K-theory

Rufus Willett

University of Hawaiʻi at Mānoa

GOALS, University of Montana, July 27th 2022

A story ...

Several prominent results from the 1970s:

- Classification of AF algebras (Bratelli, Elliott).
- Classification of almost normal operators (Brown-Douglas-Fillmore).
- The covering index theorem (Atiyah-Singer).

None were (explicitly) proved using K-theory originally ... but K-theory made them whole.

This lecture: another example (we'll define the K_0 and K_1 groups along the way).

s: the *unilateral shift* on $\ell^2(\mathbb{N})$, $s : \delta_n \mapsto \delta_{n+1}$.

$\mathcal{T} := C^*(s)$, the *Toeplitz C^*-algebra*.

$1 - ss^*$ is the projection onto $\mathrm{span}(\delta_0)$, and for any $n, m \in \mathbb{N}$

$$s^m(1 - ss^*)(s^*)^n = e_{m,n} : \delta_n \mapsto \delta_m.$$

So: $\mathcal{T}$ contains $\mathcal{K} = \mathcal{K}(\ell^2(\mathbb{N}))$ (as an ideal), and

$$\mathcal{T}/\mathcal{K} = C^*(\bar{s}) \cong C(E)$$

for some non-empty $E \subseteq S^1$.

For $z \in S^1$, define $u_z \in \mathcal{U}(\ell^2(\mathbb{N}))$ by $u_z : \delta_n \mapsto z^n \delta_n$. Then

$$u_z^* s u_z = zs.$$

Hence $z \cdot E = E$, so $E = S^1$.

We have proved a classical theorem:

Theorem (Coburn, 1967)

There is a short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \xrightarrow{\sigma} C(S^1) \longrightarrow 0 \, .$$

In particular: a is invertible in $\mathcal{T}/\mathcal{K}$ if and only if $\sigma(a) : S^1 \to \mathbb{C}$ takes values in $GL(\mathbb{C})$.

Step back:

Theorem (Atkinson, 1951)

For an operator $a \in \mathcal{B}(H)$, the following are equivalent:

- *a is invertible in $\mathcal{B}(H)/\mathcal{K}(H)$;*
- *$\ker(a)$ and $\mathrm{im}(a)^{\perp}$ are finite dimensional, and the range of a is closed.*

The class of operators in the above theorem are called *Fredholm*, and the associated *index* is

$$\mathrm{Index}(a) := \dim(\ker(a)) - \dim(\ker(a^*)) \in \mathbb{Z}.$$

For example, $\mathrm{Index}(s) = -1$, and more generally

$$\mathrm{Index}(s^n) = -n, \quad \mathrm{Index}((s^*)^n) = n.$$

Theorem (Dieudonné, 1943)

(Dieudonné, 1943) The index of a Fredholm operator a depends only on the path component of a in $GL(\mathcal{B}/\mathcal{K})$.

Back to

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \xrightarrow{\sigma} C(S^1) \longrightarrow 0\ .$$

For Fredholm $a \in \mathcal{T}$, Index(a) only depends on

$$[\sigma(a)] \in \pi_0(C(S^1)^\times) = [S^1, \mathbb{C}^\times].$$

Theorem ((essentially) Noether, 1921)

For any Fredholm operator $a \in \mathcal{T}$,

$$\mathit{Index}(a) = -(\mathit{winding\ number})(\sigma(a)).$$

Proof.

$$\begin{array}{ccc} [S^1, GL(\mathbb{C})] & \xrightarrow{\text{Index}} & \mathbb{Z}\ . \\ \text{wind-}\# \downarrow \cong & \nearrow & \\ \mathbb{Z} & & \end{array}$$

Computing both compositions on the unilateral shift, "⇢" is "$\times -1$". □

(Important point I skipped over: Index is a homomorphism).

Look again at:

$$[S^1, GL(\mathbb{C})] \xrightarrow{\text{Index}} \mathbb{Z} \ .$$

This is a special case of a general map

$$K_1(B) \xrightarrow{\partial} K_0(I)$$

associated to *any* short exact sequence of C^*-algebras

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0 \ .$$

Remaining goal of this talk: explain this.

Let A be a unital C^*-algebra (or a unital ring; more generally, A could have an approximate unit of projections).

Let $\mathcal{I}_n(A)$ be the set of idempotents in $M_n(A)$, and define

$$\mathcal{I}_\infty(A) := \bigsqcup_{n=1}^{\infty} \mathcal{I}_n(A).$$

Let $\sim$ be the equivalence relation on $\mathcal{I}_\infty(A)$ generated by

- $e \sim f$ if e and f are in the same path component of some $\mathcal{I}_n(A)$.
- $\mathcal{I}_n(A) \ni a \sim \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{I}_{n+1}(A)$.

Define $V(A) := \mathcal{I}_\infty(A)/\sim$.

Define a binary operation $\oplus$ on $I_\infty(A)$ by

$$e \oplus f := \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix},$$

so $\mathcal{I}_\infty(A)$ becomes a semigroup.

Some basic properties:

- $\oplus$ descends to a well-defined binary operation on $V(A)$.
  This operation makes $V(A)$ a commutative monoid with identity element $[0]$.
  (e.g. $[0, \pi/2] \ni t \mapsto \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$ shows that $e \oplus f \sim f \oplus e$).
- Every class in $V(A)$ contains a projection.
  Moreover, for projections, $p \sim q$ if and only if there exists v such that $vv^* = p$ and $v^*v = q$ (partially proved in the problem session).
  So:

$$V(A) := \frac{P_\infty(A)}{\text{Murray von Neumann equivalence}}$$

Examples:

- $A = \mathcal{B}(H)$. $p \sim q$ if and only if $\text{rank}(p) = \text{rank}(q)$
  (as $p \sim q$ if and only if there is an isometry $v : \text{im}(p) \overset{\cong}{\to} \text{im}(q)$).

  So:

  $$V(A) \cong \{0, 1, 2, ..., \} \text{ if } \dim(H) \text{ is finite (and throw in all the cardinal numbers up to } \dim(H) \text{ if not)} .$$

  with the usual addition
  (as $\text{rank}(p \oplus q) = \text{rank}(p) + \text{rank}(q)$).

- Similarly, (for $H \neq 0$) $V(\mathcal{K}(H)) \cong \mathbb{N}$ via $p \mapsto \text{rank}(p)$.

Definition (sort of)

$K_0(A)$ is the unique abelian group equipped with a monoid homomorphism $V(A) \to K_0(A)$, with the property that for any other monoid homomorphism $V(A) \to G$ to an abelian group, the diagram

$$\begin{array}{ccc} V(A) & \longrightarrow & K_0(A) \\ & \searrow & \downarrow \exists! \\ & & G \end{array}$$

can be filled in with a group homomorphism.

(To show this makes sense, take $K_0(A)$ to be e.g. the free abelian group generated by the elements of $V(A)$, modulo the relations $[p \oplus q] = [p] + [q]$).

Examples:

- $K_0(\mathcal{B}(H)) \cong \begin{cases} \mathbb{Z} & 0 < \dim(H) < \infty \\ 0 & \text{otherwise} \end{cases}$ (as $\alpha + \beta = \beta$ for any cardinals $\alpha \leqslant \beta$ with β infinite, so $\alpha = 0$ as we are in a group).

- $\mathcal{K}_0(\mathcal{K}(H) \cong \mathbb{Z}$.

If A is non-unital, $K_0(A)$ is the subgroup of elements of $K_0(\tilde{A})$ that go to zero under the canonical map $K_0(\tilde{A}) \to K_0(\mathbb{C})$.

For a unital C^*-algebra (or Banach algebra) A, define

$$GL_\infty(A) := \bigsqcup_{n=1}^{\infty} GL_n(A)$$

Let $\sim$ be the equivalence relation on $GL_\infty(A)$ generated by

- $a \sim b$ if they are in the same path component of some $GL_n(A)$.
- $GL_n(A) \ni a \sim \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in GL_{n+1}(A)$.

Define $K_1(A) := GL_\infty(A)/\sim$.

Example:

$$K_1(C(S^1)) = \lim_{n\to\infty} \left[[S^1, GL_n(\mathbb{C})]\ ,\ a \mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right].$$

We already saw that $[S^1, GL_1(\mathbb{C})] \cong \mathbb{Z}$, and geometric topology ("Schubert cells") tells us that all the maps in the limit above are isomorphisms.
As $[S^1, GL_1(\mathbb{C})] \to [S^1, GL_n(\mathbb{C})]$ is split by the determinant map, we have

$$K_1(C(S^1)) \xrightarrow{\cong} \mathbb{Z}, \quad [u] \mapsto \text{wind-}\#(\det \circ u).$$

Lemma (Whitehead)

Let $\pi : A \to B$ be a surjection of unital C^*-algebras (or rings). Then for any invertible element u of B, there is an invertible element u of $M_2(A)$ lifting $\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$.

(u need not lift to an invertible element of A. e.g. $A = C(\text{unit disk in } \mathbb{C})$, $B = C(\text{unit circle})$, π equals restriction, and $u(z) = z$).

Proof.

In $M_2(B)$

$$\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -u^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The right hand side lifts to an invertible element of $M_2(A)$. □

Consider now a short exact sequence

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0 \; .$$

Given $u \in M_n(B)$ invertible, lift $\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$ to invertible $v \in M_{2n}(\mathbb{C})$. The *boundary* of u is

$$\partial(u) := \left[v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1} \right] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right]$$

in $K_0(I)$.

This process induces a well-defined boundary homomorphism

$$\partial : K_1(B) \to K_0(I).$$

It does not depend on the choice of v.

Back to

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \longrightarrow C(S^1) \longrightarrow 0 \; .$$

(what follows generalizes to any C^*-algebra containing $\mathcal{K}$ as an "essential" ideal).

Let $a \in \mathcal{T}$ be Fredholm, and choose $b \in \mathcal{T}$ such that $ab = 1 - e$ and $ba = 1 - f$ with e the projection onto $\ker(a^*) = \mathrm{im}(a)^\perp$, and f is projection onto $\ker(a)$.

One computes (exercise!) that with v the "Whitehead lift" defined using a and b

$$v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1} = \begin{pmatrix} 1 - e & 0 \\ 0 & f \end{pmatrix}$$

so we get

$$\partial[\sigma(a)] = \left[v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1} \right] - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = [f] - [e] \text{``}=\text{''} \, \mathrm{Index}(a).$$

K-theoretic picture of $\partial : K_1(C(S^1)) \to K_0(\mathcal{K})$ immediately lets us deduce

Theorem (Gohberg-Kreĭn, 1958)

Let $a \in M_n(\mathcal{T})$ be a Fredholm Toeplitz operator on $\ell^2(\mathbb{N})^{\oplus n}$. Then

$$Index(a) = -(wind\text{-}\#)(det \circ \sigma(a)).$$

In general, we get functors K_0 and K_1

$$\left(C^*\text{-algebras}\ ,\ *\text{-hom.s}\right) \to \left(\text{Abelian groups}\ ,\ \text{hom.s}\right)$$

Given a short exact sequence $0 \to I \to A \to B \to 0$, there is a functorial "six-term exact sequence"

$$\begin{array}{ccccc} K_0(I) & \longrightarrow & K_0(A) & \longrightarrow & K_0(B) \\ {\scriptstyle \partial}\uparrow & & & & \downarrow{\scriptstyle \exp} \\ K_1(B) & \longleftarrow & K_1(A) & \longleftarrow & K_1(I) \end{array} .$$

The deepest fact that goes into this is (a slightly stronger version of)

Theorem (Bott, 1956)

$$\lim_{n\to\infty}\left[[S^k, GL_n(\mathbb{C})]\ ,\ a \mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right] \cong \begin{cases} \mathbb{Z} & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

(discussed for $k = 1$ already). This is used to define "exp".