New hyperfinite subfactors with infinite depth

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Joint work with Dietmar Bisch

Motivation

1. What are all hyperfinite subfactors $N \subset M$ with small index?
2. What is $\{[M : N], N \subset M \text{ hyperfinite}, N' \cap M = \mathbb{C}\}$?

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- $[M : N] \leq 4$: ADE classification
- $4 < [M : N]$: Completely classified finite depth up to index 5.25. (Small index classification)

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Conjecture

Every index of a hyperfinite finite depth irreducible subfactor is the index of a hyperfinite irreducible A_∞ subfactor.

Commuting squares

- $E_{A_{1,0}} E_{A_{0,1}} = E_{A_{0,0}}$ (Commuting square)
- $GK = HL$ and $G^t H = KL^t$ (non-degenerate)

$$\begin{array}{ccc} A_{1,0} & \overset{K}{\subset} & A_{1,1} \\ \cup_G & & \cup_L \\ A_{0,0} & \overset{H}{\subset} & A_{0,1} \end{array}$$

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$$\begin{array}{ccccccccc}
A_{1,0} & \overset{K}{\subset} & A_{1,1} & \subset & A_{1,2} & \subset & \cdots & \subset & A_{1,\infty} \\
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We like these because:

- $[A_{1,\infty} : A_{0,\infty}] = \|G\|^2 = \|L\|^2$
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Lots of examples constructed in [Sch13].

Embedding theorem

Using Ocneanu's compactness, some facts about Pimsner-Popa basis and loop algebra formulas from [JP11] we prove the following:

Theorem

Let $P_\bullet$ be the subfactor planar algebra associated to $A_{0,\infty} \subset A_{1,\infty}$ and $\mathrm{GPA}(G)_\bullet$ the graph planar algebra associated to the Bratelli diagram of $A_{0,0} \subset A_{1,0}$. Then $P_\bullet$ embeds into $\mathrm{GPA}(G)_\bullet$.

Fusion graphs and embeddings

Another embedding theorem:

Theorem (GMPPS'18)

Suppose $P_\bullet$ is a finite depth subfactor planar algebra. Let $\mathcal{C}$ denote the unitary multifusion category of projections in $P_\bullet$, with distinguished object $X = \mathrm{id}_{1,+} \in P_{1,+}$, and the standard unitary pivotal structure with respect to X. There is an equivalence between:

1. *Planar algebra embeddings $P_\bullet \to \mathrm{GPA}(G)_\bullet$, where $\mathrm{GPA}(G)_\bullet$ is the graph planar algebra associated to a finite connected bipartite graph G, and*
2. *indecomposable finitely semisimple pivotal left $\mathcal{C}$-module C^* categories $\mathcal{M}$ whose fusion graph with respect to X is G.*

Main Idea

Let $N \subset M$ be a finite depth hyperfinite subfactor with unitary multifusion category $\mathcal{C}$ and $\{A_{ij},\ i,j = 0,1\}$ a commuting square.

If G isn't a fusion graph for any ${}_{\mathcal{C}}\mathcal{M}$, then $A_{0,\infty} \subset A_{1,\infty}$ isn't isomorphic to $N \subset M$.

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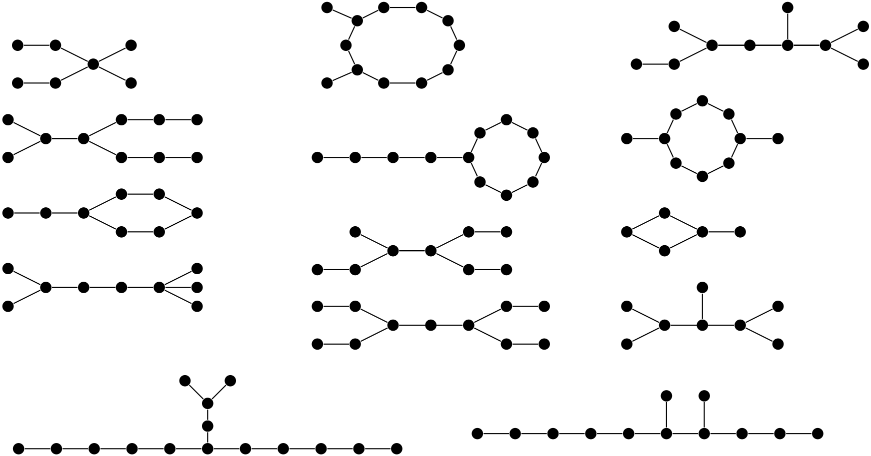
We know a lot about the left $\mathcal{C}$-module C^* categories $\mathcal{M}$ when $\mathcal{C}$ comes from:

- [Pet10]: Haagerup subfactor (3 graphs)
- [GMP$^+$18]: Extended Haagerup subfactor (4 graphs)
- [GS16] & [GIS18]: Asaeda-Haagerup subfactor (14 graphs)

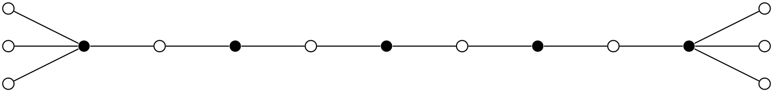
Fusion graphs for Asaeda-Haagerup



Principal graph

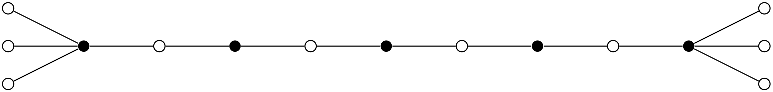
Dual principal graph

Double brooms

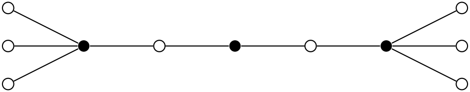


Large double broom - $\|\cdot\|^2 = \frac{5+\sqrt{17}}{2}$

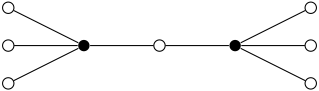
Double brooms



Large double broom - $\| \cdot \|^2 = \frac{5+\sqrt{17}}{2}$



Medium double broom - $\| \cdot \|^2 = 3 + \sqrt{3}$



Small double broom - $\| \cdot \|^2 = 5$

Bi-unitary connections

$$\begin{array}{ccc} A_{1,0} & \overset{K}{\subset} & A_{1,1} \\ \cup_G & \text{c.s} & \cup_G \\ A_{0,0} & \overset{H}{\subset} & A_{0,1} \end{array} \Leftrightarrow$$

Existence of a unitary u satisfying the bi-unitary condition

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That is $u = \bigoplus_{(p,s)} u^{(p,s)}$ and $v = \bigoplus_{(q,r)} v^{(q,r)}$ such that

- $u^{(p,s)} = \left(u^{(p,s)}_{q,r}\right)_{q,r}$
- $v^{(q,r)} = \left(v^{(q,r)}_{p,s}\right)_{p,s}$
- $v^{(q,r)}_{p,s} = \sqrt{\frac{\lambda(p)\eta(s)}{\lambda(q)\eta(r)}}(u^{(p,s)}_{q,r})^t$

Main result

Theorem

If G is one of the double brooms, there exist H and K for which we can construct a bi-unitary connection.

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If G is one of the double brooms, there exist H and K for which we can construct a bi-unitary connection.

In particular, we have constructed an irreducible hyperfinite subfactor with infinite depth and index $\frac{5+\sqrt{17}}{2}$ (the same as Asaeda-Haagerup), by classification it has to have trivial standard invariant.

Remark

Unlike the commuting squares in [Sch13], K is never a polynomial in $G^t G$.

Another approach

- In [Kaw23] it is proven that given a finite depth subfactor $N \subset M$, there are only countably many non-equivalent commuting squares associated to it.
- By classification of small index subfactors we have finitely many finite depth subfactors at the indices $\frac{5+\sqrt{17}}{2}$, $3+\sqrt{3}$, $\frac{5+\sqrt{21}}{2}$, 5 and $3+\sqrt{5}$.

More connections!

Let $G = S(i,i,j,j)$, the 4-star with two pairs of legs of equal length. It's been proved in [Sch13] that there exists bi-unitary connections for inclusions of the form:

$$\begin{array}{ccc} A_{1,0} & \overset{G^t}{\subset} & A_{1,1} \\ \cup_G & & \cup_{G^t} \\ A_{0,0} & \overset{G}{\subset} & A_{0,1} \end{array}$$

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We proved there exists a 1-parameter family of bi-unitary connections for all i, j!

Indices of $S(i, i, j, j)$

$j \backslash i$	1	2	3	4	$\cdots$	∞
1	4					
2	$\frac{5+\sqrt{17}}{2}$	5				
3	$3+\sqrt{3}$	5.1249	$3+\sqrt{5}$			
4	$\frac{5+\sqrt{21}}{2}$	5.1642	5.2703	$\frac{7+\sqrt{13}}{2}$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	
∞	$2+2\sqrt{2}$	5.1844	5.2870	5.3184		$\frac{16}{3}$

Hence we have infinite depth at $\frac{5+\sqrt{17}}{2}$, $3+\sqrt{3}$, $\frac{5+\sqrt{21}}{2}$, 5 and $3+\sqrt{5}$. All but the last must have A_∞ standard invariant.

Future work

- Can we construct a hyperfinite A_∞ subfactor with index $4.3772\ldots$ (Extended Haagerup index)?
- Are the A_∞ subfactors obtained from the Large double broom and $S(1,1,2,2)$ the same?
- Are all the infinite depth subfactors coming from a 1-parameter family of connections the same?

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