Introduction to Kazhdan's Property (T), I

Rigidity in Von Neumann algebras:

The von Neumann algebra "remembers" properties of the object that it's constructed from.

Property (T) is a rigidity property.

First defined for groups in terms of group representations.

(We only consider countable groups here.)

<u>Def</u> (Kazhdan 1967) A group G has <u>property (T)</u> if $\exists F \subseteq G$ finite subset and $\varepsilon > 0$, such that for any unitary representation $(\pi, \mathcal{H})$ of G, if π has a unit (F, ε)-invariant vector ($\xi \in \mathcal{H}$ with $\|\xi\| = 1$, $\max_{g \in F} \|\pi(g)\xi - \xi\| < \varepsilon$) then π has a nonzero invariant vector ($\eta \in \mathcal{H}$ with $\pi(g)\eta = \eta$ for all $g \in G$).

($\Leftrightarrow$ Any unitary rep. with G-almost invariant vectors ((ξ_i) with $\|\xi_i\| = 1$, $\lim_i \|g \cdot \xi_i - \xi_i\| = 0$, $\forall g \in G$) has a nonzero G-invariant vector.)

<u>Def</u> (Margulis 1982) $H < G$. The pair (G, H) has relative property (T) if any unitary representation of G with G-almost invariant vectors has a nonzero H-invariant vector.

Some examples of property (T) groups:

① Finite groups.

② $SL_n(\mathbb{Z})$, $n \geq 3$. ($SL_n(\mathbb{Z}) = \{A \in M_n(\mathbb{Z}) \mid \det(A) = 1\}$).

Remark. ① Property (T) is inherited by quotient.

② Property (T) groups are finitely generated.

Consider the group factor of a property (T) group.

Thm (Connes 1980) The group factor of a property (T) icc group is full, has discrete outer automorphism group and countable fundamental group.

(M is full: $Inn(M) = \{\varphi \in Aut(M) \mid \varphi(x) = uxu^*, \forall x \in M$ for some $u \in U(M)\}$ is closed in $Aut(M)$)

To define property (T) for vN algebras we need the notion of bimodules / Correspondences.

Def Let M, N be vN algebras. A M-N-bimodule (or a Correspondence from M to N) is a Hilbert space $\mathcal{H}$ with normal representations of M and N^{op}, s.t. $(x \cdot \xi) \cdot y = x \cdot (\xi \cdot y)$, $\forall x \in M$, $\xi \in \mathcal{H}$, $y \in N$.

Property (T) is defined for II_1 factors so that an icc group G has property (T) $\Leftrightarrow$ $L(G)$ has property (T).

<u>Def</u> (Connes, 1982) A II_1 factor M has <u>property (T)</u> if $\exists (F, \varepsilon)$ where $F \subset M$ finite subset and $\varepsilon > 0$, s.t. for all M-M-bimodule $\mathcal{H}$, if $\mathcal{H}$ has a (F, ε)-central unit vector ξ ($\max_{x \in F} \|x\xi - \xi x\| < \varepsilon$) then $\mathcal{H}$ has a nonzero M-central vector η ($x\eta = \eta x$, $\forall x \in M$)

<u>Thm</u> (Connes and Jones, 1985) G is an icc group. Then G has property (T) $\Leftrightarrow$ $M = L(G)$ has property (T).

<u>Pf</u> "$\Rightarrow$" Let (F, ε) be a Kazhdan pair of G. For a M-M-bimod $\mathcal{H}$, Define the unitary representation $\pi: G \to \mathcal{U}(\mathcal{H})$ $\pi(g)(\xi) = \lambda_g \xi \lambda_{g^{-1}}$. If $\mathcal{H}$ has a unit vector ξ satisfying $\max_{g \in F} \|\lambda_g \xi - \xi \lambda_g\| < \varepsilon$, then ξ is a (F, ε)-invariant vector of $(\pi, \mathcal{H})$. So $\exists\, 0 \neq \eta \in \mathcal{H}$, s.t. $\lambda_g \eta \lambda_{g^{-1}} = \eta$, $\forall g \in G$, i.e., η is a M-central vector in the M-M-bimodule $\mathcal{H}$.

"$\Leftarrow$" Let $(\pi, \mathcal{H})$ be a unitary representation of G. Construct a M-M-bimodule: $\ell^2(G) \otimes \mathcal{H}$ with left M-action $L(M) \otimes \pi$ (for $x = \sum_{g \in G} x_g \lambda_g \in M$, $\pi(x) = \sum_g x_g \pi(g)$) and right M-action $R(M) \otimes \mathrm{Id}$. Let (F, ε) be from property (T) of M.

We find (F', ε') for G such that if π has a unit (F', ε')-invariant vector then it has a nonzero invariant vector.

For $x \in F$, write $x = \sum_{g \in G} x_g \lambda_g$. Let $\xi \in \mathcal{H}$, then for $\delta_e \otimes \xi \in \ell^2(G) \otimes \mathcal{H}$,

$$\|x(\delta_e \otimes \xi) - (\delta_e \otimes \xi)x\|^2 = \sum_{g \in G} |x_g|^2 \|g\xi - \xi\|^2$$

$$= \underbrace{\sum_{g \in F_x} |x_g|^2 \|g\xi - \xi\|^2}_{\mathrm{I}} + \underbrace{\sum_{g \notin F_x} |x_g|^2 \|g\xi - \xi\|^2}_{\mathrm{II}}.$$

We can choose a finite subset $F_x \subset G$ so that $\sum_{g \notin F_x} |x_g|^2$ is small (so $\mathrm{II} < \varepsilon/2$). Then we include F_x in F' and set ε' so that if ξ is a (F, ε')-invariant unit vector then $\mathrm{I} < \varepsilon/2$. So if ξ is a (F', ε')-invariant unit vector we have that $\delta_e \otimes \xi$ is a (F, ε)-invariant unit vector. M has property (T) $\Rightarrow$ there is a M-central vector $\eta = \sum_{g \in G} \lambda_g \otimes \eta_g \in \ell^2(G) \otimes \mathcal{H}$. Then η_e is a $\pi(G)$-invariant vector.

<u>$\eta_e \neq 0$?</u> η_e will be nonzero if η is close enough to $\delta_e \otimes \xi$.

<u>Continuity constant</u>: the M-central vector is required to be close to the (F, ε)-central vector when defining property (T) for tracial vN algebras. But we get this for free for II_1 factors. See exercise 4.

□

Amenability and Property (T).

Recall a group G is <u>amenable</u> if there is a left invariant mean on $\ell^\infty(G)$ (a state m on $\ell^\infty(G)$ s.t. $\forall f \in \ell^\infty(G), g \in G$, $m(f) = m(gf)$, where $gf \in \ell^\infty(G)$, $gf(h) = f(g^{-1}h)$, $\forall h \in G$)

($\Leftrightarrow$ the left regular representation of G has almost invariant vectors. See §13 of C^* notes for other equivalent formulations)

Observe that an amenable property (T) group must be finite.

Some examples of amenable groups.

① Finite groups.

② Locally finite groups ($G = \cup G_n$ where G_n finite).

e.g. $S_\infty = \cup S_n$, and this is an icc non property (T) group.

③ Abelian groups.

We also have the notion of amenability for tracial vN algebras. and a group G is amenable $\Leftrightarrow L(G)$ is amenable.

<u>Thm</u> (Classification of injective factors, Connes, 1976)

All amenable II_1 factors are isomorphic to the hyperfinite II_1 factor R.

<u>Conjecture</u> (Connes, 1982) For icc property (T) groups, if $G_1 \not\cong G_2$, then $L(G_1) \not\cong L(G_2)$.