AMENABLE GROUPS

BRENT NELSON

1. Amenable Groups

Throughout[1] we let Γ be a discrete group. For $f\colon \Gamma \to \mathbb{C}$ and each $s \in \Gamma$ we define the left translation action by $(s.f)(t) = f(s^{-1}t)$.

Definition 1.1. A group Γ is **amenable** is there exists a state μ on $l^\infty(\Gamma)$ which is invariant under the left translation action: for all $s \in \Gamma$ and $f \in l^\infty(\Gamma)$, $\mu(s.f) = \mu(f)$.

Example 1.2. Finite groups are amenable: take the state which sends $\chi_{\{s\}}$ to $\frac{1}{|\Gamma|}$ for each $s \in \Gamma$. One can also see that abelian groups are amenable through the Markov-Kakutani fixed point theorem. Furthermore, the class of amenable groups is closed under taking subgroups, extensions, quotients, and inductive limits. Hence we can construct further examples from finite and abelian groups.

Example 1.3. Suppose Γ is finitely generated by $S = \{s_1, \ldots, s_d\}$ (and that $S^{-1} = S$). One can then consider the Cayley graph of Γ where vertices are group elements and edges connecting two group elements imply they differ by one of the generators in S. We place a metric on this graph by letting $d(s,t)$ by counting the "word length" of $s^{-1}t$ (with the generators as our usable letters). A property of interest is how $|B(e,r)|$, the ball centered at the identity element of radius r, varies with r; that is, the **growth rate of the group**. It turns out that groups with subexponential growth are always amenable.

Example 1.4. $\Gamma = \mathbb{F}_2$ is non-amenable: let $a, b \in \mathbb{F}_2$ be the two generators then let A^+ is the set of all words starting with a. A^- is the set of all words starting with a^{-1}, and we define $B^\pm$ similarly. Lastly, we set $C := \{1, b, b^2, \ldots\}$. We note that we can decompose $\mathbb{F}_2$ in the three following ways:

$$\begin{aligned}\mathbb{F}_2 &= A^+ \sqcup A^- \sqcup (B^+ \setminus C) \sqcup (B^- \cup C)\\ &= A^+ \sqcup aA^-\\ &= b^{-1}(B^+ \setminus C) \sqcup (B^+ \cup C).\end{aligned}$$

If we had a state μ on $l^\infty(\Gamma)$ which was invariant under left translation then we would obtain:

$$\begin{aligned}1 = \mu(1) &= \mu(\chi_{A^+} + \chi_{A^-} + \chi_{B^+\setminus C} + \chi_{B^-\cup C}) = \mu(\chi_{A^+}) + \mu(\chi_{A^-}) + \mu(\chi_{B^+\setminus C}) + \mu(\chi_{B^-\cup C})\\ &= \mu(\chi_{A^+}) + \mu(a.\chi_{A^-}) + \mu(b^{-1}.\chi_{B^+\setminus C}) + \mu(\chi_{B^-\cup C}) = \mu(\chi_{A^+} + \chi_{aA^-}) + \mu(\chi_{b^{-1}B^+\setminus C} + \chi_{B^-\cup C})\\ &= \mu(1) + \mu(1) = 2,\end{aligned}$$

a contradiction.

Our goal is to prove the following theorem:

Theorem 1.5. *For Γ a discreet group, the following are equivalent:*

1. Γ *is amenable;*
2. Γ *has an approximate invariant mean;*
3. Γ *satisfies the Følner condition;*
4. *The trivial representation τ_0 is weakly contained in the the regular representation λ (i.e., there exist unit vectors $\xi_i \in l^2(\Gamma)$ such that $\|\lambda_s(\xi_i) - \xi_i\|_2 \to 0$ for all $s \in \Gamma$);*
5. *there exists a net (φ) of finitely supported positive definite functions on Γ, with $\varphi_i(e) = 1$ for each i, such that $\varphi_i \to 1$ pointwise;*
6. $C^*(\Gamma) = C^*_\lambda(\Gamma)$*;*
7. $C^*_\lambda(\Gamma)$ *has a character (i.e., one-dimensional representation);*

[1]These notes were adapted from [1], to which we direct the reader for more information and less detail.

(8) *for any finite subset $E \subset \Gamma$, we have*

$$\left\| \frac{1}{|E|} \sum_{s\in E} \lambda_s \right\| = 1.$$

The main obstacle to proving this theorem is that we don't understand what most of it is saying. Consequently we'll parse the theorem as we go (rather than drowning the reader in definitions). The plan is to prove the cycle (1 2 3 4 5 6 7) and then (4 8). We shall additionally prove (4) $\Rightarrow$ (6) in case the reader finds condition (5) distasteful.

Definition 1.6. For a discrete group Γ, let $\mathrm{Prob}(\Gamma)$ be the space of all probability measures on Γ:

$$\mathrm{Prob}(\Gamma) := \{\mu \in l^1(\Gamma) \colon \mu \geq 0 \text{ and } \sum_{t\in\Gamma} \mu(t) = 1\}.$$

Then we say Γ has an **approximate invariant mean** if for any finite subset $E \subset \Gamma$ and $\epsilon > 0$, there exists $\mu \in \mathrm{Prob}(\Gamma)$ such that

$$\max_{s\in E} \|s.\mu - \mu\|_1 < \epsilon.$$

Proof of (1) $\Rightarrow$ (2). Let μ be an invariant mean on $l^\infty(\Gamma)$. We claim there is a net $(\mu_i) \subset \mathrm{Prob}(\Gamma)$ which converges to μ weak* as elements of $l^\infty(\Gamma)^*$. Suppose not, then $\mu \notin \overline{\mathrm{Prob}(\Gamma)}^{w*}$ and since $\mathrm{Prob}(\Gamma)$ is convex the Hahn-Banach separation theorem implies there is some $f \in l^\infty(\Gamma)$ and $t < s \in \mathbb{R}$ such that

$$\mathrm{Re}\ [\nu(f)] < t < s < \mathrm{Re}\ [\mu(f)],$$

for all $\nu \in \mathrm{Prob}(\Gamma)$. Upon replacing f with $\frac{f+\bar{f}}{2}$ we obtain $\nu(f) < t < s < \mu(f)$. Then replacing f with $f + \|f\|_\infty$ (and t, s with $t' = t + \|f\|_\infty, s' = s + \|f\|_\infty$) ensures that $f \geq 0$. Consequently $\sup\{\nu(f) \colon \nu \in \mathrm{Prob}(\Gamma)\} = \|f\|_\infty$ and yet

$$\|f\|_\infty = \sup\{\nu(f) \colon \nu \in \mathrm{Prob}(\Gamma)\} \leq t < s < \mu(f) \leq \|f\|_\infty,$$

a contradiction.

Hence we can find a net (μ_i) in $\mathrm{Prob}(\Gamma)$ which converges to μ in the weak* topology. Thus for each $s \in \Gamma$ and $f \in l^\infty(\Gamma)$ we know $s.\mu_i(f) - \mu_i(f) \to s.\mu(f) - \mu(f) = \mu(f) - \mu(f) = 0$. But since the $\mu_i \in l^1(\Gamma)$, this is equivalent to saying they converge weakly to zero in $l^1(\Gamma)$. Thus for any finite $E \subset \Gamma$, the weak closure of the convex subset $\bigoplus_{s\in E}\{s.\mu - \mu \colon \mu \in \mathrm{Prob}(\Gamma)\}$ contains 0. As a convex set, the weak and norm closures coincide by the Hahn-Banach theorem. Hence given $\epsilon > 0$ we can find $\nu \in \mathrm{Prob}(\Gamma)$ such that

$$\sum_{s\in E} \|s.\nu - \nu - 0\|_1 < \epsilon.$$

Hence we have an approximate invariant mean. $\square$

Definition 1.7. We say Γ satisfies the **Følner condition** if for any finite $E \subset \Gamma$ and $\epsilon > 0$, there exists a finite subset $F \subset \Gamma$ such that

$$\max_{s\in E} \frac{|sF \triangle F|}{|F|} < \epsilon.$$

That is, the action of E does not move F around "too much."

Furthermore, a sequence of finite sets $F_n \subset \Gamma$ such that

$$\frac{|sF_n \triangle F_n|}{|F_n|} \to 0$$

is called a **Følner sequence**.

Proof of (2) $\Rightarrow$ (3). Fix a finite subset $E \subset \Gamma$ and $\epsilon > 0$. Since we have an approximate invariant mean we can find $\mu \in \mathrm{Prob}(\Gamma)$ such that

$$\sum_{s\in E} \|s.\mu - \mu\|_1 < \epsilon.$$

Given a positive function $f \in l^1(\Gamma)$ and $r \geq 0$, we define a set $F(f,r) := \{t \in \Gamma \colon f(t) > r\}$. Now, note that for a pair of positive functions $f, h \in l^1(\Gamma)$ and $t \in \Gamma$, $|\chi_{F(f,r)}(t) - \chi_{F(h,r)}(t)| = 1$ if and only if r lies between the numbers $f(t)$ and $h(t)$. Furthermore, if f and h are bounded above by 1 then it follows that

$$|f(t) - h(t)| = \int_0^1 |\chi_{F(f,r)}(t) - \chi_{F(h,r)}(t)| \; dr.$$

We apply this to μ and $s.\mu$ to get

$$\begin{aligned}
\|s.\mu - \mu\|_1 &= \sum_{t\in\Gamma} |s.\mu(t) - \mu(t)| = \sum_{t\in\Gamma} \int_0^1 |\chi_{F(s.\mu,r)}(t) - \chi_{F(\mu,r)}(t)| \; dr \\
&= \int_0^1 \sum_{t\in\Gamma} |\chi_{F(s.\mu,r)}(t) - \chi_{F(\mu,r)}(t)| \; dr = \int_0^1 |F(s.\mu, r) \triangle F(\mu, r)| \; dr \\
&= \int_0^1 |sF(\mu, r) \triangle F(\mu, r)| \; dr.
\end{aligned}$$

Also, we have

$$\int_0^1 |F(\mu, r)| \; dr = \int_0^1 \sum_{t\in\Gamma} \chi_{F(\mu,r)}(t) \; dr = \sum_{t\in\Gamma} \int_0^1 \chi_{F(\mu,r)}(t) \; dr = \sum_{t\in\Gamma} \int_0^\mu (t) 1 \; dr = \sum_{t\in\Gamma} \mu(t) = 1.$$

Thus

$$\epsilon \int_0^1 |F(\mu, r)| \; dr = \epsilon > \sum_{s\in E} \|s.\mu - \mu\|_1 = \int_0^1 \sum_{s\in E} |sF(\mu, r) \triangle F(\mu, r)| \; dr.$$

So for some r we must have

$$\sum_{s\in E} |sF(\mu, r) \triangle F(\mu, r)| < \epsilon |F(\mu, r)|.$$

Letting $F := F(\mu, r)$ we are done. □

For the next implication we will ignore the first version of the statement and instead focus on the later, equivalent condition. We only need to understand the left regular representation. This is a homomorphism $\lambda \colon \Gamma \to \mathcal{U}(l^2(\Gamma))$ where the image of $s \in \Gamma$ is denoted λ_s and for $f \in l^2(\Gamma)$ $(\lambda_s f)(t) = f(s^{-1}t)$.

Proof of (3) ⇒ (4). From the Følner condition build a Følner sequence (F_i) (by letting $\epsilon = 1, \frac{1}{2}, \frac{1}{3}, \ldots$) set $\xi_i := |F_i|^{-1/2}\chi_{F_i}$. Then $\xi \in l^2(\Gamma$ are unit vectors and

$$\begin{aligned}
\|\lambda_s(\xi_i) - \xi_i\|_2^2 &= \sum_{t\in\Gamma} |\lambda_s(\xi_i)(t) - \xi_i(t)|^2 = \sum_{t\in\Gamma} \left| \frac{1}{|F_i|^{1/2}} \chi_{F_i}(s^{-1}t) - \frac{1}{|F_i|^{1/2}} \chi_{F_i}(t) \right|^2 \\
&= \frac{1}{|F_i|} \sum_{t\in\Gamma} |\chi_{sF_i}(t) - \chi_{F_i}(t)|^2 = \frac{|sF_i \triangle F_i|}{|F_i|} \to 0
\end{aligned}$$

□

Definition 1.8. A function $\varphi \colon \Gamma \to \mathbb{C}$ is called **positive definite** if the matrix

$$[\varphi(s^{-1}t)]_{s,t\in F} \in M_{|F|\times|F|}(\mathbb{C}),$$

is positive for every finite set $F \subset \Gamma$.

Proof (4) ⇒ (5). For a unit vector $\xi \in l^2(\Gamma)$, define $\varphi(s) := \langle \lambda_s \xi, \xi \rangle_{l^2(\Gamma)}$. Then we claim that φ is positive definite. Indeed,

$$[\varphi(s^{-1}t)] = [\langle \lambda_{s^{-1}t} \xi, \xi \rangle] = [\langle \lambda_s^* \lambda_t \xi, \xi \rangle] = [\langle \lambda_t \xi, \lambda_s \xi \rangle],$$

Let $n = |F|$, $F = \{t_1, \dots, t_n\}$, and fix $v = (z_1, \dots, z_n) \in \mathbb{C}^n$. It suffices to show $\langle [\varphi(s^{-1}t)]v, v\rangle \geq 0$. We compute

$$[\langle \lambda_t\xi, \lambda_s\xi\rangle]v = \begin{bmatrix} \langle \lambda_{t_1}\xi, \lambda_{t_1}\xi\rangle & \cdots & \langle \lambda_{t_n}\xi, \lambda_{t_1}\xi\rangle \\ \vdots & \ddots & \vdots \\ \langle \lambda_{t_1}\xi, \lambda_{t_n}\xi\rangle & \cdots & \langle \lambda_{t_n}\xi, \lambda_{t_n}\xi\rangle \end{bmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} \langle \lambda_{t_1}\xi, \lambda_{t_1}\xi\rangle z_1 + \cdots + \langle \lambda_{t_n}\xi, \lambda_{t_1}\xi\rangle z_n \\ \vdots \\ \langle \lambda_{t_1}\xi, \lambda_{t_n}\xi\rangle z_1 + \cdots + \langle \lambda_{t_n}\xi, \lambda_{t_n}\xi\rangle z_n \end{pmatrix}$$
$$= \begin{pmatrix} \langle z_1\lambda_{t_1}\xi, \lambda_{t_1}\xi\rangle + \cdots + \langle z_n\lambda_{t_n}\xi, \lambda_{t_1}\xi\rangle \\ \vdots \\ \langle z_1\lambda_{t_1}\xi, \lambda_{t_n}\xi\rangle + \cdots + \langle z_n\lambda_{t_n}\xi, \lambda_{t_n}\xi\rangle \end{pmatrix}$$
$$= \begin{pmatrix} \sum_{i=1}^n \langle z_i\lambda_{t_i}\xi, \lambda_{t_1}\xi\rangle \\ \vdots \\ \sum_{i=1}^n \langle z_i\lambda_{t_i}\xi, \lambda_{t_1}\xi\rangle \end{pmatrix}$$

and so

$$\langle [\langle \lambda_t\xi, \lambda_s\xi\rangle]v, v\rangle = \begin{pmatrix} \sum_{i=1}^n \langle z_i\lambda_{t_i}\xi, \lambda_{t_1}\xi\rangle \\ \vdots \\ \sum_{i=1}^n \langle z_i\lambda_{t_i}\xi, \lambda_{t_1}\xi\rangle \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \sum_{j=1}^n\sum_{i=1}^n \langle z_i\lambda_{t_i}\xi, \lambda_{t_j}\xi\rangle \overline{z_j} = \sum_{j=1}^n\sum_{i=1}^n \langle z_i\lambda_{t_i}\xi, z_j\lambda_{t_j}\xi\rangle$$
$$= \left\langle \sum_{i=1}^n z_i\lambda_{t_i}\xi, \sum_{j=1}^n z_j\lambda_{t_j}\xi \right\rangle = \left\| \sum_{i=1}^n z_i\lambda_{t_i}\xi \right\|^2 \geq 0.$$

Hence φ is positive definite. So letting (ξ_i) be the unit vectors from condition (4) and setting $\varphi_i(s) := \langle \lambda_s\xi_i, \xi_i\rangle$ we know $\varphi_i(e) = 1$ and from our above work that these functions are positive definite. From condition (4) we also know that they converge pointwise to 1. In order to make them finitely supported we need merely replace the ξ_i with finitely supported elements. $\square$

Starting with a discrete group Γ we can consider the group algebra $\mathbb{C}[\Gamma] = \{\sum_{i=1}^n \alpha_i \cdot t_i \colon n \in \mathbb{N},\ \alpha_t \in \mathbb{C},\ t_i \in \Gamma\}$ with addition and multiplication defined in the obvious ways and an involution defined by

$$\left(\sum_{i=1}^n \alpha_i \cdot t_i\right)^* = \sum_{i=1}^n \overline{\alpha_i} \cdot t_i^{-1}$$

We want to extend this into a C^*-algebra, but there are multiple norms we use. On the one hand we can extend the left regular representation λ to a $*$-representation of $\mathbb{C}[\Gamma]$ on $l^2(\Gamma)$, still denoted by λ, by

$$\lambda\left(\sum_{i=1}^n \alpha_i \cdot t_i\right) := \sum_{i=1}^n \alpha_i\lambda_{t_i} \in \mathcal{B}(l^2(\Gamma)).$$

The **reduced** C^*-algebra is then what we obtain by taking the closure of $\lambda(\mathbb{C}[\Gamma])$ with respect to $\|\cdot\|_{\mathcal{B}(l^2(\Gamma))}$; we denote it by $C^*_\lambda(\Gamma)$. On the other hand, the left regular representation $\lambda\colon \Gamma \to \mathcal{U}(l^2(\Gamma))$ is merely one representation of our group. Hence we can consider the norm

$$\|x\|_u = \sup\{\|\pi(x)\|_{\mathcal{B}(\mathcal{H})} \colon \pi\colon \Gamma \to \mathcal{U}(\mathcal{H}) \text{ is a } *\text{-representation}\}.$$

This easily satisfies the C^*-identity. The **full** (or universal) C^*-algebra of Γ is the closure of $\mathbb{C}[\Gamma]$ with respect to $\|\cdot\|_u$ and is denoted $C^*(\Gamma)$.

Thus assuming (5) we'll need to show that these two C^*-algebras coincide. But we first note that since $\|\lambda(x)\| \leq \|x\|_u$ for $x \in \mathbb{C}[\Gamma]$, λ extends to $C^*(\Gamma)$ (which we still denote λ). It is clear that this is onto $C^*_\lambda(\Gamma)$ (since $\mathbb{C}[\Gamma] \subset C^*(\Gamma)$). We'll need the following:

Definition 1.9. Let $\varphi\colon \Gamma \to \mathbb{C}$ be a function. The associated **multiplier** $m_\varphi\colon \mathbb{C}[\Gamma] \to \mathbb{C}[\Gamma]$ is defined by

$$m_\varphi\left(\sum_{t\in\Gamma} \alpha_t \cdot t\right) := \sum_{t\in\Gamma} \varphi(t)\alpha_t \cdot t.$$

We also define $\tilde{m}_\varphi \colon \lambda(\mathbb{C}[\Gamma]) \to \lambda(\mathbb{C}[\Gamma])$ by

$$\tilde{m}_\varphi\left(\lambda\left(\sum_{t\in\Gamma}\alpha_t\cdot t\right)\right) = \tilde{m}_\varphi\left(\sum_{t\in\Gamma}\alpha_t\lambda_t\right) = \sum_{t\in\Gamma}\varphi(t)\alpha_t\lambda_t$$

Lemma 1.10. *Suppose φ is finitely supported, positive definite, and $\varphi(e) = 1$. Then m_φ extends to a continuous map on $C^*(\Gamma)$ and $\tilde{m}_\varphi$ extends to a continuous map on $C^*_\lambda(\Gamma)$ both with norm one.*

Proof. First consider the case when $\varphi = \delta_e$ (i.e. $\varphi(t) = 1$ if $t = e$ and $\varphi(t) = 0$ otherwise). Let $\tau(x) = \langle\lambda(x)\delta_e, \delta_e\rangle_{l^2(\Gamma)}$ for $x \in C^*(\Gamma)$, then τ is a tracial state. For $x \in \mathbb{C}[\Gamma]$ we compute:

$$\|m_\varphi(x)\|_u = \|\tau(x)\cdot e\|_u = |\tau(x)|\|e\|_u = |\tau(x)| \le \|x\|_u.$$

Hence we can extend m_φ to $C^*(\Gamma)$ with norm one.

Let $\tilde{\tau}(T) = \langle T\delta_e, \delta_e\rangle_{l^2(\Gamma)}$ for $T \in \mathcal{B}(l^2(\Gamma))$, then $\tilde{\tau}$ is a tracial state. For $x \in \mathbb{C}[\Gamma]$ we compute:

$$\|\tilde{m}_\varphi(\lambda(x))\| = \|\tilde{\tau}(\lambda(x))\lambda_e\| = |\tilde{\tau}(\lambda(x))|\|\lambda_e\| = |\tilde{\tau}(\lambda(x))| \le \|\lambda(x)\|.$$

so that $\tilde{m}_\varphi$ extends to $C^*_\lambda(\Gamma)$ with norm one.

Next consider the case $\varphi = \delta_t$ for $t \in \Gamma$. Since

$$\begin{aligned}\|m_\varphi(x)\|_u &= \|\langle\lambda(x)\delta_e, \delta_t\rangle\cdot t\|_u = \|\langle\lambda(x)\delta_e, \lambda_t\delta_e\rangle\cdot t\|_u = \|\langle\lambda_{t^{-1}}\lambda(x)\delta_e, \delta_e\rangle\cdot t\|_u\\ &= \|\tau(t^{-1}x)\cdot t\|_u = |\tau(t^{-1}x)|\|t\|_u \le \|t^{-1}x\|_u \le \|x\|_u,\end{aligned}$$

we see that m_φ again extends to $C^*(\Gamma)$ with norm one. A similar computation for $\tilde{m}_\varphi$ involving $\tilde{\tau}$ yields an extension in the reduced C^*-algebra case as well.

Thus for a finitely supported φ we can write $\varphi = \sum_{t\in\Gamma}\varphi(t)\delta_t$ and so $m_\varphi = \sum_{t\in\Gamma}\varphi(t)m_{\delta_t}$. Extending each of the finitely many m_{δ_t} yields an extension for m_φ. But since φ is positive definite, m_φ is positive and hence attains its norm at the identity: $\|m_\varphi\| = \|m_\varphi(e)\| = |\varphi(e)| = 1$. A similar argument applies in the reduced C^*-algebra case. ☐

Proof of (5) $\Rightarrow$ (6). By our previous comments, we know $\lambda\colon C^*(\Gamma) \to C^*_\lambda(\Gamma)$ is onto and hence it remains to show λ is injective.

Let (φ_i) be the net in condition (5). By the above lemma, we can define multipliers m_{φ_i} and $\tilde{m}_{\varphi_i}$ on $C^*(\Gamma)$ and $C^*_\lambda(\Gamma)$ respectively, each with norm one. We note that $\lambda\circ m_{\varphi_i} = \tilde{m}_{\varphi_i}\circ\lambda$ on $C^*(\Gamma)$ since both functions are continuous and agree on the dense subspace $\mathbb{C}[\Gamma]$. Now, since $\varphi_i \to 1$ pointwise on Γ, $m_{\varphi_i}(x) \to x$ for $x \in \mathbb{C}[\Gamma]$. Since the norms of the m_{φ_i} are uniformly bounded by one and $\mathbb{C}[\Gamma]$ is dense in $C^*(\Gamma)$, this limit holds for $x \in C^*(\Gamma)$ as well.

Now, suppose $x \in C^*(\Gamma)$ and $\lambda(x) = 0$. Then

$$\lambda(m_{\varphi_i}(x)) = \tilde{m}_{\varphi_i}(\lambda(x)) = 0,$$

for every i. But since φ_i is finitely supported we know $m_{\varphi_i}(x) \in \mathbb{C}[\Gamma]$ and hence $\lambda(m_{\varphi_i}(x)) = 0$ implies $m_{\varphi_i}(x) = 0$. Hence $x = \lim_i m_{\varphi_i}(x) = 0$ and so λ is injective. ☐

Proof of (6) $\Rightarrow$ (7). $C^*(\Gamma)$ always has a one-dimensional representation since the trivial representation $\mathbb{C}[\Gamma] \ni \sum_t\alpha_t\cdot t \mapsto \sum_t\alpha_t \in \mathbb{C}$ is always subordinate to $\|\cdot\|_u$ (as all $*$-representations are). Hence $C^*_\lambda(\Gamma) = C^*(\Gamma 0$ has a character. ☐

We require a lemma:

Lemma 1.11. *Let A be a unital C^*-algebra and $\varphi\colon A \to \mathbb{C}$ a state. If $x \in A$ satisfies $\varphi(x^*x) = |\varphi(x)|^2$ then for all $y \in A$*

$$\varphi(xy) = \varphi(x)\varphi(y) = \varphi(yx).$$

Proof. Let $(\pi_\varphi, \xi, \mathcal{H})$ be a GNS representation. Then

$$\|\pi_\varphi(x)\xi\| = \langle\pi_\varphi(x)\xi, \pi_\varphi(x)\xi\rangle = \langle\pi_\varphi(x^*x)\xi, \xi\rangle = \varphi(x^*x) = |\varphi(x)|^2 = |\langle\pi_\varphi(x)\xi, \xi\rangle|^2 \le \|\pi_\varphi(x)\xi\|^2,$$

where the last inequality comes from applying Cauchy-Schwarz. However, we in fact have equality and we know that equality occurs in Cauchy-Schwarz only when the two vectors are scalar multiples of one another. Hence $\pi_\varphi(x)\xi = \lambda\xi$ for some $\lambda \in \mathbb{C}$. In fact,

$$\lambda = \langle\lambda\xi, \xi\rangle = \langle\pi_\varphi(x)\xi, \xi\rangle = \varphi(x).$$

Now, for any $y \in A$ we have

$$\varphi(yx) = \langle \pi_\varphi(yx)\xi, \xi\rangle = \langle \lambda\pi_\varphi(y)\xi, \xi\rangle = \lambda\varphi(y) = \varphi(x)\varphi(y).$$

A similar computation using the fact that $\varphi(x^*) = \overline{\varphi(x)} = \overline{\lambda}$ show this is also equal to $\varphi(xy)$. $\square$

Proof of (7) $\Rightarrow$ (1). Let $\tau\colon C^*_\lambda(\Gamma) \to \mathbb{C}$ be a character. Then in particular it is also a state. Thus using Hahn-Banach we can extend it as a linear functional to $\mathcal{B}(l^2(\Gamma))$ in a norm preserving manner. But then, since it attains it's norm at the identity (since τ did initially) it is positive. Denote this extension by φ. We note that for all $s \in \Gamma$, $\varphi(\lambda_s^*\lambda_s) = \varphi(1) = 1 = |\tau(\lambda_s)|^2 = |\varphi(\lambda_s)|^2$. Now, given $f \in l^\infty(\Gamma)$, we note that $s.f = \lambda_s f \lambda_s^*$ as operators on $l^2(\Gamma)$. Indeed, let $\xi \in l^2(\Gamma)$ then

$$[(\lambda_s f\lambda_s^*)\xi](t) = [(f\lambda_s^*)\xi](s^{-1}t) = f(s^{-1}t)[\lambda_s^*\xi](s^{-1}t) = f(s^{-1}t)\xi(t) = (s.f)(t)\xi(t) = [s.f\xi](t),$$

for all t. Hence $(\lambda_s f\lambda_s^*)\xi = s.f\xi$. Consequently,

$$\varphi(s.f) = \varphi(\lambda_s f\lambda_s^*) = \varphi(\lambda_s)\varphi(f)\varphi(\lambda_s^*) = \varphi(f),$$

where we have applied the above lemma in the second equality. This means $\varphi|_{l^\infty(\Gamma)}$ is a state on $l^\infty(\Gamma)$ which is invariant under the left translation action; that is, Γ is amenable. $\square$

Proof of (4) $\Leftrightarrow$ (8). First assume the conditions in (4). Let $E \subset \Gamma$ be a finite subset. Then

$$\left\| \frac{1}{|E|}\sum_{s\in E}\lambda_s\right\|_\infty \leq \frac{1}{|E|}\sum_{s\in E}\|\lambda_s\|_\infty = 1,$$

and

$$1 = \lim_i \left\|\frac{1}{|E|}\sum_{s\in E}\xi_i\right\|_2 = \lim_i \left\|\frac{1}{|E|}\sum_{s\in E}\lambda_s\xi_i\right\|_2 \leq \left\|\frac{1}{|E|}\sum_{s\in E}\lambda_s\right\|_\infty,$$

so (8) holds.

Conversely, suppose (8) holds. Let E be a finite and symmetric set ($E = E^{-1}$). Then the operator $S := \frac{1}{|E|}\sum_{s\in E}\lambda_s$ has norm one and is self-adjoint. Hence for any $\epsilon > 0$ we can find a unit vector $\xi \in l^2(\Gamma)$ such that $|\langle S\xi, \xi\rangle| > 1-\epsilon$. Let $|\xi|$ be the pointwise absolute value of ξ, then

$$\begin{aligned} 1-\epsilon < |\langle S\xi,\xi\rangle| = \left|\sum_{t\in\Gamma}(S\xi)(t)\overline{\xi(t)}\right| \leq \sum_{t\in\Gamma}|(S\xi)(t)||\xi(t)| \leq \sum_{t\in\Gamma}(S|\xi|)(t)|\xi|(t) &= \langle S|\xi|, |\xi|\rangle \\ &= \frac{1}{|E|}\sum_{s\in E}\langle \lambda_s|\xi|, |\xi|\rangle. \end{aligned}$$

Since each $\langle\lambda_s|\xi|, |\xi|\rangle \leq 1$ and $|E|$ is fixed, for sufficiently small ϵ all the numbers $\langle\lambda_s|\xi|, |\xi|\rangle$ must be close to 1. And since

$$\|\lambda_s|\xi| - |\xi|\|^2 = \|\lambda_s|\xi|\|^2 + \||\xi|\|^2 - 2\langle\lambda_s|\xi|, |\xi|\rangle = 2 - 2\langle\lambda_s|\xi|, |\xi|\rangle,$$

we can make $\|\lambda_s|\xi| - |\xi|\|$ small. Let $\xi_{(E,\epsilon)}$ be $|\xi|$ as above, corresponding to E and ϵ. Ordering the net $\{\xi_{(E,\epsilon)}$ by $(E,\epsilon) \leq (F,\delta)$ iff $E \subset F$ and $\epsilon \geq \delta$, gives us (4). $\square$

This finishes the proof of the theorem. We finish with a way to prove (4) $\Rightarrow$ (6), if the reader finds condition (5) unnecessary. We first need a lemma:

Lemma 1.12 (Fell's Absorption Principle)**.** *Let π be a unitary representation of Γ on $\mathcal{H}$. Then, $\lambda\otimes\pi$ is unitarily equivalent to $\lambda\otimes 1_{\mathcal{H}}$.*

Here $\lambda\otimes\pi(t)$ is an operator on $l^2(\Gamma,\mathcal{H})$, i.e. l^2 functions on Γ with image in $\mathcal{H}$. Such functions are usually written as $\sum_t \delta_t\otimes\xi_t$ where $\delta_t \in l^2(\Gamma)$ and $\xi_t \in \mathcal{H}$. Hence $(\lambda\otimes\pi(s))(\sum_t\delta_t\otimes\xi_t) = \sum_t(\lambda_s\delta_t)\otimes(\pi(s)\xi_t)$.

Proof. Define a unitary U on $l^2(\Gamma)\otimes\mathcal{H}$ by

$$\sum_{t\in\Gamma}\delta_t\otimes\xi_t \mapsto \sum_t \delta_t\otimes\pi(t)\xi_t.$$

This is unitary since π is a unitary representation. We compute

$$\begin{aligned} U(\lambda \otimes 1(s))\left(\sum_t \delta_t \otimes \xi_t\right) &= U\left(\sum_t (\lambda_s \delta_t) \otimes \xi_t\right) = U\left(\sum_t \delta_{st} \otimes \xi_t\right) \\ &= U\left(\sum_r \delta_r \otimes \xi_{s^{-1}r}\right) = \sum_r \delta_r \otimes \pi(r)\xi_{s^{-1}r}, \end{aligned}$$

and

$$(\lambda \otimes \pi(s))U\left(\sum_t \delta_t \otimes \xi_t\right) = (\lambda \otimes \pi(s))\left(\sum_t \delta_t \otimes \pi(t)\xi_t\right) = \sum_t \delta_{st} \otimes \pi(st)\xi_t = \sum_r \delta_r \pi(r)\xi_{s^{-1}r}.$$

□

Proof (4) $\Rightarrow$ (6). Let $\pi \colon \Gamma \to \mathcal{U}(\mathcal{H})$ be a $*$-representation. Let (ξ_i) be the net from condition (4). For $x \in \mathbb{C}[\Gamma]$ and $\eta \in \mathcal{H}$ a unit vector we have

$$\langle \pi(x)\eta, \eta \rangle = \lim_i \langle (\pi \otimes \lambda(x))(\eta \otimes \xi_i, \eta \otimes \xi_i) \rangle \leq \|\pi \otimes \lambda(x)\| = \|1 \otimes \lambda(x)\| = \|\lambda(x)\|,$$

where we have used Fell's absorption principle in the second to last inequality. Thus

$$\|\pi(x)\| = \sup_{\|\eta\|=1} |\langle \pi(x)\eta, \eta \rangle| \leq \|\lambda(x)\|.$$

As π was an arbitrary $*$-representation, this shows $\|x\|_u \leq \|\lambda(x)\|$. □

References

[1] N.P. Brown and N. Ozawa; *C^*-algebras and finite-dimensional approximations*, Grad. Stud. Math., 88, Amer. Math. Soc., Providence, RI, 2008