylindrically hyperbolic groups	F. Dahmani, V. Guirardel, and D. Osin (2011)
Free Burnside groups $B(m, n)$ for $m \geq 2$ and n odd and large	A.Y. Olshanskii and D.V. Osin (2014)

Many (40+) years of work shows that the converse almost always holds.

C*-simple and unique trace equivalent to no non-trivial normal subgroups	**Authors**
Free groups $\mathbb{F}_n$ for $n \geq 2$	Powers (1975)
$\vdots$	$\vdots$
Linear groups	T. Poznansky (2008)
Groups with non-zero first ℓ^2-Betti number	J. Peterson and A. Thom (2010)
Acylindrically hyperbolic groups	F. Dahmani, V. Guirardel, and D. Osin (2011)
Free Burnside groups $B(m, n)$ for $m \geq 2$ and n odd and large	A.Y. Olshanskii and D.V. Osin (2014)

All the above results were proved using variants of Powers' ideas.

Problem

Are C*-simplicity and the unique trace property always equivalent to having no non-trivial amenable normal subgroups?

A characterization of C*-simplicity

Definition (Furstenberg 1973)

A compact G-space X is a G-**boundary** if for every probability measure $\mu \in \mathcal{P}(X)$, the weak* closure of the orbit $G\mu$ contains the point masses $\{\delta_x \mid x \in X\}$.

Definition (Furstenberg 1973)

A compact G-space X is a G-**boundary** if for every probability measure $\mu \in \mathcal{P}(X)$, the weak* closure of the orbit $G\mu$ contains the point masses $\{\delta_x \mid x \in X\}$.

Most "natural" topological group-theoretic boundaries are boundaries in the above sense (e.g. Gromov boundaries of non-elementary hyperbolic groups). But any non-amenable group has many boundaries.

Definition (Furstenberg 1973)

A compact G-space X is a G-**boundary** if for every probability measure $\mu \in \mathcal{P}(X)$, the weak* closure of the orbit $G\mu$ contains the point masses $\{\delta_x \mid x \in X\}$.

Most "natural" topological group-theoretic boundaries are boundaries in the above sense (e.g. Gromov boundaries of non-elementary hyperbolic groups). But any non-amenable group has many boundaries.

Example

The Gromov boundary $\partial\mathbb{F}_n$ of the Free group $\mathbb{F}_n$ can be identified with the set of infinite reduced words

$$\partial\mathbb{F}_n = \{w = w_1 w_2 w_3 \cdots \mid w_i \in \{1, \ldots, n\}\}.$$

equipped with the relative product topology.

Theorem (Kalantar-K 2014)

C-simplicity is equivalent to the existence of a free (i.e. no fixed points) action on a boundary.*

Theorem (Breuillard-Kalantar-K-Ozawa 2014)

The unique trace property is equivalent to having no non-trivial amenable normal subgroups. In particular, every C-simple group has the unique trace property.*

A characterization of the unique trace property

Theorem (Breuillard-Kalantar-K-Ozawa 2014)

The unique trace property is equivalent to having no non-trivial amenable normal subgroups.

Theorem (Breuillard-Kalantar-K-Ozawa 2014)

The unique trace property is equivalent to having no non-trivial amenable normal subgroups.

Specifically, tracial state on $\mathrm{C}^*_r(G)$ concentrate on the amenable radical $R_a(G)$, i.e. for every tracial state τ on $\mathrm{C}^*_r(G)$,

$$\tau(\lambda_s) = 0, \quad \forall s \in G \backslash R_a(G).$$

Theorem (Breuillard-Kalantar-K-Ozawa 2014)

The unique trace property is equivalent to having no non-trivial amenable normal subgroups.

Specifically, tracial state on $\mathrm{C}_r^*(G)$ concentrate on the amenable radical $R_a(G)$, i.e. for every tracial state τ on $\mathrm{C}_r^*(G)$,

$$\tau(\lambda_s) = 0, \quad \forall s \in G \backslash R_a(G).$$

Corollary

Every C-simple group has the unique trace property.*

Problem

Is C*-simplicity equivalent to having no non-trivial amenable normal subgroups, and hence equivalent to the unique trace property?

Problem

Is C*-simplicity equivalent to having no non-trivial amenable normal subgroups, and hence equivalent to the unique trace property?

Answer: No!

Problem

Is C*-simplicity equivalent to having no non-trivial amenable normal subgroups, and hence equivalent to the unique trace property?

Answer: No!

Example (Le Boudec 2015)

There are groups with no non-trivial amenable normal subgroups that are not C*-simple. Examples are constructed by embedding groups into the automorphism group of their Bass-Serre tree and enlarging.

Problem

Is C*-simplicity equivalent to having no non-trivial amenable normal subgroups, and hence equivalent to the unique trace property?

Answer: No!

Example (Le Boudec 2015)

There are groups with no non-trivial amenable normal subgroups that are not C*-simple. Examples are constructed by embedding groups into the automorphism group of their Bass-Serre tree and enlarging.

Further examples constructed by Ivanov-Omland (2017).

A new characterization of C*-simplicity

Theorem (K 2015)

A group G is C-simple if and only if the singleton $\{\tau\}$ is the only G-boundary in the state space $S(\mathrm{C}^*_\lambda(G))$.*

Theorem (K 2015)

A group G is C-simple if and only if the singleton $\{\tau\}$ is the only G-boundary in the state space $S(\mathrm{C}^*_\lambda(G))$.*

Every tracial state gives rise to a (singleton) G-boundary in $S(\mathrm{C}^*_r(G))$. But there may be larger G-boundaries in $S(\mathrm{C}^*_r(G))$.

Theorem (K 2015)

A group G is C-simple if and only if the singleton $\{\tau\}$ is the only G-boundary in the state space $S(\mathrm{C}^*_\lambda(G))$.*

Every tracial state gives rise to a (singleton) G-boundary in $S(\mathrm{C}^*_r(G))$. But there may be larger G-boundaries in $S(\mathrm{C}^*_r(G))$.

For groups with the unique trace property that are not C*-simple, e.g. Le Boudec's examples, this necessarily occurs.

Theorem (Haagerup 2015, K 2015)

A group G is C-simple if and only if it has Powers' averaging property, i.e. if and only if for every $a \in \mathrm{C}^*_r(G)$ and $\epsilon > 0$ there are $g_1, \ldots, g_n \in G$ such that*

$$\left\| \frac{1}{n} \sum \lambda_{g_i} a \lambda_{g_i^{-1}} - \tau(a) 1 \right\| < \epsilon.$$

An (intrinsic) algebraic characterization of C*-simplicity

Let $\mathcal{S}(G)$ denote the space of subgroups of G, equipped with the Chabauty topology (i.e. the product topology on $\{0,1\}^G$).

Let $\mathcal{S}(G)$ denote the space of subgroups of G, equipped with the Chabauty topology (i.e. the product topology on $\{0,1\}^G$).

Then $S(G)$ is a compact G-space with respect to conjugation,

$$g \cdot H = gHg^{-1}, \quad g \in G,\ H \in S(G).$$

Let $\mathcal{S}(G)$ denote the space of subgroups of G, equipped with the Chabauty topology (i.e. the product topology on $\{0,1\}^G$).

Then $S(G)$ is a compact G-space with respect to conjugation,

$$g \cdot H = gHg^{-1}, \quad g \in G,\ H \in S(G).$$

Definition (Glasner-Weiss 2015)

A uniformly recurrent subgroup of G is a minimal (i.e. every orbit is dense) G-subspace of $S(G)$. It is amenable if it is a subset of the (closed) set of amenable subgroups of G.

Theorem (K 2015)

A group G is C-simple if and only if it has non-trivial amenable uniformly recurrent subgroups.*

Theorem (K 2015)

A group G is C-simple if and only if it has non-trivial amenable uniformly recurrent subgroups.*

Key idea is that amenable uniformly recurrent subgroups correspond to boundaries in the state space of $\mathrm{C}^*_r(G)$.

Unwinding the definition of a uniformly recurrent subgroup gives a more intrinsic characterization of C*-simplicity.

Unwinding the definition of a uniformly recurrent subgroup gives a more intrinsic characterization of C*-simplicity.

Definition

A subgroup $H < G$ is **residually normal** if there is a finite subset $F \subseteq G \setminus \{e\}$ such that

$$F \cap gHg^{-1} \neq \emptyset \quad \forall g \in G.$$

Unwinding the definition of a uniformly recurrent subgroup gives a more intrinsic characterization of C*-simplicity.

Definition

A subgroup $H < G$ is **residually normal** if there is a finite subset $F \subseteq G \setminus \{e\}$ such that

$$F \cap gHg^{-1} \neq \emptyset \quad \forall g \in G.$$

Note: non-trivial normal subgroups are residually normal.

Unwinding the definition of a uniformly recurrent subgroup gives a more intrinsic characterization of C*-simplicity.

Definition

A subgroup $H < G$ is **residually normal** if there is a finite subset $F \subseteq G \setminus \{e\}$ such that

$$F \cap gHg^{-1} \neq \emptyset \quad \forall g \in G.$$

Note: non-trivial normal subgroups are residually normal.

Theorem (K 2015)

A group G is C-simple if and only if it has no amenable residually normal subgroups.*

Example: The Thompson groups

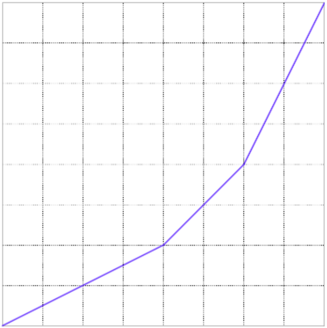
Thompson (1965) introduced three groups $F < T < V$.

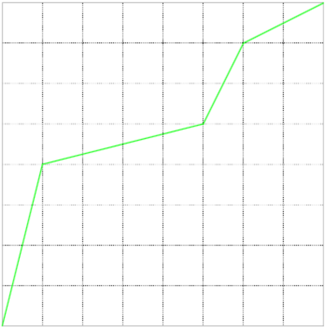
Thompson (1965) introduced three groups $F < T < V$.

The group F can be identified with the group of piecewise linear homeomorphisms of $[0, 1]$ that are differentiable, except at finitely many dyadic rationals, with derivative a power of 2 when it exists.

Thompson (1965) introduced three groups $F < T < V$.

The group F can be identified with the group of piecewise linear homeomorphisms of $[0, 1]$ that are differentiable, except at finitely many dyadic rationals, with derivative a power of 2 when it exists.





Big Open Question

Is F amenable?

Big Open Question

Is F amenable?

At a recent conference devoted to the group a poll was taken. *Is F amenable?* Twelve participants voted "yes" and twelve voted "no".

Le Boudec and Bon completely classified the residually normal subgroups of F, T, V.

Le Boudec and Bon completely classified the residually normal subgroups of F, T, V.

Theorem (Le Boudec-Bon 2016)

1. *Every non-trivial residually normal subgroup of T contains an isomorphic copy of F.*
2. *Every non-trivial residually normal subgroup of V is non-amenable.*

Le Boudec and Bon completely classified the residually normal subgroups of F, T, V.

Theorem (Le Boudec-Bon 2016)

1. *Every non-trivial residually normal subgroup of T contains an isomorphic copy of F.*
2. *Every non-trivial residually normal subgroup of V is non-amenable.*

Corollary

Thompson's group V is C-simple.*

Le Boudec and Bon completely classified the residually normal subgroups of F, T, V.

Theorem (Le Boudec-Bon 2016)

1. *Every non-trivial residually normal subgroup of T contains an isomorphic copy of F.*
2. *Every non-trivial residually normal subgroup of V is non-amenable.*

Corollary

Thompson's group V is C-simple.*

Corollary (Haagerup-Olesen 2014, Le Boudec-Bon 2016)

Thompson's group F is non-amenable if and only if T is C-simple.*

Proof.

($\Leftarrow$) It is easy to check that F is a residually normal subgroup of T. If T is C*-simple, then it has no amenable residually normal subgroups. Hence F is necessarily non-amenable. $\square$

Some recent results

Crossed Products: A C*-dynamical system (A, G) is said to have the **ideal separation property** if every ideal of $A \times_r G$ is of the form $I \times_r G$ for a G-invariant ideal I of A.

Crossed Products: A C*-dynamical system (A, G) is said to have the **ideal separation property** if every ideal of $A \times_r G$ is of the form $I \times_r G$ for a G-invariant ideal I of A.

Theorem (Kawabe 2017)

Characterization of ideal intersection property for commutative C-dynamical systems $(C(X), G)$ in terms of amenable uniformly recurrent "generalized" subgroups.*

Crossed Products: A C*-dynamical system (A, G) is said to have the **ideal separation property** if every ideal of $A \times_r G$ is of the form $I \times_r G$ for a G-invariant ideal I of A.

Theorem (Kawabe 2017)

Characterization of ideal intersection property for commutative C-dynamical systems $(C(X), G)$ in terms of amenable uniformly recurrent "generalized" subgroups.*

Corollary

If $(C(X), G)$ is minimal then $C(X) \times_r G$ is simple if and only if there are no non-trivial amenable uniformly recurrent "generalized" subgroups.

Crossed Products: A C*-dynamical system (A, G) is said to have the **ideal separation property** if every ideal of $A \times_r G$ is of the form $I \times_r G$ for a G-invariant ideal I of A.

Theorem (Kawabe 2017)

Characterization of ideal intersection property for commutative C-dynamical systems $(C(X), G)$ in terms of amenable uniformly recurrent "generalized" subgroups.*

Corollary

If $(C(X), G)$ is minimal then $C(X) \times_r G$ is simple if and only if there are no non-trivial amenable uniformly recurrent "generalized" subgroups.

Theorem (K-Schafhauser 2019)

Characterization of ideal intersection property for noncommutative C-dynamical systems (A, G) with "vanishing obstruction" in terms of amenable uniformly recurrent "generalized" subgroups.*

Measurable dynamics: Hartman-Kalantar discovered new and important connection with measurable dynamics.

Measurable dynamics: Hartman-Kalantar discovered new and important connection with measurable dynamics.

Let (A, G) be a C*-dynamical system and $\mu \in \text{Prob}(G)$ a probability measure. A state α on A is **stationary** if $\mu = \mu * \alpha = \sum_{g \in G} \mu(g)(g\alpha)$.

Measurable dynamics: Hartman-Kalantar discovered new and important connection with measurable dynamics.

Let (A, G) be a C*-dynamical system and $\mu \in \mathrm{Prob}(G)$ a probability measure. A state α on A is **stationary** if $\mu = \mu * \alpha = \sum_{g \in G} \mu(g)(g\alpha)$.

Theorem (Hartman-Kalantar 2018)

A countable group G is C-simple if there is $\mu \in \mathrm{Prob}(G)$ such that the corresponding Poisson boundary has a uniquely stationary compact model.*

Measurable dynamics: Hartman-Kalantar discovered new and important connection with measurable dynamics.

Let (A, G) be a C*-dynamical system and $\mu \in \text{Prob}(G)$ a probability measure. A state α on A is **stationary** if $\mu = \mu * \alpha = \sum_{g \in G} \mu(g)(g\alpha)$.

Theorem (Hartman-Kalantar 2018)

A countable group G is C-simple if there is* $\mu \in \text{Prob}(G)$ *such that the corresponding Poisson boundary has a uniquely stationary compact model.*

Boutonnet-Houdayer utilized these ideas to obtain a far-reaching operator algebraic superrigidity theorem, among other important results.

Theorem (Boutonnet-Houdayer 2019)

Let G be a connected simple Lie group with finite center and real rank at least 2 and $\Gamma < G$ *a lattice (e.g.* $G = SL_n(\mathbb{R})$ *and* $\Gamma = SL_n(\mathbb{Z})$ *for* $n \geq 3$*). Let* $\pi : \Gamma \to U(M)$ *be a unitary representation into a finite factor such that* $\pi(\Gamma)'' = M$*. Then either M is finite dimensional or* π *extends to a normal unital *-isomorphism* $\hat{\pi} : L(G) \to M$*.*

A few problems

Question: When are non-discrete groups C*-simple? Some examples known due to Raum (2015) and Suzuki (2016).

Question: When are non-discrete groups C*-simple? Some examples known due to Raum (2015) and Suzuki (2016).

Question: When are groupoids C*-simple? What about C*-algebras arising in other ways? Some results known, e.g. Brown-Clark-Farthing-Sims (2012), Crytser-Nagy (2018) and Clark-Exel-Pardo-Sims-Starling (2018).

Question: When are non-discrete groups C*-simple? Some examples known due to Raum (2015) and Suzuki (2016).

Question: When are groupoids C*-simple? What about C*-algebras arising in other ways? Some results known, e.g. Brown-Clark-Farthing-Sims (2012), Crytser-Nagy (2018) and Clark-Exel-Pardo-Sims-Starling (2018).

Question: Suppose G is not C*-simple. What can we say about the ideal structure of $\mathrm{C}^*_\lambda(G)$?

Thanks!