K-theory: An Elementary Introduction

Mark Tomforde

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Question What is K-theory (for Operator Algebras)?
Short Answer: A Homology Theory for C^*-algebras.

Question Why do I, as an operator algebraist, care about K-theory?
Short Answer: It provides some of the most important invariants for C^*-algebras. These invariants allow you to show that particular C^*-algebras are different, ascertain knowledge about the C^*-algebra, and sometimes (perhaps surprisingly often) show two C^*-algebras are the same.

Question: What does the K stand for?
Answer: Grothendieck used the letter K to stand for "Klasse", which means "class" in German (Grothendieck 's mother tongue).

Question Where does K-theory (for Operator Algebras) come from?
Short Answer: Algebraic/Differential Topology.

Topological K-theory	$\subseteq$	Operator K-theory	$\subseteq$	Algebraic K-theory
(cohomology for compact spaces)		(homology for C^*-algebras)		(homology for rings)

What is a homology for C^*-algebras?

First, recall that we say a sequence of objects and morphisms

$$\dots \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \dots$$

is exact at B if $\operatorname{im} f = \ker g$. We say a sequence is exact if it is exact at all locations.

A short exact sequence is an exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0.$$

Note that if A, B, and C are C^*-algebras, then $\operatorname{im} f = \ker g$, f is injective, g is surjective, A may be identified with an ideal in B, and $C \cong B/A$. So essentially any short exact sequence looks like

$$0 \longrightarrow I \xrightarrow{i} A \xrightarrow{\pi} A/I \longrightarrow 0.$$

for a C^*-algebra A and an ideal I of A.

What is a homology for C^*-algebras?

Motivation: Algebraic Topology

To begin, a homology consists of a sequence of covariant functors $H_n : \mathbf{C^*} \to \mathbf{AbGp}$ for each $n \in \mathbb{N} \cup \{0\}$.

Notation for the functor H_n:

$$\begin{array}{rcl} A & \rightsquigarrow & H_n(A) \\ f : A \to B & \rightsquigarrow & f_n : H_n(A) \to H_n(B) \end{array}$$

We require each H_n functor to be half-exact: For each $n \in \mathbb{N} \cup \{0\}$, whenever we have a short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

we may apply H_n to get a sequence

$$H_n(A) \xrightarrow{f_n} H_n(B) \xrightarrow{g_n} H_n(C)$$

that is exact at $H_n(B)$. (But typically not at $H_n(A)$ or $H_n(C)$.)

What is a homology for C^*-algebras?

Thus, when we have a short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

we may apply each H_n to get

$$H_0(A) \xrightarrow{f_0} H_0(B) \xrightarrow{g_0} H_0(C)$$

$$H_1(A) \xrightarrow{f_1} H_1(B) \xrightarrow{g_1} H_1(C)$$

$$H_2(A) \xrightarrow{f_2} H_2(B) \xrightarrow{g_2} H_2(C)$$

$$\vdots \qquad\qquad \vdots$$

For each n we require a connecting homomorphism $\delta_n : H_n(C) \to H_{n+1}(A)$ that makes a long exact sequence when inserted above. That is . . .

What is a homology for C^*-algebras?

$$
\begin{array}{ccccc}
H_0(A) & \xrightarrow{f_0} & H_0(B) & \xrightarrow{g_0} & H_0(C) \\
 & \swarrow_{\delta_0} & & & \\
H_1(A) & \xrightarrow{f_1} & H_1(B) & \xrightarrow{g_1} & H_1(C) \\
 & \swarrow_{\delta_1} & & & \\
H_2(A) & \xrightarrow{f_2} & H_2(B) & \xrightarrow{g_2} & H_2(C) \\
 & \swarrow_{\delta_2} & & & \\
 & \vdots & & \vdots &
\end{array}
$$

We usually write this long exact sequence horizontally.

$$H_0(A) \xrightarrow{f_0} H_0(B) \xrightarrow{g_0} H_0(C) \xrightarrow{\delta_0} H_1(A) \xrightarrow{f_1} H_1(B) \xrightarrow{g_1} H_1(C) \xrightarrow{\delta_1} \dots$$

What is a homology for C^*-algebras?

In topology (when we assign long exact sequences of abelian groups to topological spaces), one can build the H_n-groups in different ways.

However, there is an axiomatization of a "unique" homology. One can prove that if the Eilenberg-Steendrod Axioms are satisfied, then the H_n-groups you get are the same (at least, on large classes of spaces).

In a cohomology one uses contravariant functors, and you "reverse the arrows".

Our homology for C^*-algebras is called K-theory and we'll use the symbol K_n, in place of H_n, for our functors.

How do we build/define our K_n-groups? We look to topological K-theory, which was developed first, for motivation and inspiration.

Motivation: Topological K-theory

Topological K-theory is a cohomology for compact Hausdorff spaces.

The Big Idea: Fix a compact Hausdorff space X. The 0^{th} K-group for X is constructed using vector bundles over X, and the other groups are obtained by "suspending"; i.e., the n^{th} group is the 0^{th} group of the n^{th} suspension $S^n X$.

How do we generalize to C^*-algebras (and rings)?
Noncommutative topology: We use the following functor

$$\begin{aligned} X &\quad \rightsquigarrow \quad C(X) \\ f : X \to Y &\quad \rightsquigarrow \quad f^* : C(Y) \to C(X) \\ &\qquad\qquad \text{where } f^*(g) := g \circ f \end{aligned}$$

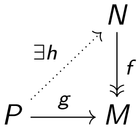
Note: This functor is contravariant.

Motivation: Topological K-theory

Swan's Theorem: The category of vector bundles over a compact space X is equivalent (i.e., isomorphic in the category sense) to the category of finitely-generated projective modules over $C(X)$.

Finitely-generated: has a finite spanning set.

Projective: A module M is *projective* if for every surjective module homomorphism $f : N \to M$ and every module homomorphism $g : P \to M$, there exists a module homomorphism $h : P \to N$ such that $f \circ g = h$.



(This is the definition of projective module, but it is equivalent to a handful of other properties.)

Motivation: Topological K-theory

Topological K-theory for a locally compact space X

0^{th} group formed using (isomorphism classes of) Vector Bundles over X.
Higher groups obtained by "suspending".

Operator (resp. Algebraic) K-theory for a C^*-algebra (resp. ring) R

0^{th} group formed using (isomorphism classes) of Finitely-Generated Projective Modules over R.

Higher groups obtained by "suspending".

Motivation: Topological K-theory

Let R be a C^*-algebra, and let M be a projective module over R.

Then M is a direct summand of a free module; i.e., there exists N such that $M \oplus N$ is free. If M is finitely generated, this free module can be chosen of finite rank; i.e., there exists $n \in \mathbb{N}$ such that

$$M \oplus N \cong R^n.$$

This means M is a subspace of R^n. But, as you know, $\operatorname{End} R^n \cong M_n(R)$, and we can identify the subspace M with the image of the projection $p \in M_n(R)$ onto M.

Q: When will two subspaces of R^n be isomorphic?
A: When there is an isomorphism (i.e., a partial isometry) between them. If p and q are the associated projections, this occurs iff there exists $v \in M_n(R)$ with $p = vv^*$ and $q = v^*v$. Murray-von Neumann equivalence!

Motivation: Topological K-theory

Topological K-theory for a locally compact space X

0^{th} group formed using (isomorphism classes of) Vector Bundles over X. Higher groups obtained by "suspending".

Operator (resp. Algebraic) K-theory for a C^*-algebra (resp. ring) R

0^{th} group formed using (isomorphism classes) of Finitely-Generated Projective Modules over R.

. . . or equivalently . . .

0^{th} group constructed using Murray-von Neumann equivalence classes of projections (resp. idempotents) in square matrices over the C^*-algebra (resp. ring).

Higher groups obtained by "suspending".

Let's focus on constructing K_0 for C^*-algebras and go through details.

Constructing the K_0-group

Let A be a C^*-algebra. If p and q are projections in A, then $p+q$ may not be a projection. (It is precisely when $p \perp q$.)

However, in $M_2(A)$ we can identify p with $\left(\begin{smallmatrix} p & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and we can identify q with $\left(\begin{smallmatrix} q & 0 \\ 0 & 0 \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} 0 & 0 \\ 0 & q \end{smallmatrix}\right)$.

We can then define a sum

$$p \oplus q := \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}.$$

Likewise for $p \in M_n(A)$ and $q \in M_k(A)$, we can define

$$p \oplus q := \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \in M_{n+k}(A)$$

.

The K_0-group for Unital C^*-algebras

Let A be a unital C^*-algebra. Embed $M_n(A)$ in $M_{n+1}(A)$ by $x \mapsto \left(\begin{smallmatrix} x & 0 \\ 0 & 0 \end{smallmatrix}\right)$.
Define

$$M_\infty(A) := \bigcup_{n=1}^{\infty} M_n(A).$$

Note: $M_\infty(A)$ is the non-closed $*$-algebra of infinite matrices that have only finitely many nonzero entries. (Also, $\overline{M_\infty(\mathbb{C})} = \mathcal{K}(\mathcal{H})$.)
Define

$$V(A) := \{[p] : p \in \operatorname{Proj} M_\infty(A)\}$$

with

$$[p] + [q] := \left[\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \right].$$

(The symbol V is a historical carryover — it stands for "vector bundle".)
Fact: $V(A)$ is an abelian semigroup with identity (i.e., an abelian monoid).
We want a group.

The Grothendieck Group of a Semigroup

Let $(V, +)$ be an abelian semigroup with identity.

Consider a pair $(h, k) \in V \times V$ and "think of it" representing $h - k$.

Define an equivalence relation $\equiv$ on $V \times V$ by

$$(h_1, k_1) \equiv (h_2, k_2) \iff \exists x \in V \text{ s.t. } h_1 + k_2 + x = h_2 + k_1 + x.$$

Why the x? To get transitivity.

The **Grothendieck Group** is the set of equivalence classes

Groth $V := \{[(h, k)] : h, k \in V\}$ w/ $[(h_1, k_1)]+[(h_2, k_2)] = [(h_1+h_2, k_1+k_2)]$.

We often write $[(h, k)]$ as the formal difference $h - k$.
But keep in mind: $h_1 - k_1 = h_2 - k_2$ iff $\exists x$ s.t. $h_1 + k_2 + x = h_2 + k_1 + x$.

Groth V is an abelian group and universal for V in the following sense: We can "include" $V \to$ Groth V by $h \mapsto (h, 0)$, (this isn't always injective). If G is a group and there is a homomorphism $\phi : V \to G$, then ϕ extends to $\widetilde{\phi}$: Groth $V \to G$ by $\widetilde{\phi}(h - k) = \phi(h) - \phi(k)$.

Examples:

Let $V = \{0, 1, 2, 3, \ldots\}$ with $+$. Then Groth $V \cong \mathbb{Z}$.

Let $V = \{0, 1, 2, 3, \ldots\} \cup \{\infty\}$ with $+$. Then Groth $V \cong 0$.
(b/c $x + \infty = y + \infty$ for all x, y)

Let $V = \{1, 2, 3, \ldots\}$ with $\times$. Then Groth $V \cong \mathbb{Q}^+$.

Constructing the K_0-group

Back to $K_0(A)$ . . .

A is a unital C^*-algebra.

$$V(A) := \{[p] : p \in \operatorname{Proj} M_\infty(A)\} \text{ with } [p] + [q] = \left[\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \right].$$

We then define

$$K_0(A) := \operatorname{Groth} V(A) = \{[p] - [q] : p, q \in \operatorname{Proj} M_\infty(A)\}.$$

Also, we want K_0 to be a functor, so if $h : A \to B$ is a $*$-homomorphism, we define $h_0 : K_0(A) \to K_0(B)$ by

$$h_0([p] - [q]) = [h(p)] - [h(q)].$$

Constructing the K_0-group

What about when A is nonunital? Let A be a nonunital C^*-algebra. Let A^1 be its (minimal) unitization. We have a short exact sequence

$$0 \longrightarrow A \xrightarrow{\ i\ } A^1 \xrightarrow{\ \pi\ } \mathbb{C} \longrightarrow 0$$

and both A^1 and $\mathbb{C}$ are unital, so using our prior definition we obtain $\pi_0 : K_0(A^1) \to K_0(\mathbb{C}) \cong \mathbb{Z}$. We then define

$$K_0(A) := \ker \pi_0.$$

Fact: It turns out, that $K_0(A^1) \cong K_0(A) \oplus \mathbb{Z}$ when A is nonunital.

Fact: If A has a countable approximate unit consisting of projections, then

$$K_0(A) \cong \operatorname{Groth} V(A) = \{[p] - [q] : p, q \in \operatorname{Proj} M_\infty(A)\}.$$

Examples of K_0

$\mathbb{C}$, $M_n(\mathbb{C})$, and $\mathcal{K}(\mathcal{H})$

Projections in $M_\infty(\mathbb{C})$ are finite rank, so $V(\mathbb{C}) \cong \{0, 1, 2, \ldots\}$ and

$$K_0(\mathbb{C}) \cong \mathbb{Z}.$$

Likewise, $M_\infty(M_n(\mathbb{C})) = M_\infty(\mathbb{C})$, and projections in $\mathcal{K}(\mathcal{H})$ and $M_\infty(\mathcal{K}(\mathcal{H}))$ are finite rank, so $V(M_n(\mathbb{C})) \cong V(\mathcal{K}(\mathcal{H})) \cong \{0, 1, 2, \ldots\}$ and

$$K_0(M_n(\mathbb{C})) \cong \mathbb{Z} \qquad \text{and} \qquad K_0(\mathcal{K}(\mathcal{H})) \cong \mathbb{Z}.$$

$B(\mathcal{H})$

In $M_\infty(B(\mathcal{H})) \cong B(\mathcal{H})$ all projections are either finite rank or have countably infinite rank. So $V(B(\mathcal{H})) \cong \{0, 1, 2, \ldots\} \cup \{\infty\}$ and

$$K_0(B(\mathcal{H})) \cong \{0\}.$$

$\mathcal{C}(\mathcal{H}) := B(\mathcal{H})/\mathcal{K}(\mathcal{H})$

In $\mathcal{C}(\mathcal{H})$ and $M_\infty(\mathcal{C}(\mathcal{H}))$ all finite-rank projections are equivalent, so $V(\mathcal{C}(\mathcal{H})) = \{0, \infty\}$ and

$$K_0(\mathcal{C}(\mathcal{H})) \cong \{0\}.$$

A Note on Equivalence in the K_0-group

Let p and q be projections in A. We say p and q are . . .

Murray-von Neumann equivalent, denoted $p \sim q$ if there exists $v \in A$ with $p = vv^*$ and $q = v^*v$.

unitarily equivalent, denoted $p \sim_u q$, if there exists unitary $u \in A^1$ with $p = u^*qu$.

homotopic, denoted $p \sim_h q$, when p and q are connected by a norm-continuous path of projections in A.

Facts:

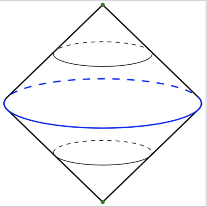
$p \sim_h q \implies p \sim_u q \implies p \sim q$

$p \sim q \implies \left(\begin{smallmatrix} p & 0 \\ 0 & 0 \end{smallmatrix}\right) \sim_u \left(\begin{smallmatrix} q & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $p \sim_u q \implies \left(\begin{smallmatrix} p & 0 \\ 0 & 0 \end{smallmatrix}\right) \sim_h \left(\begin{smallmatrix} q & 0 \\ 0 & 0 \end{smallmatrix}\right)$

So in $M_\infty(A)$ (and hence in $K_0(A)$) the Murray-von Neumann equivalence classes, unitary equivalence classes, and homotopy equivalence classes coincide.

The Higher K-groups

In topology, the suspension of a topological space X is intuitively obtained by stretching X into a cylinder and then collapsing both end faces to points. One views X as "suspended" between these end points.



Suspension of a circle. The original space is in blue, and the collapsed end points are in green.

The noncommutative version: If A is a C^*-algebra,

$$SA := \{f \in C([0,1], A) : f(0) = f(1) = 0\}.$$

Equivalent descriptions:

$$SA \cong C_0((0,1), A) \cong C_0(\mathbb{R}, A) \cong \{f \in C(\mathbb{T}, A) : f(1) = 0\}$$

The Higher K-groups

Higher K-groups are defined inductively. Given $K_0(A)$, we define

$$K_{n+1}(A) := K_n(SA) \quad \text{for } n = 0, 1, 2, \ldots$$

So inductively we obtain $K_n(A) := K_0(S^n A)$.

Although the K_1-group is defined as $K_1(A) := K_0(SA)$, we can also obtain a description in terms of unitaries . . .

The K_1-group

Define

$$A^+ := \begin{cases} A^1 & \text{if } A \text{ is nonunital} \\ A & \text{if } A \text{ is unital.} \end{cases}$$

Let $U_n(A^+)$ denote set of unitariies in $M_n(A^+)$. We can embed $U_n(A^+)$ in $U_{n+1}(A^+)$ by $x \mapsto \left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and we define $U_\infty(A^+) := \bigcup_{n=1}^\infty U_n(A^+)$.

We say $u, v \in U_n(A^+)$ are **homotopic** if there is a norm-continuous path $\gamma : [0,1] \to U_n(A^+)$ with $\phi(0) = u$ and $\phi(1) = v$.

Given $u, v \in U_\infty(A^+)$ with $u \in U_n(A^+)$ and $v \in U_m(A^+)$, we define $u \sim_h v$ if $\exists\ k \geq \max\{m, n\}$ s.t. $\left(\begin{smallmatrix} u & 0 \\ 0 & 1_{k-n} \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} v & 0 \\ 0 & 1_{k-m} \end{smallmatrix}\right)$ are homotopic.

We define

$$K_1(A) := U_\infty(A^+)/\sim_h \qquad \text{with} \qquad [u]_h + [v]_h := [\left(\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right)]_h$$

Fact: $K_1(A)$ is an abelian group; moreover $-[u]_h = [u^*]_h$.

K_1 is a functor: If $\phi : A \to B$, it extends to $\widetilde{\phi} : M_\infty(A^+) \to M_\infty(B^+)$ and we define $\phi_1 : K_1(A) \to K_1(B)$ by $\phi_1([u]_h) = [\widetilde{\phi}(u)]_h$

Examples of K_1

The K_1-group is a bit harder to compute at this stage. But with some work, one can prove that all unitaries in $U_\infty(\mathbb{C})$ and $U_\infty(B(\mathcal{H}))$ are homotopic, giving

$$K_1(\mathbb{C}) \cong K_1(M_n(\mathbb{C})) \cong K_1(\mathcal{K}(\mathcal{H})) \cong K_1(B(\mathcal{H})) \cong \{0\}.$$

We'll show some tricks for computing more K_1-groups later.

The Index Maps

At this point we have our functors K_n, but to obtain a homology we also need connecting maps (sometimes called index maps); i.e., for each C^*-algebra A and each ideal I of A, we need to construct a map

$$\delta_n : K_n(A/I) \to K_{n+1}(I) \quad \text{for each } n = 0, 1, \ldots$$

I'll spare you the details, but the index maps do exist. Moreover, it can be proven that each is unique up to sign, so despite what may seem to be a complicated or unmotivated construction, we are assured we have obtained the correct map in the end.

Thus for any ideal I in A, we map apply K-theory to the short exact sequence $0 \to I \to A \to A/I \to 0$ to obtain a long exact sequence

$$K_0(I) \longrightarrow K_0(A) \longrightarrow K_0(A/I) \xrightarrow{\delta_0} K_1(I) \longrightarrow K_1(A) \longrightarrow K_1(A/I) \xrightarrow{\delta_1} \ldots$$

In addition, a truly remarkable fact emerges during the construction of the index maps . . .

Bott Periodicity

It turns out that $K_0(A) \cong K_2(A)$ for any C^*-algebra A. (Wow!)

This implies all the higher K-groups after K_1 are redundant. For instance,

$$K_3(A) := K_2(SA) \cong K_0(SA) = K_1(A).$$

Inductively, we obtain

$$K_0(A) \cong K_2(A) \cong K_4(A) \cong K_6(A) \cong \dots$$

and

$$K_1(A) \cong K_3(A) \cong K_5(A) \cong K_7(A) \cong \dots$$

Thus there are really only two distinct K-groups: $K_0(A)$ and $K_1(A)$.

Also, since the K_0-group and the K_2-group of any C^*-algebra agree, for any short exact sequence $0 \to I \to A \to A/I \to 0$, the corresponding long exact sequence

$$K_0(I) \longrightarrow K_0(A) \longrightarrow K_0(A/I) \xrightarrow{\delta_0} K_1(I) \longrightarrow K_1(A) \longrightarrow K_1(A/I) \xrightarrow{\delta_1} \dots$$

wraps around on itself . . .

Theorem (The Cyclic 6-term Exact Sequence)

For any C^*-algebra A and any ideal I of A, applying K-theory to the short exact sequence

$$0 \longrightarrow I \xrightarrow{\ i\ } A \xrightarrow{\ \pi\ } A/I \longrightarrow 0$$

yields the cyclic 6-term exact sequence

$$\begin{array}{ccccc} K_0(I) & \xrightarrow{i_0} & K_0(A) & \xrightarrow{\pi_0} & K_0(A/I) \\ \delta_1 \uparrow & & & & \downarrow \delta_0 \\ K_1(A/I) & \xleftarrow{\pi_1} & K_1(A) & \xleftarrow{i_1} & K_1(I) \end{array}$$

Topological K-theory also has Bott periodicity of period 2. Algebraic K-theory does not have Bott periodicity.

Fun Fact: If you work over $\mathbb{R}$ instead of $\mathbb{C}$ in Topological or Operator K-theory, you get period 8 and a cyclic 24-term exact sequence.

The 6-term exact sequence can be useful for computing K-groups.

Example: We know the K-groups for $\mathcal{K}(\mathcal{H})$ and $B(\mathcal{H})$. We can use them to calculate the K-groups of the Calkin algebra $\mathcal{C}(\mathcal{H}) := B(\mathcal{H})/\mathcal{K}(\mathcal{H})$. Applying K-theory to $0 \to \mathcal{K}(\mathcal{H}) \to B(\mathcal{H}) \to \mathcal{C}(\mathcal{H}) \to 0$ we get

$$\begin{array}{ccccc} K_0(\mathcal{K}(\mathcal{H})) & \longrightarrow & K_0(B(\mathcal{H})) & \longrightarrow & K_0(\mathcal{C}(\mathcal{H})) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{C}(\mathcal{H})) & \longleftarrow & K_1(B(\mathcal{H})) & \longleftarrow & K_1(\mathcal{K}(\mathcal{H})) \end{array}$$

Substituting known values yields

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & K_0(\mathcal{C}(\mathcal{H})) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{C}(\mathcal{H})) & \longleftarrow & 0 & \longleftarrow & 0 \end{array}$$

So $K_1(\mathcal{C}(\mathcal{H})) \cong \mathbb{Z}$ and $K_0(\mathcal{C}(\mathcal{H})) \cong \{0\}$.

A covariant functor F from $\mathbf{C^*}$ to $\mathbf{AbGp}$ is . . .

- **Half Exact** when every short exact sequence $0 \to A \to B \to C \to 0$ is taken to an exact sequence $FA \to FB \to FC$.
- **Homotopy Invariant** If $\alpha : A \to B$ and $\beta : A \to B$ are homotopic (i.e., there exists a path of morphisms $\gamma_t : A \to B$, $t \in [0,1]$ such that $t \mapsto \gamma_t(a)$ is norm continuous for all $a \in A$ and with $\gamma_0 = \alpha$ and $\gamma_1 = \beta$), then $\alpha_* = \beta_*$.
- **Stable** For any C^*-algebra A and any rank 1 projection $p \in \mathcal{K}(\mathcal{H})$, the morphism $a \mapsto a \otimes p$ from A to $A \otimes \mathcal{K}(\mathcal{H})$ induces an isomorphism from $F(A)$ onto $F(A \otimes \mathcal{K}(\mathcal{H}))$.
- **Continuous** if whenever $\{A_n, \phi_n\}_{n=1}^\infty$ is a countable directed sequence, then $F(\varinjlim(A_n, \phi_n)) = \varinjlim(F(A_n), \phi_{n*})$

K_0 and K_1 are half exact, homotopy invariant, stable, and continuous.

Theorem: If F is a functor that is half exact, homotopy invariant, stable, and continuous with $F(\mathbb{C}) = \mathbb{Z}$ and $F(S\mathbb{C}) = 0$ then F is K_0.

Theorem: If F is a functor that is half exact, homotopy invariant, stable, and continuous with $F(\mathbb{C}) = 0$ and $F(S\mathbb{C}) = \mathbb{Z}$ then F is K_1.

Other K-theory Results

Direct Sums: If A and B are C^*-algebras, then

$$K_0(A \oplus B) \cong K_0(A) \oplus K_0(B) \quad \text{and} \quad K_1(A \oplus B) \cong K_1(A) \oplus K_1(B).$$

Split exact sequences: If we have a split exact sequence

$$0 \longrightarrow I \xrightarrow{i} A \underset{\pi}{\overset{s}{\rightleftarrows}} A/I \longrightarrow 0$$

then K_0 an K_1 each take it to a split exact sequence

$$0 \longrightarrow K_0(I) \xrightarrow{i_0} K_0(A) \underset{\pi_0}{\overset{s_0}{\rightleftarrows}} K_0(A/I) \longrightarrow 0 \qquad 0 \longrightarrow K_1(I) \xrightarrow{i_1} K_1(A) \underset{\pi_1}{\overset{s_1}{\rightleftarrows}} K_1(A/I) \longrightarrow 0$$

Tensor Products: The Künneth Theorem says that if A and B are nuclear and their K-groups are all torsion free, then

$$K_0(A \otimes B) \cong (K_0(A) \otimes K_0(B)) \oplus (K_1(A) \otimes K_1(B))$$
$$K_1(A \otimes B) \cong (K_0(A) \otimes K_1(B)) \oplus (K_1(A) \otimes K_0(B))$$

Pimsner-Voiculescu Exact Sequence for crossed products by $\mathbb{Z}$

If A is a unital C^*-algebra and α is a $*$-automorphism of A, we may form the crossed product $A \times_\alpha \mathbb{Z}$. If we let $i : A \hookrightarrow A \times_\alpha \mathbb{Z}$ denote the natural embedding, then there is an exact sequence

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{id-\alpha_0} & K_0(A) & \xrightarrow{i_0} & K_0(A \times_\alpha \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(A \times_\alpha \mathbb{Z}) & \xleftarrow{i_1} & K_1(A) & \xleftarrow{id-\alpha_1} & K_1(A) \end{array}$$

Note: This 6-term sequence does *not* come from a short exact sequence.

Application: If A is an $n \times n$ matrix and $\mathcal{O}_A$ is the associated Cuntz-Krieger algebra, (a dual version of) the above sequence can be used to obtain

$$\begin{array}{ccccc} \mathbb{Z}^n & \xrightarrow{I-A^t} & \mathbb{Z}^n & \longrightarrow & K_0(\mathcal{O}_A) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{O}_A) & \longleftarrow & 0 & \longleftarrow & 0 \end{array}$$

So $\quad K_0(\mathcal{O}_A) \cong \operatorname{coker}(I - A^t) \quad$ and $\quad K_1(\mathcal{O}_A) \cong \ker(I - A^t)$.

Relation with Topological K-theory

If X is a compact Hausdorff space, the n^{th} topological K-group of X is isomorphic to $K_n(C(X))$.

AF-algebras

If A is an AF-algebra, $A = \varinjlim(A_n, \phi_n)$, with each A_n finite-dimensional. Thus each A_n is a direct sum of matrix algebras, and by the continuity of K-theory and the fact K-theory distributes over direct sums

$$K_0(A) = \varinjlim(K_0(A_n), (i_n)_0) = \varinjlim(K_0(A_n), (i_n)_0) = \varinjlim(\mathbb{Z}^{k_n}, (i_n)_0)$$

and

$$K_1(A) = \varinjlim(K_1(A_n), (i_n)_1) = \varinjlim(0, (i_n)_1) = \{0\}.$$

Therefore, when A is an AF-algebra, $K_1(A) = 0$. Also, $K_0(A)$ is a direct limit of $\mathbb{Z}^{n_k}$'s and, in particular, $K_0(A)$ has no torsion.

BREAK TIME

Stabilization and Morita Equivalence

A C^*-algebra is stable if $A \otimes \mathcal{K}(\mathcal{H}) \cong A$.

For any C^*-algebra A, the stabilizaion of A is defined to be $A \otimes \mathcal{K}(\mathcal{H})$. The stabilization $A \otimes \mathcal{K}(\mathcal{H})$ is stable because $\mathcal{K}(\mathcal{H}) \otimes \mathcal{K}(\mathcal{H}) \cong \mathcal{K}(\mathcal{H})$, so

$$(A \otimes \mathcal{K}(\mathcal{H})) \otimes \mathcal{K}(\mathcal{H}) \cong A \otimes (\mathcal{K}(\mathcal{H}) \otimes \mathcal{K}(\mathcal{H})) \cong A \otimes \mathcal{K}(\mathcal{H}).$$

Another way to view the stabilization: Since $\overline{M_\infty(\mathbb{C})} = \mathcal{K}(\mathcal{H})$, we have

$$A \otimes \mathcal{K}(\mathcal{H}) \cong A \otimes \overline{M_\infty(\mathbb{C})} \cong \overline{A \otimes M_\infty(\mathbb{C})} \cong \overline{M_\infty(A)}.$$

We say A and B are stably isomorphic when $A \otimes \mathcal{K}(\mathcal{H}) \cong B \otimes \mathcal{K}(\mathcal{H})$

Theorem: If A and B have countable approximate units (e.g., they are unital or separable), then A and B are Morita equivalent if and only if A and B are stably isomorphic.

K-theory as an Invariant

Our groups K_0 and K_1 are stable:

$$K_0(A) \cong K_0(M_n(A)) \cong K_0(A \otimes \mathcal{K}(\mathcal{H}))$$
$$K_1(A) \cong K_1(M_n(A)) \cong K_1(A \otimes \mathcal{K}(\mathcal{H}))$$

Thus K-theory only "sees" a C^*-algebra up to Morita equivalence; i.e., if A and B are Morita equivqlent, then $K_0(A) \cong K_0(B)$ and $K_1(A) \cong K_1(B)$. In other words, K-theory is a Morita equivalence invariant.

K-theory can therefore be used to show two C^*-algebras are "different", where "different" means "not Morita equivalent". For example,

$$K_0(\mathcal{O}_n) \cong \mathbb{Z}/n\mathbb{Z}.$$

Thus the Cuntz algebra $\mathcal{O}_n$ is not Morita equivalent to $\mathcal{O}_m$ when $n \neq m$.

In some cases, K-theory can also be used to show two C^*-algebras are "the same", where "the same" sometimes means "Morita equivalent" and sometimes means "isomorphic". In these situations, we say K-theory is a complete invariant.

Classification of AF-algebras

Let A be an AF-algebra. Recall $K_1(A) = 0$, so all K-theory info is in the K_0-group. Since A has a countable approximate unit of projections,

$$K_0(A) = \{[p] - [q] : p, q \in \operatorname{Proj} M_\infty(A)\}.$$

We define the positive elements of $K_0(A)$ to be

$$K_0(A)^+ = \{[p] : p \in \operatorname{Proj} M_\infty(A)\}.$$

Defining $a \leq b$ iff $b - a \in K_0(A)^+$ gives a partial ordering on $K_0(A)$. We define the scale of $K_0(A)$ to be

$$\Sigma(A) = \{[p] : p \in \operatorname{Proj}(A)\}.$$

Theorem (Elliott)

Let A and B be AF-algebras.

(1) *A is Morita equivalent to B iff $(K_0(A), K_0(A)^+) \cong (K_0(B), K_0(B)^+)$.*

(2) *$A \cong B$ iff $(K_0(A), K_0(A)^+, \Sigma(A)) \cong (K_0(B), K_0(B)^+, \Sigma(B))$. Moreover, when A (respectively, B) is unital, we may replace $\Sigma(A)$ by $[1_A]$ (respectivly, we may replace $\Sigma(B)$ by $[1_B]$).*

Classification of Purely Infinite, Simple C^*-algebras

Let A be a C^*-algebra that is purely infinite and simple. Then $K_0(A) = K_0(A)^+ = \{[p] : p \in \operatorname{Proj} M_\infty(A)\}$. If A is also unital, then $K_0(A) = \Sigma(A) = \{[p] : p \in \operatorname{Proj}(A)\}$.

Theorem (Kirchberg and Phillips)

Let A and B be purely infinite, simple C^*-algebras that are also separable and nuclear.[1]

(1) *If A and B are nonunital, the following are equivalent:*
   - (a) *A is Morita equivalent to B.*
   - (b) *A is isomorphic to B.*
   - (c) *$K_0(A) \cong K_0(B)$ and $K_1(A) \cong K_1(B)$.*

(2) *If A and B are unital, then*
   - (i) *A is Morita equivalent to B iff $K_0(A) \cong K_0(B)$ and $K_1(A) \cong K_1(B)$.*
   - (ii) *A is isomorphic to B iff $(K_0(A), [1_A]) \cong (K_0(B), [1_B])$ and $K_1(A) \cong K_1(B)$.*

[1] Technically, we also need A and B to be in the bootstrap class to which the UCT applies, but let's not get into that.

Classification of simple nuclear C^*-algebras

Elliott conjectured that all simple, separable, nuclear C^*-algebras can be classified up to Morita equivalence by an invariant $\mathrm{Ell}(A)$ that includes the ordered K_0-group, the K_1-group, and other data provided by K-theory.

Counterexamples showed the conjecture is not true for *all* simple, separable, nuclear C^*-algebras — one needs an additional hypothesis, which may be formulated in various ways. TFAE:

(i) A has finite nuclear dimension.
(ii) A is $\mathcal{Z}$-stable; i.e., $A \cong A \otimes \mathcal{Z}$ where $\mathcal{Z}$ is the Jiang-Su algebra.
(iii) A has strict comparison of positive elements.[1]

Theorem (By many hands)

Let A and B be simple, separable, nuclear C^*-algebras satisfing one (and hence all) of the above three conditions. Then $A \cong B$ if and only if $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$.

[1] As Kristin Courtney graciously pointed out, (1) $\iff$ (2) has been established and (1) $\iff$ (2) $\iff$ (3) is known in many cases (e.g., when the trace space of the C^*-algebra has finitely many extreme points) but has yet to be proven in general.

What about non-simple C^*-algebras?

Elliott's Theorem applies to non-simple AF-algebras. Some progress has also been made for purely infinite C^*-algebras.

Far-reaching results have also been obtained for graph C^*-algebras (which contain the Cuntz-Krieger algebras and the AF-algebras as subclasses).

Theorem (Eilers and T)

Let A be a separable graph C^*-algebra with exactly one ideal I. Then A is classified up to Morita equivalence by the 6-term exact sequence

$$\begin{array}{ccccc} K_0(I) & \xrightarrow{i_0} & K_0(A) & \xrightarrow{\pi_0} & K_0(A/I) \\ {\scriptstyle \delta_1}\uparrow & & & & \downarrow{\scriptstyle \delta_0} \\ K_1(A/I) & \xleftarrow{\pi_1} & K_1(A) & \xleftarrow{i_1} & K_1(I) \end{array}$$

where the K_0-groups in the invariant are considered as ordered groups.

A complete classification up to Morita equivalence has been obtained for C^*-algebras of finite graphs.

The invariant, called ordered, filtered K-theory includes the 6-term exact sequences of every ideal and subquotient of A.

Theorem (Eilers, Restorff, Ruiz, and Sorensen)

Let A be a separable graph C^*-algebra of a finite graph. Then A is classified up to Morita equivalence by its ordered, filtered K-theory.

Generalizations of K-theory

Using extensions, it is possible to create a contravariant theory, called K-homology that assigns groups $K^0(A)$ and $K^1(A)$ to a C^*-algebra A.

KK-theory is a bivariant functor that takes a pair of C^*-algebra (A, B) and assigns an abelian group $KK(A, B)$.

It turns out that

- $KK(\mathbb{C}, A) \cong K_0(A)$
- $KK(S\mathbb{C}, A) \cong K_1(A)$
- $KK(A, \mathbb{C}) \cong K^0(A)$
- $KK(A, S\mathbb{C}) \cong K^1(A)$

Recall: $S\mathbb{C} = C_0(\mathbb{R})$.

So KK-theory simultaneously generalizes K-theory and K-homology, and can be viewed as a bivariant pairing between the two theories.

There is also a variant of KK-theory, known as E-theory, that was developed to get more (and better) exact sequences.

Table of K-groups

A	$K_0(A)$	$K_1(A)$
$\mathbb{C}$	$\mathbb{Z}$	0
$\mathbb{M}_n$	$\mathbb{Z}$	0
$\mathbb{K}$	$\mathbb{Z}$	0
$\mathbb{B}$	0	0
$\mathbb{B}/\mathbb{K}$	0	$\mathbb{Z}$
$C_0(\mathbb{R}^{2n})$	$\mathbb{Z}$	0
$C_0(\mathbb{R}^{2n+1})$	0	$\mathbb{Z}$
$C(\mathbb{T}^n)$	$\mathbb{Z}^{2^{n-1}}$	$\mathbb{Z}^{2^{n-1}}$
$C(S^{2n})$	$\mathbb{Z}^2$	0
$C(S^{2n+1})$	$\mathbb{Z}$	$\mathbb{Z}$
$\mathcal{T}$	$\mathbb{Z}$	0
$\mathcal{O}_n$	$\mathbb{Z}/(n-1)$	0
A_θ	$\mathbb{Z}^2$	$\mathbb{Z}^2$
II_1-factor	$\mathbb{R}$	0

To learn more about K-theory, visit your local library . . .

Introductory Textbooks

- "K-theory and C^*-algebras. A friendly approach" by N.E. Wegge-Olsen.
- "An introduction to K-theory for C^*-algebras" by M. Rørdam, F. Larsen, and N. Laustsen

Harder Textbook

- "K-theory for operator algebras", Second Edition, by B. Blackadar

A crash course on the K_0-group and Elliott's theorem for AF-algebras appears in Sec. III and Sec. IV of Davidson's book.

- "C^*-algebras by example" by K. Davidson.


THANK YOU