The Kaplansky conjecture and higher index theory

Γ: torsion free, finitely presented group (e.g. $\Gamma = \mathbb{Z}$)

Kaplansky's conjecture: $\mathbb{C}\Gamma$ contains no non-trivial idempotents (non-trivial: not 0 or 1).

(Note: if $\{e\} \neq \Lambda \leq \Gamma$ is finite, then $\mathbb{C}\Lambda \subseteq \mathbb{C}\Gamma$;

$$\mathbb{C}\Lambda \cong \bigoplus_{i=1}^{n} M_{k_i}(\mathbb{C})$$

hence $\mathbb{C}\Gamma$ does contain idempotents)

Goal: <u>sketch</u> a proof of this for amenable groups.

<u>Step 1</u>: convert to a C^*-algebra problem.

Kadison-Kaplansky conjecture: $C^*_r\Gamma$ has no non-trivial projections.

(Example: $\Gamma = \mathbb{Z}$, $C^*_r\Gamma \cong C(S^1)$; S^1 is connected so no non-trivial $\{0,1\}$-valued functions.)

By the problems, Kad.-Kap. $\Rightarrow$ Kap., so s.t.p. Kad.-Kap.

Step 2: convert to a problem about traces.

The canonical trace

$$\tau : C_r^* \Gamma \to \mathbb{C}, \quad a \mapsto \langle \delta_e, a\delta_e \rangle$$

is faithful, i.e. $a \geq 0$, $\tau(a) = 0 \Rightarrow a = 0$.

If $p \in C_r^*\Gamma$ is a non-trivial projection, then

$\tau(p) > 0$, $\tau(1-p) > 0 \Rightarrow \tau(p) \in (0,1)$.

So: s.t.p. $\tau(p) \in \mathbb{Z}$ for all projections $p \in C_r^*\Gamma$.

Step 3: convert to a problem about K-theory ☺

If τ is a tracial state on a C^*-algebra A,

then $\tau_n : M_n(A) \to \mathbb{C}$
$$(a_{ij}) \mapsto \sum \tau(a_{ii})$$

is a trace too.

The maps $\mathcal{P}_n(A) \to \mathbb{C}$ (actually $\mathbb{R}$)
$$e \mapsto \tau_n(e)$$

respect the equivalence relation defining $V(A)$, so induce

$$\tau_* : V(A) \longrightarrow \mathbb{R}$$

Universal property $\rightsquigarrow$ $\tau_* : K_0(A) \longrightarrow \mathbb{R}$.

Back to Γ: for Kaplansky, s.t.p.

$$\tau_* : K_0(C^*_r\Gamma) \longrightarrow \mathbb{R}$$

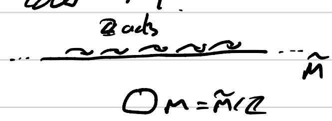
takes values in $\mathbb{Z}$.

<u>Step 4</u>: prove that $\tau_*(\alpha) \in \mathbb{Z}$ for any "geometric" class $\alpha \in K_0(C^*_r\Gamma)$.

"Geometric classes" are ones arising like this:

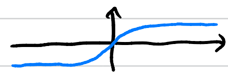
M: closed manifold with $\pi_1 M = \Gamma$ and universal cover $\tilde{M}$.

(e.g. $\Gamma = \mathbb{Z}$, $M = S^1$, $\tilde{M} = \mathbb{R}$)



D: "elliptic" differential operator on $C^\infty(M)$ with lift $\tilde{D}$ to $\tilde{M}$

(e.g. $D = -i\frac{d}{d\theta}$, $\tilde{D} = -i\frac{d}{dx}$)

Replace D with $\chi(D) \in \mathcal{B}(L^2M)$, $\chi =$ 

PDE theory implies D is invertible modulo $\mathcal{K}(L^2(M))$

$\rightsquigarrow$ "$\partial[D]$" $\in K_*(\mathcal{K}) \cong \mathbb{Z}$.

More of the same: $\tilde{D}$ is invertible modulo $C^*_r(\Gamma) \otimes \mathcal{K}(L^2(M))$ on $L^2(\tilde{M}) \cong L^2(\Gamma \times M) \cong \ell^2\Gamma \otimes L^2M$.

$\rightsquigarrow (\partial[\tilde{D}]) \in K_*(C^*_r\Gamma)$.

Any class arising like $\partial[\tilde{D}]$ is "geometric".

Theorem (Atiyah-Singer covering index theorem):

$$\mathbb{R} \ni \tau_*(\partial[\tilde{D}]) \text{ "="} \; \partial[D] \in \mathbb{Z}.$$

Proof: localize, and compute.

<u>Step 5</u>: prove that every class in $K_0(C^*_r\Gamma)$ is "geometric".

This step, which is part of the "Baum-Connes conjecture", is not known for all groups, but it is for amenable groups, hyperbolic groups, ...

Very, very rough idea of a proof for amenable groups:

- Build a group of "geometric cycles" $K_0^{geo}(\Gamma)$, and an "assembly map"

$$\begin{array}{ccc} K_0^{geo}(\Gamma) & \xrightarrow{\mu} & K_0(C_r^*\Gamma) \\ (M, D) & \longmapsto & \partial[\tilde{D}] \end{array}$$

- Show that Γ acts nicely on a Hilbert space E, and use an infinite-dimensional version of Bott periodicity to replace μ with an equivalent map

$$K_0^{geo}(\Gamma; \text{"}C^*(E)\text{"}) \xrightarrow{\mu_E} K_0(C^*(E) \rtimes_r \Gamma)$$

- Use induction-restriction from representation theory to reduce this to showing that the following is an isomorphism

$$K_0(C^*(E)) \xrightarrow{id} K_0(C^*(E))$$