Dana Williams "crossed products of C^*-algebras"

Definition:

A topological group is a group $(\Gamma, \cdot)$ together with a topology, τ s.t.,

i) points are closed in (Γ, τ)

ii) The map $G \times G \to G$, $(s, r) \mapsto s r^{-1}$ is cont.

Consequence: multiplication and inverse maps are cont.

topological groups are regular Hausdorff

Def:

A topological space is <u>localy cpt</u> if for every point $x \in X$ There is a compact subspace $K \subseteq X$ that contains a neighborhood of x.

"infinite dim Hilbert spaces are not locally cpt"

Def: A <u>locally compact group</u> is a topological group that is locally cpt

our primary example is discrete groups.

Haar's Theorem:

Let Γ be a locally cpt group. Up to a positive multiplicitive constant There is a unique, contably additive, Left Translation invarient, nontrivial measure μ s.t. for every compact $K \subseteq \Gamma$ $\mu(K) < \infty$

Q: if Γ is discrete what is μ?

from this we obtain a bounded linear functional

$I: C_c(\Gamma) \to \mathbb{C}$ $\quad f \mapsto \int_\Gamma f(s)\, d\mu(s)$

the L^1-norm: for $f \in C_c(\Gamma)$

$$\|f\|_1 := \int_\Gamma |f(s)|\, d\mu(s)$$

we complete $C_c(\Gamma)$ under the norm $\|\cdot\|_1$ to get $L^1(\Gamma, \mu)$

– L^1-spaces are C^*-alg's iff they are 1-D –

Next, Let A be a C^*-alg and consider $f \in \underline{C_c(\Gamma, A)}$

Then $\tilde{f}: s \mapsto \|f(s)\|_A$ is in $C_c(\Gamma)$ so we define

$$\|f\|_1 := \int_\Gamma \|f(s)\|_A\, d\mu(s) \leq \|f\|_\infty\, \mu(\mathrm{supp}(f)) < \infty$$

if Γ is discrete we have $\|f\|_1 = \sum_{s \in \Gamma} \|f(s)\|_A$

1.91

Lemma: Let A be a C^*-alg and Γ a localy cpt group

Then there is a linear map

$\Phi: C_c(\Gamma, A) \to A$, $\quad \Phi: f \mapsto \int_\Gamma f(s)\, d\mu(s)$

(Note that if Γ is discrete This is Just a finite Sum of elements of A)

Φ is completely determined by the condition

$\forall\ \varphi \in A^*$

$$\varphi\left(\int_\Gamma f(s)\, d\mu(s)\right) = \int_\Gamma \varphi(f(s))\, d\mu(s)$$

in The discrete case this Just means φ is linear

Def: Let Γ be a locally compact group and let A be a C^*-alg.
An action Γ on A is a group homomorphism
$$\alpha : \Gamma \to \mathrm{Aut}(A)$$
A C^*-Alg equipped with a Γaction is called a Γ-C^*-alg
The triple (A, Γ, α) is called a <u>C^*-dynamical system</u>

We define multiplication on $C_c(\Gamma, A)$ for (A, Γ, α) by
$(f *_\alpha g)(s) := \int_\Gamma f(r)\alpha_r(g(r^{-1}s))\,d\mu(r)$ (α-Twisted convolution)
and involution as
$f^*(s) := \Delta(s^{-1})\alpha_{s^{-1}}(f(s)^*)$
Δ This is called the modular function ($\equiv 1$ when Γ is discrete)

discrete: $(f *_\alpha g)(s) = \sum_{r\in\Gamma} f(r)\alpha_r(g(r^{-1}s))$, $f^*(s) = \alpha_{s^{-1}}(f(s)^*)$

Def: A <u>covarient representation</u> (u, π, H) of the Γ-C^*-alg A is a unitary representation $u : \Gamma \to \mathcal{U}(H)$ $s \mapsto u_s$
and a $*$-representation $\pi : A \to \mathcal{B}(H)$ $a \mapsto \pi(a)$
Such that $u_s \pi(a) u_s^* = \pi(\alpha_s(a))$ $*$

Prop 2.23: Suppose that (u, π, H) is a covarient rep of the C^*-dynamical system (A, Γ, α) Then $(\pi \rtimes u)(f) := \int_\Gamma \pi(f(s))u_s\,d\mu(s)$ $\left(\sum_{s\in\Gamma}\pi(f(s))u_s\right)$
is an L^1-norm decreasing $*$-rep of $C_c(\Gamma, A)$ on H.
this is called the integrated form of (u, π, H). (Pf as exercise)

$\hat{f} : s \mapsto \pi(f(s))u_s$ then $\hat{f} \in C_c(\Gamma, \mathcal{B}(H))$

Convolution in the discrete case:
Suppose we have a covariant rep (u, π, H) of a C^*-dynamical system (A, Γ, α), Γ discrete
Let $f, g \in C_c(\Gamma, A)$ Then $(u_s^* = u_{s^{-1}})$
$$\left(\sum_{s\in\Gamma}\pi(f_s)u_s\right)\left(\sum_{r\in\Gamma}\pi(g(r))u_r\right)$$
$$= \sum_{s,r\in\Gamma}\pi(f_s)u_s\pi(g_r)u_r$$
$$= \sum_{s,r\in\Gamma}\pi(f_s)\underline{u_s\pi(g_r)u_s^*}u_s u_r$$
$$= \sum_{s,r\in\Gamma}\pi(f_s)\pi(\alpha_s(g_r))u_{sr} \qquad \begin{aligned} sr &= t \\ r &= s^{-1}t \end{aligned}$$
$$= \sum_{s,t\in\Gamma}\pi(f_s\alpha_s(g_{s^{-1}t}))u_t$$
recall that $C_c(\Gamma, A)$ is a $*$-alg. So we complete $C_c(\Gamma, A)$ under the norm
$\|f\| = \|(\pi \rtimes u)(f)\|_{\mathcal{B}(H)}$ to obtain the C^*-alg $A \rtimes_{\alpha,\pi} \Gamma$

We write the elements of $C_c(\Gamma, A) \subseteq A \rtimes_{\alpha,\pi} \Gamma$ as formal sums
$f = \sum_{s\in\Gamma} f_s\, s$ with the condition that $s a s^* = \alpha_s(a)$ $(s^* = s^{-1})$

So we here made the action of Γ on A an inner (by conj) action.
Also, $A \rtimes_{\alpha,\pi} \Gamma$ encodes the action of Γ on A, $A \subseteq A \rtimes_{\alpha,\pi}\Gamma$ & (if A is unital) $\Gamma \subseteq A \rtimes_{\alpha,\pi} \Gamma$
In group theory this is called the semi-direct product. unfortunatly our analogy ends there
Note if π is faithful then $\mathbb{C} \rtimes_{\alpha,r} \Gamma = C_r^*[\Gamma]$

But, $\mathbb{C} \not\subseteq C_r^*[\Gamma]$ Γ non-trivial