Strongly hyperbolic second order Einstein's evolution equations

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BSSN-type evolution equations are discussed. The name refers to the Baumgarte, Shapiro, Shibata, and Nakamura version of the Einstein evolution equations, without introducing the conformal-traceless decomposition but keeping the three connection functions and including a densitized lapse. It is proved that a pseudodifferential first order reduction of these equations is strongly hyperbolic. In the same way, densitized Arnowitt-Deser-Misner evolution equations are found to be weakly hyperbolic. In both cases, the positive densitized lapse function and the spacelike shift vector are arbitrary given fields. This first order pseudodifferential reduction adds no extra equations to the system and so no extra constraints.

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I. INTRODUCTION

Einstein's equation determines geometries; hence its solutions are equivalent classes under space-time diffeomorphisms of metric tensors. It is this invariance, however, which imposes a particular aftermath on every initial value formulation for Einstein's equation. The geometrical equation must be first converted into a system having a well posed Cauchy problem, and so without the diffeomorphism invariance. A preferred foliation of spacelike hypersurfaces on the space-time is usually introduced in order that adapted coordinates break this invariance. Einstein's equation is then decomposed into constraint equations on the foliation hypersurfaces and evolution equations. While the constraints are uniquely determined by this procedure, the evolution equations are not. Some of these evolution equations turn out to be hyperbolic. This is in accordance with a main aspect of general relativity, that of causal propagation of the gravitational field.

Hyperbolicity refers to algebraic conditions on the principal part of the equations which imply well posedness for the Cauchy problem, that is, the existence of a unique continuous map between solutions and initial data. There are several notions of hyperbolicity, which are related to different algebraic conditions. Some notions imply well posedness for the Cauchy problem in constant coefficient equations but not in more general systems, such as quasilinear equations. See [1,2] for reviews intended for researchers on general relativity. Regarding quasilinear systems, strong hyperbolicity is one of the more general notions of hyperbolicity that implies well posedness for the Cauchy problem. The proof involves pseudodifferential analysis [3,4]. Symmetric hyperbolic systems are a particular case of strongly hyperbolic systems where well posedness can be proved without using pseudodifferential techniques. Several equations from physics can be cast into this symmetric hyperbolic form [5]. Finally, weak hyperbolicity is a less rigid notion than strong hyperbolicity but it does not imply well posedness for quasilinear equations.

A definition of strong hyperbolicity for pseudodifferential first order systems is introduced in Sec. II. Differential first order strongly hyperbolic systems known in the literature are included. A reason for this definition is that it incorporates precisely the hypothesis needed for proving well posedness. The proof involves standard pseudodifferential techniques. If an *m*-order differential system has a first order, differential or pseudodifferential, strongly hyperbolic reduction, then it is well posed. See the end of this section for an example of a first order pseudodifferential reduction of the wave equation. Also see the Appendix for a brief and self-contained introduction to the subject of pseudodifferential operators.

Although strongly hyperbolic systems are at the core of the various proofs of well posedness for the Cauchy problem in general relativity, they have played, until recently, no similar role in numerical relativity [6]. Finite difference schemes have been implemented for non-strongly-hyperbolic equations. However, Lax's equivalence theorem does not hold in these situations [7]. It has been shown that discretization schemes standard in numerical relativity are not convergent when applied to weakly hyperbolic and ill posed systems [8,9]. As more complicated situations are studied numerically, the interest in strongly hyperbolic reductions of Einstein's equation is increasing. There is also much experience and a vast literature in numerical schemes based on well posed formulations coming from inviscid hydrodynamics [10,11]. This experience can be transferred into numerical relativity when strongly hyperbolic reductions are used.

Early numerical schemes to solve Einstein's equation were based on variants of the Arnowitt-Deser-Misner (ADM) decomposition [12]. Only recently has it been proven that a first order differential reduction of the ADM evolution equations is weakly hyperbolic [13]. This is the reason for some of the instabilities observed in ADM-based numerical schemes [8,9,14]. In the first part of this work the ADM evolution equations are reviewed. A densitized lapse function

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is introduced, with the density exponent held as a free parameter. The evolution equations are reduced to first order in a pseudodifferential way. It is found that the resulting system is weakly hyperbolic for every prescription of a positive density lapse and a spacelike shift vector. This is summarized in Theorem 1. Pseudodifferential techniques and mode decomposition are at the core of a proof directed to computing the eigenvalues and eigenvectors of the principal symbol of the evolution equations, and to check that the eigenvectors do not span the whole eigenspace. The mode decomposition also helps to understand why the addition of the Hamiltonian constraint into the system does not produce a strongly hyperbolic system.

Baumgarte-Shapiro-Shibata-Nakamura- (BSSN-)type systems are introduced in the second part of this work. They are essentially the densitized ADM evolution equations where some combination of connection coefficients of the three-metric is introduced as a new variable. The mode decomposition of the densitized ADM evolution equations and previous work on the linearized ADM equations [15] suggest the introduction of this variable. It turns out to be related with the variable $\tilde{\Gamma}^i$ of the BSSN system, defined by Eq. (21) in [14]. (See also [16].) Similar variables have been introduced in [17–21]. Their evolution equation is obtained, as in the BSSN system, from commuting derivatives and then adding the momentum constraint. It is shown here that the addition of the momentum constraint transforms a weakly hyperbolic system into a strongly hyperbolic one. This is the main result of this work, and it is presented in Theorem 2. The first part of the proof follows the previous one for the densitized ADM evolution equations. Once the eigenvectors are computed, and it is verified that they do span the whole eigenspace, the proof continues with the construction of the symmetrizer. This construction is carried out with the eigenvectors. Finally, the smooth properties of the symmetrizer are verified.

The hyperbolicity of a family of BSSN-type evolution equations has previously been studied with a different technique [22]. The equations were reduced to a differential system of first order in time and in space derivatives. The lapse was densitized, the eigenvalues and eigenvectors of the principal symbol were computed, and it was verified that the latter do span their eigenspace. A smooth symmetrizer was computed for a subfamily of systems, showing strong hyperbolicity in this case.

All notions of hyperbolicity mentioned here require rewriting the evolution equations as a first order system. This can be done in a differential or pseudodifferential way. Some pseudodifferential reductions to first order have the advantage that no extra equations are added into the system, so there are no extra constraints. This reduces the algebra needed to compute the symmetrizer. These techniques are well known in the field of pseudodifferential calculus. They were first used in general relativity in [15], where linearized ADM evolution equations were proved to be weakly hyperbolic. The example below presents the wave equation as a toy model to understand how the pseudodifferential first order reduction works. See Appendix A, and also Sec. 5.3 in [4], for other possible first order reductions. Consider the wave equation on $\mathbb{R}^4$ for a function *h*, written as a first order system in time, in appropriate coordinates, that is,

$$\partial_t h = k, \quad \partial_t k = \Delta h,$$

with $t \in [0,\infty)$, $x^i \in \mathbb{R}^3$, and $\Delta = \delta^{ij}\partial_i\partial_j$ the flat Laplacian. Here $\delta^{ij} = \mathrm{diag}(1,1,1)$. Fourier transform the system in x^i,

$$\partial_t \hat{h} = \hat{k}, \quad \partial_t \hat{k} = -|\omega|^2 \hat{h}, \tag{1}$$

where $\hat{h}(t,\omega_i)$ is the Fourier transform of $h(t,x^i)$ in the space variables, as defined in the Appendix. The function $\hat{k}$ is defined in an analogous way, and $|\omega|^2 = \delta^{ij}\omega_i\omega_j$. The key step is to rewrite Eqs. (1) as a first order system by introducing the unknown $\hat{\ell} := i|\omega|\hat{h}$, where $i|\omega|$ is the symbol of the pseudodifferential operator square root of the Laplacian. One gets

$$\partial_t \hat{\ell} = i|\omega|\hat{k}, \quad \partial_t \hat{k} = i|\omega|\hat{\ell}.$$

The system so obtained is a reduction to first order of the original first order in time wave equation. Notice that there is no increase in the number of unknowns, just a replacement of $\hat{h}$ by $\hat{\ell}$, and correspondingly no extra constraints are introduced. For the wave equation the result is a symmetric hyperbolic system. In the case of the ADM equations the resulting system is weakly hyperbolic, with or without densitizing the lapse function, while for the BSSN-type equations one gets a strongly hyperbolic system.

In Sec. II a precise definition for well posedness is introduced for first order quasilinear pseudodifferential systems. Strongly hyperbolic systems are also defined and the main theorem asserting well posedness for these systems is reviewed. Section III A is dedicated to reviewing the densitized ADM equations. The main result here is Theorem 1, asserting that the resulting evolution equations are weakly hyperbolic. The role of adding the Hamiltonian constraint is briefly discussed. Section III B is dedicated to introducing the BSSN-type system modifying the densitized ADM equations. The main result of this work, Theorem 2, asserts that this BSSN-type system is strongly hyperbolic for some choices of the free parameters. The key point is the introduction of the momentum constraint into the evolution equations. Section IV summarizes these results briefly. The Appendix is an introduction to pseudodifferential calculus. It summarizes the main ideas and highlights the main results. It is intended for physicists interested in learning the subject. It provides all the background knowledge to follow the calculations presented in this work. A summary of this type of pseudodifferential calculus was not found by the authors in the specialized literature.

II. WELL POSEDNESS

Hadamard first introduced the concept of well posedness for a Cauchy problem. It essentially says that a well posed problem should have a solution, that this solution should be unique, and that it should depend continuously on the data of the problem. The first two requirements are clear, but the last one needs additional specifications. First, there is no unique way to prescribe this notion of continuity. Although a topological space is all that is needed to introduce it, Banach spaces are present in most definitions of well posedness. This refinement simplifies the analysis while still including a large class of problems. Second, as nonlinear systems lack, in general, global in time solutions, one can at most expect a local in time notion of well posedness. Discussions on well posedness can be found in [10,11] and [3,4]. See [1] for a summary. This section is dedicated to reviewing the minimum set of definitions and results on well posedness for quasilinear pseudodifferential strongly hyperbolic systems which are needed to describe the equations coming from general relativity. It assumes that the reader is acquainted with the notions from functional analysis and pseudodifferential calculus given in the Appendix.

Consider the Cauchy problem for a quasilinear first order pseudodifferential system

$$\partial_t u = p(t,x,u,\partial_x)u, \quad u|_{t=0} = f, \tag{2}$$

where u,f are m-dimensional vector valued functions, $m \geqslant 1$, and x represents Cartesian coordinates in $\mathbb{R}^n$, $n \geqslant 1$. Here $p(t,x,v,\partial_x)$ is a smooth family of pseudodifferential operators in $\psi^1_{\rm cl}$, parametrized by $t \in \mathbb{R}^+$ and $v \in \mathbb{R}^m$. Let $p(t,x,v,\omega)$ and $\underline{p}_1(t,x,v,\omega)$ be their symbols and principal symbols, respectively.

If p is a differential operator with analytic coefficients, then the Cauchy-Kowalewski theorem asserts that there exists a unique solution for every analytic data f. However, solutions corresponding to smooth data behave very differently depending on the type of operator p. For example, write the flat Laplace equation in $\mathbb{R}^{n+1}$ and the flat wave equation in $\mathbb{R}^{n+1}$ as first order systems in the form $\partial_t u = A^i \partial_i u$. The matrices A^i are skew symmetric for the Laplacian, and symmetric for the wave operator. Therefore, solutions of the form $u(t,x) = \hat{u}(t) e^{i\omega \cdot x}$ for the corresponding Cauchy problems behave very differently at the high frequency limit. The solutions of the Cauchy problem for the Laplace equation diverge in the limit $|\omega| \to \infty$, while the solutions of the wave equation do not diverge in that limit.[1]

[1] An explicit example in $\mathbb{R}^2$, presented by Hadamard in [28], may clarify this. Consider the functions

$$v(t,x) = \sin(nt)\sin(nx)/n^{p+1},$$
$$w(t,x) = \sinh(nt)\sin(nx)/n^{p+1},$$

with $p \geqslant 1, n$, constants, defined on $t \geqslant 0$, $x \in [0,1]$. They are solutions of the Cauchy problem for wave equation and the Laplace equation, respectively, with precisely the same Cauchy data on $t = 0$, that is,

$$v_{tt} - v_{xx} = 0, \quad v|_{t=0} = 0, \quad v_t|_{t=0} = \sin(nx)/n^p,$$
$$w_{tt} + w_{xx} = 0, \quad w|_{t=0} = 0, \quad w_t|_{t=0} = \sin(nx)/n^p.$$

As $n \to \infty$, the Cauchy data converge to zero in $C^{p-1}([0,1])$. In this limit, the solution of the wave equation converges to zero, while the solution of the Laplace equation diverges. The concept of well posedness is introduced in order to capture this behavior of the wave equation's solution under high frequency perturbations on its Cauchy data.

Let $B(\mathbb{R}^n)$ be a Banach space with norm $\| \, \|$, whose elements are vector valued functions from $\mathbb{R}^n$ to $\mathbb{R}^m$. The Cauchy problem (2) is well posed in $B(\mathbb{R}^n)$ if given initial data $f(x) \in B(\mathbb{R}^n)$ there exists a solution $u(t,x)$ which is unique in $B(\mathbb{R}^n)$ for each $t \in [0,T)$, for some $T > 0$; and given any number $\epsilon > 0$ there exists $\delta > 0$ such that, for every data $\tilde{f}(x) \in B(\mathbb{R}^n)$ satisfying $\|\tilde{f} - f\| < \delta$ there exists a unique solution $\tilde{u}(t,x) \in B(\mathbb{R}^n)$ for $t \in [0,\tilde{T}) \times \mathbb{R}^n$ for some $\tilde{T} > 0$, with $|\tilde{T} - T| < \epsilon$, and satisfying $\|\tilde{u}(t) - u(t)\| < \epsilon$, for all $t \in [0, \min(\tilde{T}, T))$. This means that the solution depends continuously on the data in the norm $\| \, \|$.

Well posedness is essentially a statement about the behavior of the solutions of a Cauchy problem under high frequency perturbations of the initial data. Here is where pseudodifferential calculus is most useful to study solutions of the Cauchy problem. The high frequency part of the solution can be determined by studying the higher order terms in the asymptotic expansion of symbols.

A wide class of operators with well posed Cauchy problem is called strongly hyperbolic. A first order pseudodifferential system (2) is strongly hyperbolic if $p \in \psi^1_{\rm cl}$ and the principal symbol is symmetrizable. This means that there exists a positive definite, Hermitian operator $H(t,x,\omega)$ homogeneous of degree zero in ω, smooth in all its arguments for $\omega \neq 0$, such that

$$(H\underline{p}_1 + \underline{p}_1^* H) \in S^0,$$

where $\underline{p}_1^*$ is the adjoint of the principal symbol $\underline{p}_1$.

The definition summarizes all the hypotheses on quasilinear systems needed to prove well posedness. It is the definition given in Sec. 3.3.1 in [10] for linear variable coefficient systems, and the so-called symmetrizable quasilinear systems given in Sec. 5.2 in [4].

Consider first order differential systems of the form $\partial_t u = A^i(t,x)\partial_i u + B(t,x)u$. The symbol is $p(t,x,\omega) = iA^j(t,x)\omega_j + B(t,x)$, and the principal symbol is $\underline{p}_1(t,x,\omega) = iA^j\omega_j$. If the matrices A^i are all symmetric, then the system is called symmetric hyperbolic. The symmetrizer H is the identity, and $\underline{p}_1 + \underline{p}_1^* = 0$. The wave equation on a fixed background, written as a first order system is an example of a symmetric hyperbolic system. Well posedness for symmetric hyperbolic systems can be shown without pseudodifferential calculus. The basic energy estimate can be obtained by integration by parts in space-time.

If the matrices A^i are symmetrizable, then the differential system is called strongly hyperbolic. The symmetrizer $H = H(t,x,\omega)$ is assumed to depend smoothly on ω. Every symmetric hyperbolic system is strongly hyperbolic. Pseudodifferential calculus must be used to show well posedness for variable coefficient strongly hyperbolic systems that are not symmetric hyperbolic [3]. The definition given two paragraphs above is more general because the symbol does not need to be a polynomial in ω. The definition given above includes first order pseudodifferential reductions of second order differential systems. These type of reductions are performed with operators like Λ, λ, or ℓ, defined in the Appendix.

In the particular case of constant coefficient systems there exists in the literature a more general definition of strong hyperbolicity [10,11]. The principal symbol $\underline{p_1}$ must have only imaginary eigenvalues, and a complete set of linearly independent eigenvectors. The latter must be uniformly linear independent in $\omega \neq 0$ over the whole integration region. Kreiss's matrix theorem (see Sec. 2.3 in [10]) says that this definition is equivalent to the existence of a symmetrizer H. Nothing is known about the smoothness of H with respect to t, x, and ω. The existence of this symmetrizer is equivalent to well posedness for constant coefficient systems. However, the proof of well posedness for variable coefficient and quasilinear systems does require the smoothness of the symmetrizer. There are examples showing that this smoothness does not follow from the previous hypothesis on eigenvalues and eigenvectors of $\underline{p_1}$. Because it is not known what additional hypothesis on the latter could imply this smoothness, one has to include it into the definition of strong hyperbolicity for nonconstant coefficient systems.

A more fragile notion of hyperbolicity is called weak hyperbolicity, where the operator $\underline{p_1}$ has imaginary eigenvalues, but nothing is required of its eigenvectors. Quasilinear weakly hyperbolic systems are not well posed. The following example gives an idea of the problem. The 2×2 system $\partial_t u = A\partial_x u$ with $t,x \in \mathbb{R}$ and

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

is weakly hyperbolic. Plane wave solutions of the form $u(t,x) = \hat{u}(t)e^{i\omega\cdot x}$ satisfy $|\hat{u}(t)| \leq |\hat{u}(0)|(1+|\omega|t)$. Therefore, plane wave solutions to a weakly hyperbolic system do not diverge exponentially in the high frequency limit (as in the case of Cauchy problem for the Laplace equation) but only polynomically. This divergence causes solutions to variable coefficient weak hyperbolic systems to be unstable under perturbations in the lower order terms of the operator, as well as in the initial data.

The main theorem about well posedness for strongly hyperbolic systems is the following. The Cauchy problem (2) for a strongly hyperbolic system is well posed with respect to the Sobolev norm $\| \, \|_s$ with $s > n/2+1$. The solution belongs to $C([0,T),H^s)$, and $T>0$ depends only on $\|f\|_s$.

In the case of strongly hyperbolic differential systems, this is Theorem 5.2.D in [4]. The proof for pseudodifferential strongly hyperbolic systems is essentially the same. One builds an estimate for the solution in a norm, defined using the symmetrizer, equivalent to the Sobolev norm H^s. Then the argument follows the standard proof for differential systems. The construction of the symmetrizer is basically the one carried out in [23].

III. ADM DECOMPOSITION OF EINSTEIN'S EQUATION

The ADM decomposition of Einstein's equation is reviewed. The densitized lapse function is introduced in Sec.III A. Theorem 1 says that the resulting evolution equations are weakly hyperbolic, for every choice of a positive densitized lapse function and spacelike shift vector as given fields. The BSSN-type system is introduced in Sec. III B. It is essentially the system given in [14,16] without conformal-traceless decomposition, keeping the three connection functions and densitizing the lapse function. It is reduced to a first order pseudodifferential system like the ADM evolution equations. The main result, Theorem 2, asserts that BSSN-type equations are strongly hyperbolic, for every positive densitized lapse function and spacelike shift vector.

Let (M,g_{ab}) be a space-time solution of Einstein's equation. That is a four-dimensional, smooth, orientable manifold M, and a smooth, Lorentzian metric g_{ab} solution of

$$G_{ab} = \kappa T_{ab},$$

with $G_{ab} = R_{ab} - Rg_{ab}/2$ the Einstein tensor, T_{ab} the stress-energy tensor, and $\kappa = 8\pi$. Ricci's tensor is R_{ab} and R denotes Ricci's scalar. Latin indices a,b,c,d denote abstract indices, and they are raised and lowered with g^{ab} and g_{ab}, respectively, with $g_{ac}g^{cb} = \delta_a{}^b$. The unique torsion-free metric connection is denoted by ∇_a. The conventions throughout this work are $2\nabla_{[a}\nabla_{b]}v_c = R_{abc}{}^d v_d$ for Riemann's tensor and $(-,+,+,+)$ for the metric signature.

Prescribe on M a foliation of spacelike hypersurfaces by introducing a time function t, which is a scalar function satisfying the condition that $\nabla_a t$ is everywhere timelike. Denote the foliation by S_t, and by n_a the unit normal to S_t such that $n^a = g^{ab}n_b$ is future directed. Therefore, $n_a = -N\nabla_a t$ for some positive function N. Fixing the foliation determines its first and second fundamental forms $h_{ab} = g_{ab} + n_a n_b$ and $k_{ab} = -h_a{}^c\nabla_c n_b$, respectively. Decompose Einstein's equation into evolution equations (3), (4) and constraint equations (5), (6), as follows:

$$\mathcal{L}_n h_{ab} = -2k_{ab}, \tag{3}$$

$$\mathcal{L}_n k_{ab} = {}^{(3)}R_{ab} - 2k_a{}^c k_{bc} + kk_{ab} - (D_a D_b N)/N - \kappa S_{ab}, \tag{4}$$

$$D_b k_a{}^b - D_a k = \kappa j_a, \tag{5}$$

$${}^{(3)}R + k^2 - k_{ab}k^{ab} = 2\kappa\rho, \tag{6}$$

where $\mathcal{L}_n$ denotes the Lie derivative along n^a, and $k = k_a{}^a$. Here ${}^{(3)}R_{ab}$, ${}^{(3)}R$, and D_a are, respectively, the Ricci tensor, the Ricci scalar, and the Levi-Cività connection of h_{ab}, while h^{ab} denotes its inverse. The stress-energy tensor is decomposed as $S_{ab} = (h_a{}^c h_b{}^d T_{cd} - Th_{ab}/2)$, with $T = T_{ab}g^{ab}$, $j_a = -h_a{}^c n^d T_{cd}$, and $\rho = T_{ab}n^a n^b$.

Introduce on M a future-directed timelike vector field t^a. Impose the additional condition $t^a\nabla_a t = 1$, that is, the integral lines of t^a are parametrized precisely by t. This condition implies that the orthogonal decomposition of t^a with respect to S_t has the form $t^a = Nn^a + \beta^a$, with $n_a\beta^a = 0$. N is called the lapse function and β^a the shift vector. The integral lines of t^a determine a diffeomorphism among the hypersurfaces S_t. This, in turn, determines a coordinate system on M from a coordinate system on S_0. Lie derivatives with respect to n^a can be rewritten in terms of t^a and β^a. The resulting equations are called the ADM decomposition of Einstein's equation.

A. Densitized ADM equations

Consider the ADM decomposition of Einstein's equation. Let x^μ be a coordinate system adapted to the foliation S_t, where $x^0 = t$ and x^i are intrinsic coordinates on each S_t that remain constant along the integral lines of t^a. Greek indices take values 0, 1, 2, 3, and latin indices i,j,k,l, take values 1, 2, 3. In these coordinates, $t^\mu = \delta_0{}^\mu$, $n_\mu = -N\delta_\mu{}^0$; then $\beta^\mu = \delta_i{}^\mu \beta^i$ and $n^\mu = (\delta_0{}^\mu - \beta^\mu)/N$. The components of the space-time metric have the form

$$g_{\mu\nu} = -N^2 \delta_\mu{}^0 \delta_\nu{}^0 + h_{ij}(\beta^i \delta_\mu{}^0 + \delta_\mu{}^i)(\beta^j \delta_\nu{}^0 + \delta_\nu{}^j).$$

In these coordinates

$$ {}^{(3)}R_{ij} = \frac{1}{2} h^{kl}[-\partial_k \partial_l h_{ij} - \partial_i \partial_j h_{kl} + 2\partial_k \partial_{(i} h_{j)l}] + \gamma_{ikl}\gamma_j{}^{kl} - \gamma_{ij}{}^k \gamma_{kl}{}^l,$$

where $\gamma_{\mu\nu}{}^\sigma = h_\mu{}^{\mu'} h_\nu{}^{\nu'} h_{\sigma'}{}^\sigma \Gamma_{\mu'\nu'}{}^{\sigma'}$ are the spatial components of the Christoffel symbols of $g_{\mu\nu}$, and $\gamma_i{}^{jk} := \gamma_{il}{}^k h^{jl}$.

Densitize the lapse function, that is, write it as $N = (\mathbb{h})^b Q$, where $\mathbb{h} := \sqrt{\det(h_{ij})}$, b is a constant, and Q is a given, positive function. This modifies the principal part of Eq. (4). New terms containing second spatial derivatives of h_{ij} come from $(D_i D_j N)/N$.

Summarizing, the unknowns for the densitized ADM equations are h_{ij} and k_{ij}. The evolution equations, Eqs. (3), (4), have the form

$$\mathcal{L}_{(t-\beta)} h_{ij} = -2N k_{ij}, \tag{7}$$

$$\mathcal{L}_{(t-\beta)} k_{ij} = (N/2) h^{kl}[-\partial_k \partial_l h_{ij} - (1+b)\partial_i \partial_j h_{kl} + 2\partial_k \partial_{(i} h_{j)l}] + B_{ij}, \tag{8}$$

where $\mathcal{L}_{(t-\beta)} h_{ij} = \partial_t h_{ij} - (\beta^k \partial_k h_{ij} + 2 h_{k(i}\partial_{j)}\beta^k)$, and the same holds for k_{ij}. The nonprincipal part terms are grouped in

$$B_{ij} = N[\gamma_{ikl}\gamma_j{}^{kl} - \gamma_{ij}{}^k \gamma_{kl}{}^l - 2k_i{}^l k_{jl} + k_{ij} k_l{}^l - A_{ij} - \kappa S_{ij}],$$

$$A_{ij} = a_i a_j - \gamma_{ij}{}^k a_k + \partial_i \partial_j (\ln Q) + 2b\,\gamma_{ikl}\gamma_j{}^{(kl)},$$

with $a_\mu = n^\nu \nabla_\nu n_\mu = D_\mu(\ln N)$. The relations $\ln N = b \ln \mathbb{h} + \ln Q$ and $(D_i D_j N)/N = (b/2) h^{kl} \partial_i \partial_j h_{kl} + A_{ij}$ were used.

The following result asserts that densitized ADM evolution equations are weakly hyperbolic.

Theorem 1. Fix any positive function Q, a vector field β^i, and first and second fundamental forms h_{ij}, k_{ij} on S_0. If $b \geqslant 0$, then Eqs. (7), (8) are weakly hyperbolic. If $b<0$, these equations are not hyperbolic.

The proof has two steps: first, to write down Eqs. (7), (8) as an appropriate first order pseudodifferential system, Eqs. (9), (10); second, to split the corresponding principal symbol into orthogonal parts with respect to the Fourier variable ω_i, and then to explicitly compute the associated eigenvalues and eigenvectors.

Proof.

First order reduction. Compute the symbol associated with the second order operator given by Eqs. (7), (8), that is,

$$\partial_t h_{ij} = \int_{S_t} \{-2N\hat{k}_{ij} + i\omega_k \beta^k \hat{h}_{ij} + 2\hat{h}_{l(i}\partial_{j)}\beta^l\} e^{i\omega x} \bar{d}\omega,$$

$$\partial_t k_{ij} = \int_{S_t} \{(N/2)[|\omega|_h^2 \hat{h}_{ij} + (1+b)\omega_i \omega_j h^{kl}\hat{h}_{kl} - 2\omega^k \omega_{(i}\hat{h}_{j)k}] + i\omega_k \beta^k \hat{k}_{ij} + \tilde{B}_{ij}\} e^{i\omega x} \bar{d}\omega,$$

where $\hat{h}_{ij}$ and $\hat{k}_{ij}$ denote the Fourier transforms in x^i of h_{ij} and k_{ij}, and

$$\tilde{B}_{ij} = \hat{B}_{ij} + 2\hat{k}_{l(i}\partial_{j)}\beta^l$$

denotes the terms not in the principal symbol. Here $\bar{d}\omega = d\omega/(2\pi)^{3/2}$, $|\omega|_h^2 = \omega_i \omega_j h^{ij}$, and we will use the convention $\omega^i = \omega_j h^{ij}$. Transform this second order symbol into a first order one via $\hat{\ell}_{ij} = i|\omega|_\delta \hat{h}_{ij}$, where $|\omega|_\delta^2 = \omega_i \omega_j \delta^{ij}$, with $\delta^{ij} = \mathrm{diag}(1,1,1)$. The associated first order system is then

$$\partial_t \ell_{ij} = \int_{S_t} \{i|\omega|_h[-(2N/\alpha)\hat{k}_{ij} + \tilde{\beta}\hat{\ell}_{ij}] + 2\hat{\ell}_{k(i}\partial_{j)}\beta^k\} e^{i\omega x}\bar{d}\omega, \tag{9}$$

$$\partial_t k_{ij} = \int_{S_t} \{i|\omega|_h[-(N\alpha/2)(\hat{\ell}_{ij} + (1+b)\tilde{\omega}_i \tilde{\omega}_j h^{kl}\hat{\ell}_{kl} - 2\tilde{\omega}^k \tilde{\omega}_{(i}\hat{\ell}_{j)k}) + \tilde{\beta}\hat{k}_{ij}] + \tilde{B}_{ij}\} e^{i\omega x}\bar{d}\omega, \tag{10}$$

with $\alpha = |\omega|_h / |\omega|_\delta$, $\tilde{\omega}_i = \omega_i / |\omega|_h$, $\tilde{\beta} := \tilde{\omega}_k \beta^k$, and $\ell_{ij} = \int_{S_t} i|\omega|_\delta \hat{h}_{ij} e^{i\omega x}\bar{d}\omega$. Then the symbol of equations above can be written as

$$p(t,x,u,i\omega) = i|\omega|_h \underline{p}_1(t,x,u,\omega) + \mathbb{B}(t,x,u,\omega), \tag{11}$$

where $(\mathbb{B}\hat{u})^T := (2\hat{\ell}_{k(i}\partial_{j)}\beta^k, \tilde{B}_{ij})$, $\hat{u}^T := (\hat{\ell}_{ij}, \hat{k}_{ij})$, with the upper index T meaning transpose. The principal part operator $\underline{p}_1$ can be read out from the terms inside the square brackets in Eqs. (9), (10). Notice that the definition of the principal symbol here differs from the one given in Sec. II by a factor of $i|\omega|_h$. (In particular, the eigenvalues of $\underline{p}_1$ as defined here must be real to be hyperbolic.)

Eigenvalues and eigenvectors of $\underline{p}_1$. Once the principal symbol is known, it only remains to compute its eigenvalues and eigenvectors. The assumption $\alpha = 1$ facilitates the computations. It is not a restriction since the norms $|\,|_\delta$ and $|\,|_h$ are equivalent and smoothly related, and therefore the properties of the eigenvalues and eigenvectors of the principal symbol are the same with either norm. Furthermore, one can check that if $\hat{u}^T = (\hat{\ell}_{ij}, \hat{k}_{ij})$ is an eigenvector of $\underline{p}_1(\alpha=1)$ with eigenvalue λ, then $\hat{u}^T(\alpha) = (\alpha^{-1/2}\hat{\ell}_{ij}, \alpha^{1/2}\hat{k}_{ij})$ is an eigenvector of $\underline{p}_1(\alpha)$ with the same eigenvalue λ. Therefore, from now on $\alpha = 1$ is assumed. A second suggestion for doing these calculations is to decompose the eigenvalue equation $\underline{p}_1\hat{u}=\lambda\hat{u}$ into orthogonal components with respect to $\tilde{\omega}_i$. Introduce the splitting

$$\hat{\ell}_{ij}=\tilde{\omega}_i\tilde{\omega}_j\hat{\ell}+\hat{\ell}' q_{ij}/2+2\tilde{\omega}_{(i}\hat{\ell}'_{j)}+\hat{\ell}'_{\langle ij\rangle}, \quad (12)$$

$$\hat{k}_{ij}=\tilde{\omega}_i\tilde{\omega}_j\hat{k}+\hat{k}' q_{ij}/2+2\tilde{\omega}_{(i}\hat{k}'_{j)}+\hat{k}'_{\langle ij\rangle}, \quad (13)$$

where $q_{ij}:=h_{ij}-\tilde{\omega}_i\tilde{\omega}_j$ is the orthogonal projector to $\tilde{\omega}_i$, and

$$\hat{\ell}=\tilde{\omega}^i\tilde{\omega}^j\hat{\ell}_{ij}, \quad \hat{\ell}'=q^{ij}\hat{\ell}_{ij}, \quad \hat{\ell}'_i=q_i{}^k\tilde{\omega}^l\hat{\ell}_{kl},$$

$$\hat{\ell}'_{\langle ij\rangle}=q_i{}^k q_j{}^l(\hat{\ell}_{kl}-\hat{\ell}' q_{kl}/2).$$

The same definitions hold for the $\hat{k}_{ij}$ components. This decomposition implies that $\hat{u}=\hat{u}^{(1)}+\hat{u}^{(2)}+\hat{u}^{(3)}$ where

$$\hat{u}^{(1)}=\begin{bmatrix}\tilde{\omega}_i\tilde{\omega}_j\hat{\ell}+(q_{ij}/2)\hat{\ell}'\\ \tilde{\omega}_i\tilde{\omega}_j\hat{k}+(q_{ij}/2)\hat{k}'\end{bmatrix},$$

$$\hat{u}^{(2)}=\begin{bmatrix}2\tilde{\omega}_{(i}\hat{\ell}'_{j)}\\ 2\tilde{\omega}_{(i}\hat{k}'_{j)}\end{bmatrix}, \quad \hat{u}^{(3)}=\begin{bmatrix}\hat{\ell}'_{\langle ij\rangle}\\ \hat{k}'_{\langle ij\rangle}\end{bmatrix}. \quad (14)$$

The principal symbol $\underline{p}_1$ and the eigenvalue equation $\underline{p}_1\hat{u}=\lambda\hat{u}$ can also be decomposed into the same three parts. The first part is four dimensional, corresponding to the variable $\hat{u}^{(1)}$, that is, the scalar fields, $\hat{\ell}$, $\hat{k}$, $\hat{\ell}'$, and $\hat{k}'$. The eigenvalues are

$$\tilde{\lambda}_1^{(1)}=\pm 1, \quad \tilde{\lambda}_2^{(1)}=\pm\sqrt{b},$$

where $\tilde{\lambda}:=(\lambda-\tilde{\beta})/N$, so the role of the shift vector is to displace the value of the eigenvalue by an amount $\tilde{\beta}=\tilde{\omega}_k\beta^k$, and the lapse rescales it. But a change of lapse (which here is the function Q) and shift cannot change a real eigenvalue into an imaginary one. It cannot affect the hyperbolicity of the system. The associated eigenvectors for this first part are

$$\hat{u}_{\lambda_1}^{(1)}=\begin{bmatrix}2[(1+b)\tilde{\omega}_i\tilde{\omega}_j+(1-b)q_{ij}/2]\\ \mp[(1+b)\tilde{\omega}_i\tilde{\omega}_j+(1-b)q_{ij}/2]\end{bmatrix},$$

$$\hat{u}_{\lambda_2}^{(1)}=\begin{bmatrix}2\tilde{\omega}_i\tilde{\omega}_j\\ \mp\sqrt{b}\tilde{\omega}_i\tilde{\omega}_j\end{bmatrix}.$$

Notice that for $b=1$ the two eigenvectors $\hat{u}_{\lambda_1}^{(1)}$ collapse to the two eigenvectors $\hat{u}_{\lambda_2}^{(1)}$. The conclusion for this part is that the eigenvalues are real for $b\geqslant 0$, and the four eigenvectors are linearly independent for $b\neq 0$, $b\neq 1$.

The second part is also four dimensional and corresponds to the variable $\hat{u}^{(2)}$, that is, the vector fields $\hat{\ell}'_i$ and $\hat{k}'_i$. (The vector $\hat{\ell}'_i$ has only two independent components because of the condition $\hat{\ell}'_i\tilde{\omega}^i=0$. The same holds for $\hat{k}'_i$.) The result is

$$\tilde{\lambda}_1^{(2)}=0, \quad \hat{u}_{\lambda_1}^{(2)}=\begin{bmatrix}v_j{}^A\\ 0\end{bmatrix}.$$

The eigenvalue has multiplicity 4, but there are only two linearly independent eigenvectors. Here, $v_j{}^A$ represent two linearly independent vectors, each one orthogonal to $\tilde{\omega}_i$, and labeled with the index A, which takes values 1,2. This part is the main reason why the ADM equations are weakly hyperbolic.

The last part is again four dimensional and corresponds to the variable $\hat{u}^{(3)}$, that is, the two-tensor fields $\hat{\ell}'_{\langle ij\rangle}$ and $\hat{k}'_{\langle ij\rangle}$. (The tensor $\hat{\ell}'_{\langle ij\rangle}$ has only two independent components because of the symmetry, the orthogonality to $\tilde{\omega}_i$, and the trace-free condition. The same holds for $\hat{k}'_{\langle ij\rangle}$.) The result is

$$\tilde{\lambda}_1^{(3)}=\pm 1, \quad \hat{u}_{\lambda_1}^{(3)}=\begin{bmatrix}2v_{\langle ij\rangle}{}^A\\ \mp v_{\langle ij\rangle}{}^A\end{bmatrix}.$$

The eigenvalues each have multiplicity 2, and there are four linearly independent eigenvectors. Here $v_{\langle kl\rangle}{}^A$ represent two linearly independent symmetric, traceless tensors, orthogonal to $\tilde{\omega}_i$.

At the end one gets the following picture. All eigenvalues are real for $b\geqslant 0$. Notice that $\tilde{\lambda}_2^{(1)}$ becomes imaginary for $b<0$, so the equations are not hyperbolic in this case. With respect to the eigenvectors, there are two main cases. First, $b>0$ and $b\neq 1$. Then, the eigenvectors of the first and third parts of $\underline{p}_1$ do span their associated eigenspaces; but the eigenvectors $\hat{u}_{\lambda}^{(2)}$ corresponding to the second part of $\underline{p}_1$ do not span their eigenspace. In the second case, $b=0$ or $b=1$. In this case there are linearly dependent eigenvectors even among the scalar variables. Therefore, the conclusion is that the system (7), (8) is weakly hyperbolic for $b\geqslant 0$. ■

It is interesting here to comment on the role of the Hamiltonian constraint. Suppose that a term of the form ah_{ij} times Eq. (6) is added to Eq. (4). Here a is some real constant. Can this modification alter the hyperbolicity of the ADM equations? Here a is some real constant. One might think that adding the Hamiltonian constraint to the ADM evolution equation could have a similar role as densitizing the lapse function i.e., it could keep both eigenvectors $\hat{u}_{\lambda_2}^{(1)}$ linearly independent. The fact is, it does not. Such an addition of the Hamiltonian constraint modifies only $\tilde{\lambda}_1^{(1)}$ and $\hat{u}_{\lambda_1}^{(1)}$, and does not modify $\hat{u}_{\lambda_2}^{(1)}$ and $\tilde{\lambda}_2^{(1)}$. The result is

$$\tilde{\lambda}_1^{(1)}=\pm\sqrt{1+2a},$$

$$\hat{u}_{\lambda_1}^{(1)}=\begin{bmatrix}2[(1+b+2a)\tilde{\omega}_i\tilde{\omega}_j+(1-b+2a)q_{ij}/2]\\ \mp\sqrt{1+2a}[(1+b)\tilde{\omega}_i\tilde{\omega}_j+(1-b)q_{ij}/2]\end{bmatrix}.$$

Therefore, adding the Hamiltonian constraint only helps to keep the eigenvectors $\hat{u}_{\lambda_1}^{(1)}$ independent of the $\hat{u}_{\lambda_2}^{(1)}$, so it helps only in the case $b=1$, where the former collapse onto the latter (for $a=0$). For $b\neq 1$ the addition of the Hamiltonian constraint does not contribute to make the vectors $\hat{u}^{(1)}_{\lambda_2}$ linearly independent, whereas densitizing the lapse does.

B. BSSN-type equations

Consider the densitized ADM evolution equations (7), (8). Introduce into these equations the new variable

$$f^{\mu} := h^{\nu\sigma}\gamma_{\nu\sigma}{}^{\mu}.$$

By definition $n_\mu f^\mu = 0$, that is, $f^0 = 0$, so the new variables are the components $f^i = h^{ij}[h^{kl}\partial_k h_{lj} - \partial_j(\ln \mathbb{h})]$, where $\mathbb{h} = \sqrt{\det(h_{ij})}$ as above. They are related to the three connection variables $\tilde{\Gamma}^i$ of the BSSN system defined in Eq. (21) in [14]. More precisely, $\tilde{\gamma}_{ij}\tilde{\Gamma}^j = f_i + (1/3)\partial_i(\ln \mathbb{h})$, where $\tilde{\gamma}_{ij}$ is defined in Eq. (10) of that reference. The evolution equation for f_i is obtained by taking the trace in indices ν,σ of the identity

$$h_{\mu\delta}\mathcal{L}_n\gamma_{\nu\sigma}{}^{\delta} = -2D_{(\nu}k_{\sigma)\mu} + D_\mu k_{\nu\sigma} - 2a_{(\nu}k_{\sigma)\mu} + k_{\nu\sigma}a_\mu + \frac{1}{N}h_{\mu\delta}h_\nu{}^{\nu'}h_\sigma{}^{\sigma'}\partial_{\nu'}\partial_{\sigma'}\beta^\delta,$$

where $\mathcal{L}_n\gamma_{\nu\sigma}{}^{\delta} = n^\mu\partial_\mu\gamma_{\nu\sigma}{}^{\delta} + 2\gamma_{\mu(\nu}{}^{\delta}\partial_{\sigma)}n^\mu - \gamma_{\nu\sigma}{}^{\mu}\partial_\mu n^\delta$, and adding to the result c times the momentum constraint (5). Here c is any real constant. One then gets

$$\mathcal{L}_n f_\mu = (c-2)D_\nu k_\mu{}^\nu + (1-c)D_\mu k + C_\mu,$$

where the nonprincipal terms are grouped in

$$C_\mu = -c\kappa j_\mu - 2k_{\mu\nu}a^\nu + ka_\mu - 2\gamma_{\nu\sigma\mu}k^{\nu\sigma} - 2k_{\mu\nu}f^\nu + (1/N)h_{\mu\nu}h^{\sigma\delta}\partial_\sigma\partial_\delta\beta^\nu.$$

Summarizing, the unknowns for BSSN-type systems are h_{ij}, k_{ij}, and f_i. The evolution equations are

$$\mathcal{L}_{(t-\beta)}h_{ij} = -2Nk_{ij}, \tag{15}$$

$$\mathcal{L}_{(t-\beta)}k_{ij} = \frac{N}{2}h^{kl}[-\partial_k\partial_l h_{ij} - b\partial_i\partial_j h_{kl}] + N\partial_{(i}f_{j)} + \mathcal{B}_{ij}, \tag{16}$$

$$\mathcal{L}_{(t-\beta)}f_i = N[(c-2)h^{kj}\partial_k k_{ij} + (1-c)h^{kj}\partial_i k_{kj}] + \mathcal{C}_i, \tag{17}$$

where $\mathcal{L}_{(t-\beta)}f_i = \partial_t f_i - (\beta^j\partial_j f_i + f_j\partial_i\beta^j)$, while $\mathcal{L}_{(t-\beta)}h_{ij}$ and $\mathcal{L}_{(t-\beta)}k_{ij}$ are defined below Eqs. (7), (8), and

$$\mathcal{B}_{ij} = N[2\gamma_{kl(i}\gamma_{j)}{}^{kl} + \gamma_{ikl}\gamma_j{}^{kl} - \gamma_{ijl}\gamma_k{}^{kl} - 2k_i{}^l k_{jl} + k_{ij}k_l{}^l - A_{ij} - \kappa S_{ij}],$$

$$\mathcal{C}_i = N[C_i + (c-2)(\gamma_{kj}{}^k k_i{}^j - \gamma_{ki}{}^j k_j{}^k)].$$

The constraint equations are Eqs. (5), (6) and

$$f^\mu - h^{\nu\sigma}\gamma_{\nu\sigma}{}^{\mu} = 0.$$

The main result of this work asserts that BSSN-type evolution equations are strongly hyperbolic for some choices of the free parameters.

Theorem 2. Fix any positive function Q, vector field β^i, first and second fundamental forms h_{ij}, k_{ij} on S_0. If $b>0$, $b\neq 1$, and $c>0$, then Eqs. (15)–(17) are strongly hyperbolic. Assume that $b=1$. If $c=2$, then Eqs. (15)–(17) are strongly hyperbolic; if $c\neq 2$, $c>0$, then they are weakly hyperbolic.

One can check that the system (15)–(17) remains strongly hyperbolic under a transformation of the form $F_i = f_i + d\partial_i(\ln \mathbb{h})$ for any real constant d, in particular $d=1/3$, which gives the BSSN variable $\tilde{\Gamma}^i$. (See the comment at the end of this section.)

The first part of the proof follows the argument that establishes Theorem 1. That is, from Eqs. (15)–(17) obtain the first order pseudodifferential system Eqs. (18)–(20) below. Then compute the eigenvector and eigenvalues, by splitting the principal symbol into orthogonal parts with respect to the Fourier variable ω_i. Finally, the second part of the proof is the construction of the symmetrizer.

Proof.

First order reduction. Compute the symbol associated with the second order operator given by Eqs. (15)–(17):

$$\partial_t h_{ij} = \int_{S_t}\{-2N\hat{k}_{ij} + i\omega_k\beta^k\hat{h}_{ij} + 2\hat{h}_{k(i}\partial_{j)}\beta^k\}e^{i\omega x}\bar{d}\omega,$$

$$\partial_t k_{ij} = \int_{S_t}\{(N/2)[|\omega|_h^2\hat{h}_{ij} + b\,\omega_i\omega_j h^{kl}\hat{h}_{kl}] + iN\omega_{(i}\hat{f}_{j)} + i\omega_k\beta^k\hat{k}_{ij} + \tilde{\mathcal{B}}_{ij}\}e^{i\omega x}\bar{d}\omega,$$

$$\partial_t f_i = \int_{S_t}\{iN[(c-2)\hat{k}_{ik}\omega^k + (1-c)\omega_i h^{kj}\hat{k}_{kj}] + i\omega_k\beta^k\hat{f}_i + \tilde{\mathcal{C}}_i\}e^{i\omega x}\bar{d}\omega,$$

where the terms not in the principal symbol have the forms

$$\tilde{\mathcal{B}}_{ij} = \hat{\mathcal{B}}_{ij} + 2\hat{k}_{k(i}\partial_{j)}\beta^k,$$

$$\tilde{\mathcal{C}}_i = \hat{\mathcal{C}}_i + \hat{f}_k\partial_i\beta^k.$$

Here $\bar{d}\omega = d\omega/(2\pi)^{3/2}$, $|\omega|_h^2 = \omega_i\omega_j h^{ij}$, and $\omega^i = \omega_j h^{ij}$. Introduce the unknown $\hat{\ell}_{ij} = i|\omega|_\delta\hat{h}_{ij}$, with $|\omega|_\delta^2 = \omega_i\omega_j\delta^{ij}$, where $\delta^{ij} = \mathrm{diag}(1,1,1)$. The resulting pseudodifferential system is a first order one, given by

$$\partial_t\ell_{ij} = \int_{S_t}\{i|\omega|_h[-(2N/\alpha)\hat{k}_{ij} + \tilde{\beta}\hat{\ell}_{ij}] + 2\hat{\ell}_{k(i}\partial_{j)}\beta^k\}e^{i\omega x}\bar{d}\omega, \tag{18}$$

$$\partial_t k_{ij} = \int_{S_t}\{i|\omega|_h[(N\alpha/2)(-\hat{\ell}_{ij} - b\tilde{\omega}_i\tilde{\omega}_j h^{kl}\hat{\ell}_{kl} + 2\tilde{\omega}_{(i}\hat{f}_{j)}) + \tilde{\beta}\hat{k}_{ij}] + \tilde{\mathcal{B}}_{ij}\}e^{i\omega x}\bar{d}\omega, \tag{19}$$

$$\partial_t f_i = \int_{S_t} \{ i|\omega|_h [N((c-2)\hat{k}_{ik}\tilde{\omega}^k + (1-c)\tilde{\omega}_i h^{kj}\hat{k}_{kj}) + \tilde{\beta}\hat{f}_i] + \tilde{\mathcal{C}}_i \} e^{i\omega x} \bar{d}\omega, \quad (20)$$

with $\alpha = |\omega|_h / |\omega|_\delta$, $\tilde{\omega}_i = \omega_i / |\omega|_h$, $\tilde{\beta} = \tilde{\omega}_k \beta^k$, and $\ell_{ij} = \int_{S_t} i|\omega|_\delta \hat{h}_{ij} e^{i\omega x} \bar{d}\omega$. The symbol of Eqs. (18)–(20) has the form,

$$p(t,x,u,i\omega) = i|\omega|_h \underline{p}_1(t,x,u,\omega) + \mathbb{B}(t,x,u,\omega), \quad (21)$$

where $(\mathbb{B}\hat{u})^T := (2\hat{\ell}_{k(i}\partial_{j)}\beta^k, \tilde{\mathcal{B}}_{ij}, \tilde{\mathcal{C}}_i)$, $\hat{u}^T := (\hat{\ell}_{ij}, \hat{k}_{ij}, \hat{f}_i)$, the index T denotes the transpose, and the principal symbol $\underline{p}_1$ can be read out from the terms inside the square brackets in Eqs. (18)–(20). As in the proof of Theorem 1, the definition of the principal symbol here differs from the one given in Sec. II by a factor of $i|\omega|_h$.

Eigenvalues and eigenvectors of $\underline{p}_1$. Following the proof of Theorem 1, $\alpha = 1$ is assumed. One can check in this case that, if $\hat{u}^T = (\hat{\ell}_{ij}, \hat{k}_{ij}, \hat{f}_i)$ is an eigenvector of $\underline{p}_1(\alpha=1)$ with eigenvalue λ, then $\hat{u}^T(\alpha) = (\alpha^{-2/3}\hat{\ell}_{ij}, \alpha^{1/3}\hat{k}_{ij}, \alpha^{1/3}\hat{f}_i)$ is an eigenvector of $\underline{p}_1(\alpha)$ with the same eigenvalue λ.

The orthogonal decomposition (12), (13) simplifies the calculation. In addition, decompose

$$\hat{f}_i = \tilde{\omega}_i \hat{f} + \hat{f}'_i,$$

where $\hat{f} = \tilde{\omega}_i \hat{f}^i$ and $\hat{f}'_i = q_i{}^j \hat{f}_j$. This decomposition implies that $\hat{u} = \hat{u}^{(1)} + \hat{u}^{(2)} + \hat{u}^{(3)}$ where

$$\hat{u}^{(1)} = \begin{bmatrix} \tilde{\omega}_i\tilde{\omega}_j\hat{\ell} + (q_{ij}/2)\hat{\ell}' \\ \tilde{\omega}_i\tilde{\omega}_j\hat{k} + (q_{ij}/2)\hat{k}' \\ \tilde{\omega}_i\hat{f} \end{bmatrix},$$

$$\hat{u}^{(2)} = \begin{bmatrix} 2\tilde{\omega}_{(i}\hat{\ell}'{}_{j)} \\ 2\tilde{\omega}_{(i}\hat{k}'{}_{j)} \\ \hat{f}'_i \end{bmatrix}, \quad \hat{u}^{(3)} = \begin{bmatrix} \hat{\ell}'_{\langle ij\rangle} \\ \hat{k}'_{\langle ij\rangle} \\ 0 \end{bmatrix}. \quad (22)$$

Split the principal symbol $\underline{p}_1$ and the eigenvalue equation $\underline{p}_1\hat{u} = \lambda\hat{u}$ into the same three parts. The first part is five dimensional, corresponding to the variable $\hat{u}^{(1)}$, that is, the scalar fields $\hat{f}$, $\hat{\ell}$, $\hat{k}$, $\hat{\ell}'$, and $\hat{k}'$. The eigenvalues are

$$\tilde{\lambda}_1^{(1)} = \pm 1, \quad \tilde{\lambda}_2^{(1)} = \pm\sqrt{b}, \quad \tilde{\lambda}_3^{(1)} = 0,$$

each having multiplicity 1, where again $\tilde{\lambda} := (\lambda - \tilde{\beta})/N$. Hence, the eigenvalues are real if $b \geqslant 0$. The corresponding eigenvectors are

$$\hat{u}_{\lambda_1}^{(1)} = \begin{bmatrix} 2[(b-c+1)\tilde{\omega}_i\tilde{\omega}_j + (1-b)(q_{ij}/2)] \\ \mp[(b-c+1)\tilde{\omega}_i\tilde{\omega}_j + (1-b)(q_{ij}/2)] \\ (2-c)b\tilde{\omega}_i \end{bmatrix},$$

$$\hat{u}_{\lambda_2}^{(1)} = \begin{bmatrix} 2\tilde{\omega}_i\tilde{\omega}_j \\ \mp\sqrt{b}\tilde{\omega}_i\tilde{\omega}_j \\ \tilde{\omega}_i \end{bmatrix}, \quad \hat{u}_{\lambda_3}^{(1)} = \begin{bmatrix} 2\tilde{\omega}_i\tilde{\omega}_j \\ 0 \\ (1+b)\tilde{\omega}_i \end{bmatrix}.$$

Notice that both eigenvectors $\hat{u}_{\lambda_2}^{(1)}$ collapse if $b=0$. Also see that the eigenvectors $\hat{u}_{\lambda_1}^{(1)}$ collapse to $\hat{u}_{\lambda_2}^{(1)}$ in the case $b=1$ and $c \neq 2$. Thus, in these cases Eqs. (18)–(20) are weakly hyperbolic. In the case $b=1$ and $c=2$ the eigenvalues $\pm\sqrt{b}$ collapse to ± 1. Therefore, one has $\tilde{\lambda}_1^{(1)} = \pm 1$, each with multiplicity 2, and $\tilde{\lambda}_2^{(1)} = 0$, with multiplicity 1. There are five linearly independent eigenvectors in this case,

$$\hat{u}_{\lambda_1}^{(1)} = \begin{bmatrix} 2\tilde{\omega}_i\tilde{\omega}_j \\ \mp\tilde{\omega}_i\tilde{\omega}_j \\ \tilde{\omega}_i \end{bmatrix}, \quad \hat{u}_{\lambda_1}^{(1)} = \begin{bmatrix} q_{ij} \\ \mp q_{ij}/2 \\ \tilde{\omega}_i \end{bmatrix},$$

$$\hat{u}_{\lambda_2}^{(1)} = \begin{bmatrix} \tilde{\omega}_i\tilde{\omega}_j \\ 0 \\ \tilde{\omega}_i \end{bmatrix}.$$

The second part is six dimensional and corresponds to the variables $\hat{u}^{(2)}$, that is, the vector fields $\hat{f}'_i$, $\hat{\ell}'_i$, and $\hat{k}'_i$, orthogonal to $\tilde{\omega}_i$. The eigenvalues are

$$\tilde{\lambda}_1^{(2)} = 0, \quad \tilde{\lambda}_2^{(2)} = \pm\sqrt{c/2},$$

where each one has multiplicity 2. They are real if $c \geqslant 0$. There are six linearly independent eigenvectors in the case $c > 0$, given by

$$\hat{u}_{\lambda_1}^{(2)} = \begin{bmatrix} 2\tilde{\omega}_{(i}v_{j)}{}^A \\ 0 \\ v_i{}^A \end{bmatrix}, \quad \hat{u}_{\lambda_2}^{(2)} = \begin{bmatrix} 4\tilde{\omega}_{(i}v_{j)}{}^A \\ \mp\sqrt{2c}\tilde{\omega}_{(i}v_{j)}{}^A \\ (2-c)v_i{}^A \end{bmatrix},$$

where $v_j{}^A$ represent two linearly independent vectors, each one orthogonal to $\tilde{\omega}_i$, labeled by the index A which takes values 1, 2. Here is the key role of the momentum constraint. If $c=0$, that is, the momentum constraint is not added to the system, then the two eigenvectors $\hat{u}_{\lambda_2}^{(2)}$ become linearly dependent, as occurs in the densitized ADM evolution equations.

The last part is four dimensional and is the same as in Theorem 1. It corresponds to the variables $\hat{u}^{(3)}$, that is, the tensor fields $\hat{\ell}'_{\langle ij\rangle}$ and $\hat{k}'_{\langle ij\rangle}$. The result is

$$\tilde{\lambda}_1^{(3)} = \pm 1, \quad \hat{u}_{\lambda_1}^{(3)} = \begin{bmatrix} 2v_{\langle ij\rangle}{}^A \\ \mp v_{\langle ij\rangle}{}^A \\ 0 \end{bmatrix}.$$

The eigenvalues each have multiplicity 2, and there are four linearly independent eigenvectors. The tensors $v_{\langle kl\rangle}{}^A$ represent two linearly independent symmetric, traceless tensors, orthogonals to $\tilde{\omega}_i$.

The symmetrizer. The operator $H=(T^{-1})^*(T^{-1})$ is a symmetrizer of system (21), where T is an operator whose columns correspond to the eigenvectors of $\underline{p}_1$, T^{-1} is its inverse, and * denotes the adjoint. Then it only remains to do a lengthy, although straightforward, calculation.

There are two hints that help to simplify the construction of the symmetrizer. They are based on the observation that the principal symbol has the particular form $\underline{p}_1=N\underline{\tilde{p}}_1+\tilde{\beta}\mathbb{I}$, where $\mathbb{I}$ is the identity matrix $\mathbb{I}=\mathrm{diag}(h_{(i}{}^k h_{j)}{}^l, h_{(i}{}^k h_{j)}{}^l, h_i{}^k)$, and

$$\underline{\tilde{p}}_1=\begin{bmatrix} 0 & \underline{\tilde{p}}_{1(\ell)ij}{}^{(k)kl} & 0 \\ \underline{\tilde{p}}_{1(k)ij}{}^{(\ell)kl} & 0 & \underline{\tilde{p}}_{1(k)ij}{}^{(f)k} \\ 0 & \underline{\tilde{p}}_{1(f)i}{}^{(k)kl} & 0 \end{bmatrix},$$

where the indices (ℓ), (k), and (f) indicate rows and columns, that is, equations and variables, respectively, and the matrix components are given by

$$\underline{\tilde{p}}_{1(\ell)ij}{}^{(k)kl}=-2h_{(i}{}^k h_{j)}{}^l,$$

$$\underline{\tilde{p}}_{1(k)ij}{}^{(\ell)kl}=(-1/2)[h_{(i}{}^k h_{j)}{}^l+b\tilde{\omega}_i\tilde{\omega}_j h^{kl}],$$

$$\underline{\tilde{p}}_{1(k)ij}{}^{(f)k}=\tilde{\omega}_{(i}h_{j)}{}^k,$$

$$\underline{\tilde{p}}_{1(f)i}{}^{(k)kl}=(c-2)\tilde{\omega}^{(k}h^{l)}{}_i+(1-c)\tilde{\omega}_i h^{kl}.$$

Then, the first hint is that a symmetrizer for $\underline{p}_1$ is indeed a symmetrizer for the nondiagonal elements $\underline{\tilde{p}}_1$. A second hint is that the orthogonal decomposition in Eq. (22) induces the same splitting in $\underline{\tilde{p}}_1=\underline{\tilde{p}}_1{}^{(1)}+\underline{\tilde{p}}_1{}^{(2)}+\underline{\tilde{p}}_1{}^{(3)}$ and therefore in $H=H^{(1)}+H^{(2)}+H^{(3)}$.

The result is

$$H=\begin{bmatrix} H_{(\ell)ij}{}^{(\ell)kl} & 0 & H_{(\ell)ij}{}^{(f)k} \\ 0 & H_{(k)ij}{}^{(k)kl} & 0 \\ H_{(f)i}{}^{(\ell)kl} & 0 & H_{(f)i}{}^{(f)k} \end{bmatrix},$$

where

$$H_{(\ell)ij}{}^{(\ell)kl}=H^{(1)}_{(\ell)ij}{}^{(\ell)kl}+H^{(2)}_{(\ell)ij}{}^{(\ell)kl}+H^{(3)}_{(\ell)ij}{}^{(\ell)kl},$$

$$H_{(\ell)ij}{}^{(f)k}=H^{(1)}_{(\ell)ij}{}^{(f)k}+H^{(2)}_{(\ell)ij}{}^{(f)k},$$

$$H_{(k)ij}{}^{(k)kl}=H^{(1)}_{(k)ij}{}^{(k)kl}+H^{(2)}_{(k)ij}{}^{(k)kl}+H^{(3)}_{(k)ij}{}^{(k)kl},$$

$$H_{(f)i}{}^{(\ell)kl}=H^{(1)}_{(f)i}{}^{(\ell)kl}+H^{(2)}_{(f)i}{}^{(\ell)kl},$$

$$H_{(f)i}{}^{(f)k}=H^{(1)}_{(f)i}{}^{(f)k}+H^{(2)}_{(f)i}{}^{(f)k}.$$

The matrix coefficients of each part depend on $\tilde{\omega}_i$. The scalar variable part of the symmetrizer has the form

$$H^{(1)}_{(\ell)ij}{}^{(\ell)kl}=H_{\ell\ell}\tilde{\omega}_i\tilde{\omega}_j\tilde{\omega}^k\tilde{\omega}^l+H_{\ell'\ell'}\frac{q_{ij}}{2}q^{kl}+H_{\ell\ell'}\left(\tilde{\omega}_i\tilde{\omega}_j q^{kl}+\frac{q_{ij}}{2}\tilde{\omega}^k\tilde{\omega}^l\right),$$

$$H^{(1)}_{(\ell)ij}{}^{(f)k}=H_{f\ell}\tilde{\omega}_i\tilde{\omega}_j\tilde{\omega}^k+H_{f\ell'}\frac{q_{ij}}{2}\tilde{\omega}^k,$$

$$H^{(1)}_{(k)ij}{}^{(k)kl}=H_{kk}\tilde{\omega}_i\tilde{\omega}_j\tilde{\omega}^k\tilde{\omega}^l+H_{k'k'}\frac{q_{ij}}{2}q^{kl}+H_{kk'}\left(\tilde{\omega}_i\tilde{\omega}_j q^{kl}+\frac{q_{ij}}{2}\tilde{\omega}^k\tilde{\omega}^l\right),$$

$$H^{(1)}_{(f)i}{}^{(\ell)kl}=H_{f\ell}\tilde{\omega}_i\tilde{\omega}^k\tilde{\omega}^l+H_{f\ell'}\tilde{\omega}_i q^{kl},$$

$$H^{(1)}_{(f)i}{}^{(f)k}=H_{ff}\tilde{\omega}_i\tilde{\omega}^k,$$

where the scalar functions that appear above are the following:

$$H_{\ell\ell}=\frac{1}{8b^2}[2+(1+b)^2],\quad H_{kk}=\frac{1}{2b},$$

$$H_{ff}=\frac{3}{2b^2},\quad H_{k'k'}=\frac{(b+1-c)^2+b}{2b(1-b)^2},$$

$$H_{\ell'\ell'}=\frac{1}{4b^2}\left[(c-1)^2+\frac{(b^2+1-c)^2+b^2}{2(1-b)^2}\right],$$

$$H_{\ell\ell'}=\frac{1}{4b^2}\left[c-1-\frac{(b^2+1-c)(1+b)}{2(1-b)}\right],$$

$$H_{f\ell'}=-\frac{1}{4b^2}\left[2(c-1)-\frac{(b^2+1-c)}{(1-b)}\right],$$

$$H_{kk'}=\frac{(c-1-b)}{2b(1-b)},\quad H_{f\ell}=-\frac{1}{4b^2}(b+3).$$

The symmetrizer for the vector variable part is

$$H^{(2)}_{(\ell)ij}{}^{(\ell)kl}=\frac{2(c-2)^2+1}{4c^2}2\tilde{\omega}_{(i}q_{j)}{}^{(k}\tilde{\omega}^{l)},$$

$$H^{(2)}_{(\ell)ij}{}^{(f)k}=\frac{4(c-2)-1}{2c^2}\tilde{\omega}_{(i}q_{j)}{}^k,$$

$$H^{(2)}_{(k)ij}{}^{(k)kl}=\frac{1}{2c}2\tilde{\omega}_{(i}q_{j)}{}^{(k}\tilde{\omega}^{l)},$$

$$H^{(2)}_{(f)i}{}^{(\ell)kl}=\frac{4(c-2)-1}{2c^2}\tilde{\omega}^{(k}q^{l)}{}_i\,,$$

$$H^{(2)}_{(f)i}{}^{(f)k}=\frac{9}{2c^2}q_i{}^k.$$

Finally, the two-tensor part of the symmetrizer is

$$H^{(3)}_{(\ell)ij}{}^{(\ell)kl}=\frac{1}{8}q_{\langle i}{}^k q_{j\rangle}{}^l,$$

$$H^{(3)}_{(k)ij}{}^{(k)kl}=\frac{1}{2}q_{\langle i}{}^k q_{j\rangle}{}^l.$$

One can check that the symmetrizer so defined satisfies $H(i|\omega|_h\underline{p}_1)+(i|\omega|_h\underline{p}_1)^*H=0$ and so trivially belongs to $S^0_{1,0}$. This symmetrizer $H(t,x,u,\tilde{\omega})$ is bounded for $b\neq 1$, $c>0$, and smooth in all its arguments. The case $b=1$ and $c=2$ can be computed in the same way described above, and the same conclusion holds. Then, for these two cases the system (18)–(20) is strongly hyperbolic. ■

Two further generalizations are immediate. The first one involves the Hamiltonian constraint. Suppose the term ah_{ij} times the Hamiltonian constraint (6) is added to Eq. (16). Here a is any real constant. What are the eigenvalues and eigenvectors of the resulting principal symbol? The result is, as one expects, that the only change is in the scalar variable part $\underline{p}_1{}^{(1)}$. The eigenvalues are

$$\tilde{\lambda}_1^{(1)}=\pm\sqrt{1+2a(2-c)},\quad \tilde{\lambda}_2^{(1)}=\pm\sqrt{b},\quad \tilde{\lambda}_3^{(1)}=0,$$

each one having multiplicity 1, and $\lambda_1^{(1)}$ is real provided $a(2-c)\geqslant-(1/2)$. The corresponding eigenvectors are

$$\hat{u}^{(1)}_{\lambda_1}=\begin{bmatrix}2[(b-c+1)\tilde{\omega}_i\tilde{\omega}_j+(1-b)q_{ij}/2]\\ \mp\lambda[(b-c+1)\tilde{\omega}_i\tilde{\omega}_j+(1-b)q_{ij}/2]\\ (2-c)b\tilde{\omega}_i\end{bmatrix}+a(2-c)\begin{bmatrix}2(\tilde{\omega}_i\tilde{\omega}_j+q_{ij})\\ \mp\lambda(\tilde{\omega}_i\tilde{\omega}_j+q_{ij})\\ (2c-1)\tilde{\omega}_i\end{bmatrix},$$

$$\hat{u}^{(1)}_{\lambda_3}=\begin{bmatrix}2\tilde{\omega}_i\tilde{\omega}_j\\ 0\\ (1+b)\tilde{\omega}_i\end{bmatrix}+a\begin{bmatrix}2[(3-2b)\tilde{\omega}_i\tilde{\omega}_j+bq_{ij}]\\ 0\\ 3\tilde{\omega}_i\end{bmatrix},$$

where $\lambda=\sqrt{1+2a(2-c)}$. The eigenvector $\hat{u}^{(2)}_{\lambda_2}$ does not change. The conclusion is summarized below.

Corollary 1. Consider Eqs. (15)–(17) and assume the hypothesis of Theorem 2. Assume $b>0$ and $c>0$. Assume that a term ah_{ij} times the Hamiltonian constraint (6) is added to Eq. (16), where a is a real constant satisfying $a(2-c)>-1/2$. Then the resulting principal symbol, as defined in this section, has real eigenvalues and a complete set of linearly independent eigenvectors.

The second generalization involves the transformation $F_i=f_i+d\partial_i(\ln \mathbb{h})$, where d is any real constant. That is, instead of defining the BSSN-type system with the variable f_i, define it with F_i. The new evolution equations have a different principal symbol from Eqs. (15)–(17) of the BSSN-type equations. However, one can check that it does not change the hyperbolicity of the system. Indeed, it modifies only the $\underline{p}_1{}^{(1)}$ part of the principal symbol. Its eigenvalues remain the same, namely, $\tilde{\lambda}_1^{(1)}=\pm 1$, $\tilde{\lambda}_2^{(1)}=\pm\sqrt{b}$, and $\tilde{\lambda}_3^{(1)}=0$. The associated eigenvectors are now given by

$$\hat{u}^{(1)}_{\lambda_1}=\begin{bmatrix}2[(b-c+1)\tilde{\omega}_i\tilde{\omega}_j+(1-b)(q_{ij}/2)]\\ \mp[(b-c+1)\tilde{\omega}_i\tilde{\omega}_j+(1-b)(q_{ij}/2)]\\ (2-c)(b+d)\tilde{\omega}_i\end{bmatrix},$$

$$\hat{u}^{(1)}_{\lambda_2}=\begin{bmatrix}2\tilde{\omega}_i\tilde{\omega}_j\\ \mp\sqrt{b}\tilde{\omega}_i\tilde{\omega}_j\\ (1+d)\tilde{\omega}_i\end{bmatrix},\quad \hat{u}^{(1)}_{\lambda_3}=\begin{bmatrix}2\tilde{\omega}_i\tilde{\omega}_j\\ 0\\ (1+b+d)\tilde{\omega}_i\end{bmatrix}.$$

Therefore, the hyperbolicity of the BSSN-type equations is not changed by this transformation.

IV. DISCUSSION

The first order pseudodifferential reduction performed in the space derivatives is here the main tool used to study the hyperbolicity of the BSSN-type systems. This technique is widely used in pseudodifferential analysis. It does not increase the number of equations, so there are no new constraints added to the system. It emphasizes that well posedness essentially captures the absence of divergent behavior in the high frequency limit of the solutions of a given system. This tool is applied to Eqs. (15)–(17), which have derivatives of first order in time and second order in space. They are obtained from the ADM equations by densitizing the lapse function and introducing the three connection variables f_i. Its evolution equation is obtained by adding the momentum constraint to an identity from commuting derivatives. The positive-density lapse function and the spacelike shift vector are arbitrary given functions. There are free parameters given by the exponent in the densitized lapse and the factor in the addition of the momentum constraint. The resulting first order pseudodifferential system is strongly hyperbolic for some values of the free parameters. (See Theorem 2.)

The introduction of f_i as a new variable is inspired by the variable $\tilde{\Gamma}^i$ of the BSSN system, defined by Eq. (21) in [14], and in the study of the linearized ADM evolution equations given in [15]. This variable is the crucial step that allows us to introduce the momentum constraint into the system. These two things, in turn, produce the result that the vector variable eigenvectors $\hat{u}^{(2)}_{\lambda}$ do span their eigenspace. This is the key feature that converts the weakly hyperbolic densitized ADM system into a strongly hyperbolic one. This property does not change when a term of the form $d\partial_i(\ln \mathbb{h})$ is added to the system. Then, both results suggest why the BSSN system is preferred to the ADM equations for numerical analysis. This conclusion agrees with a previous result in [22], where the hyperbolicity of the BSSN system was also studied. It is shown there that a differential reduction to first order in time and space derivatives, together with a densitization of the lapse, produce a strongly hyperbolic system. The results in the present work are also consistent with numerical studies on the evolution equations presented in [17–19]. For further developments on these systems, see [20,21].

Finally, the role of the Hamiltonian constraint, when added to the ADM and BSSN-type evolution equations, is studied. In the case where the density lapse exponent $b \neq 1$ it does not affect the hyperbolicity properties of the two systems. That is, densitized ADM evolution equations remain weakly hyperbolic, and the BSSN-type system remain strongly hyperbolic. In the case $b=1$ the addition of the Hamiltonian constraint in both systems prevents the two eigenvectors of the scalar variable block from collapsing onto each other. This keeps the BSSN-type equation strongly hyperbolic even in the case $b=1$, but is not enough to change the weakly hyperbolic character of the densitized ADM evolution equations.

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APPENDIX: ESSENTIALS OF PSEUDODIFFERENTIAL OPERATORS

1. Introduction

Pseudodifferential operators are a generalization of differential operators that make use of Fourier theory. The idea is to think of a differential operator acting upon a function as the inverse Fourier transform of a polynomial in the Fourier variable times the Fourier transform of the function. This integral representation leads to a generalization of differential operators, which correspond to functions other than polynomials in the Fourier variable, as long as the integral converges.

In other words, given a smooth complex valued function $p(x,\omega)$ from $\mathbb{R}^n \times \mathbb{R}^n$ with some asymptotic behavior at infinity, associate with it an operator $p(x,\partial_x):\mathcal{S}\to\mathcal{S}$. Here $\mathcal{S}$ is the Schwartz space, that is, the set of complex valued smooth functions in $\mathbb{R}^n$, such that the function and every derivative decay faster than any polynomial at infinity. The association $p\to p$, that is, functions into differential operators, is not unique. This is known to anyone acquainted with quantum mechanics. Different maps from functions $p(x,\omega)$ into operators $p(x,\partial_x)$ give rise to different theories of pseudodifferential calculus. Every generalization must coincide in the following: The polynomial $p(x,\omega)=\Sigma_{|\alpha|\leq m} a_\alpha(x)(i\omega)^\alpha$ where α is a multi-index in $\mathbb{R}^n$ must be associated with the operator $p(x,\partial_x)=\Sigma_{|\alpha|\leq m} a_\alpha(x)\partial_x^\alpha$, that is, with a differential operator of order m. The map $p\to p$ used in these notes is introduced in subsection 3 of the Appendix. It is the most used definition of pseudodifferential operators in the literature, and the one most studied.

The Fourier transform is used to rewrite the differential operator because it maps derivatives into multiplication, that is, $[\partial_x u(x)]\hat{} = i\omega\hat{u}(\omega)$. This property is used to solve constant coefficient partial differential equations (PDEs) by transforming the whole equation into an algebraic equation. This technique is not useful on variable coefficient PDEs, because of the inverse property, that is, $[xu(x)]\hat{} = i\partial_\omega\hat{u}(\omega)$. For example, one has $[\partial_x u + xu]\hat{} = i[\partial_\omega\hat{u} + \omega\hat{u}]$, and nothing has been simplified by the Fourier transform. That is why one looks for other ways of rewriting differential operators. The generalization to pseudodifferential operators is an additional consequence. Other transforms can be used to define different generalizations of differential operators. For example, Mellin transforms are used in [24].

The functions $p(x,\omega)$ are called symbols. Differential operators correspond to polynomial symbols in ω. They contain the main equations from physics. Even strongly hyperbolic PDEs have polynomial symbols. Why should one consider more general symbols? Because the generalization is evident, and it has proved worth doing it. The Atiyah and Singer index theorem is proved using pseudodifferential operators with smooth symbols, which are more suitable for studying homotopy invariants than polynomial symbols [25]. Techniques to prove the well posedness of the Cauchy problem for a strongly hyperbolic system require one to mollify polynomial symbols into smooth nonpolynomial ones [3]. The main application in these notes is simple: to reduce a second order partial differential equation to a first order system without adding new characteristics into the system. This is done by introducing the operator, a square root of the Laplacian, which is a first order pseudodifferential, but not differential, operator. The main idea for this type of reduction was introduced in [26].

How far should this generalization be carried? In other words, how is the set of symbols that define the pseudodifferential operators determined? The answer depends on which properties of differential operators one wants to be preserved by the general operators, and which additional properties one wants the latter to have. There are different spaces of symbols defined in the literature. Essentially all of them agree that the associated space of pseudodifferential operators is closed under taking the inverse. The inverse of a pseudodifferential operator is another pseudodifferential operator. This statement is not true for differential operators. The algebra developed in studying pseudodifferential operators is useful to compute their inverses. This is important from a physical point of view, because the behavior of solutions of PDEs can be inferred from the inverse operator. One could even say that pseudodifferential operators were created in the middle 1950s from the procedure to compute parametrices to elliptic equations. [A parametrix is a function that differs from a solution of the equation $p(u)=\delta$ by a smooth function, where p is a differential operator, and δ is Dirac's delta distribution.] To know the parametrix is essentially the same as to have the inverse operator.

Spaces of pseudodifferential operators are usually defined to be closed under composition and transpose, and to act on distribution spaces and on Sobolev spaces. They can be invariant under diffeomorphisms, so they can be defined on a manifold. This definition for pseudodifferential operators is not so simple as for differential operators, because the latter are local operators and the former are not. Pseudodifferential operators are pseudolocal. An operator p acting on a distribution u is local if $p(u)$ is smooth in the same set where u is smooth. Pseudolocal means that the set where $p(u)$ is smooth includes the set where u is smooth. This means that p could smooth out a nonsmoothness of u. Mollifiers are an example of this kind of smoothing operator. They are integral operators, which justifies the name of pseudolocal. Differential operators with smooth coefficients are an example of local operators. The proofs of all these properties of pseudodifferential operators are essentially algebraic calculations on the symbols. One could say that the main practical advantage of pseudodifferential calculus is, precisely, turning differential problems into algebraic ones.

2. Function spaces

The study of the existence and uniqueness of solutions to PDEs, as well as the qualitative behavior of these solutions, is at the core of mathematical physics. Function spaces are the basic ground for carrying on this study. The mathematical structure needed is that of the Hilbert space, or Banach space, or Fréchet space, which are complete vector spaces having, respectively, an inner product, a norm, and a particular metric constructed with a family of seminorms. Every Hilbert space is a Banach space, and every Banach space is a Fréchet space. The main examples of Hilbert spaces are the space of square integrable functions L^2, and the Sobolev spaces H^k, with k a positive integer, which consist of functions whose k derivatives belong to L^2. The Fourier transform makes it possible to extend Sobolev spaces to real indices. This generalization in the idea of the derivative is essentially the same as one uses to construct pseudodifferential operators. Examples of Banach spaces are L^p, spaces of p-power integrable functions, where L^2 is the particular case $p=2$. The main examples of Fréchet spaces are $C^\infty(\Omega)$, the set of smooth functions in any open set $\Omega\subset\mathbb{R}^n$, with a particular metric on it (the case $\Omega=\mathbb{R}^n$ is denoted C^∞), the Schwartz space of smooth functions of rapid decrease, and its dual as a Fréchet space, which is a space of distributions.

This section presents only Sobolev spaces, first with nonnegative integer index, and the generalization to a real index. The Fourier transform is needed to generalize the Sobolev spaces. Therefore, Schwarz spaces are introduced to facilitate the definition of the Fourier transform, and to extend it to L^2. The next section is dedicated to introducing pseudodifferential operators.

Let L^2 be the vector space of complex valued, square integrable functions on $\mathbb{R}^n$, that is, functions such that $\|u\|<\infty$, where $\|u\|:=\sqrt{(u,u)}$ and

$$(u,v):=\int_{\mathbb{R}^n}\overline{u}(x)v(x)dx,$$

with $\overline{u}$ the complex conjugate of u. This set is a Hilbert space, that is, a complete vector space with inner product, where the inner product is given by $(\ ,\ )$ and is complete with respect to the associated norm $\|\ \|$.

The Sobolev spaces H^k, for k a non-negative integer, are the elements of L^2 such that

$$\|u\|_k^2:=\sum_{|\alpha|\leqslant k}\|\partial^\alpha u\|^2<\infty,$$

where $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)$ is a multi-index, and for every such multi-index ∂^α denotes $\partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n}$ and $|\alpha|=\Sigma_{i=1}^n\alpha_i$. The inner product in L^2 defines an inner product in H^k given by

$$(u,v)_k:=\sum_{|\alpha|\leqslant k}(\partial^\alpha u,\partial^\alpha v).$$

Let $\mathcal{S}$ be the space of functions of rapid decrease, also called the Schwartz space, that is, the set of complex valued, smooth functions on $\mathbb{R}^n$, satisfying

$$|u|_{k,\alpha}:=\sup_{x\in\mathbb{R}^n}|(1+|x|^2)^{k/2}\partial^\alpha u|<\infty$$

for every multi-index α, and all $k\in\mathbb{N}$ natural, with $|x|$ the Euclidean length in $\mathbb{R}^n$. The Schwartz space is useful in several contexts. It is the appropriate space to introduce the Fourier transform. It is simple to check that the Fourier transform is well defined on elements in that space, in other words, the integral converges. It is also simple to check the main properties of the transformed function. More important is that the Fourier transform is an isomorphism between Schwartz spaces. As mentioned earlier, the Schwartz space provided with an appropriate metric is an example of a Fréchet space. Its dual space is the set of distributions, which generalizes the usual concept of functions.

The Fourier transform of any function $u\in\mathcal{S}$ is given by

$$\mathcal{F}[u](x)=\hat{u}(x):=\int_{\mathbb{R}^n}e^{-ix\cdot\omega}u(\omega)\bar{d}\omega,$$

where $\bar{d}\omega=d\omega/(2\pi)^{n/2}$, while $d\omega$ and $x\cdot\omega=\delta_{ij}x^i\omega^j$ are the Euclidean volume element and scalar product in $\mathbb{R}^n$, respectively. The map $\mathcal{F}:\mathcal{S}\rightarrow\mathcal{S}$ is an isomorphism. The inverse map is given by

$$\mathcal{F}^{-1}[u](x)=\check{u}(x):=\int_{\mathbb{R}^n}e^{ix\cdot\omega}u(\omega)\bar{d}\omega.$$

An important property of the Fourier transform useful in PDE theory is the following: $[\partial_x^\alpha u(x)]\hat{}=i^{|\alpha|}\omega^\alpha\hat{u}(\omega)$, and $[x^\alpha u(x)]\hat{}=i^{|\alpha|}\partial_\omega^\alpha\hat{u}(\omega)$, that is, it converts smoothness of the function into decay properties of the transformed function, and vice versa. The Fourier transform is extended to an isomorphism $\mathcal{F}:L^2\rightarrow L^2$, first proving Parseval's theorem, that is, $(u,v)=(\hat{u},\hat{v})$ for all u, $v\in\mathcal{S}$ (which gives Plancherel's formula for norms, $\|u\|=\|\hat{u}\|$, in the case that the norm comes from an inner product, with $u=v$) and second recalling that $\mathcal{S}$ is dense in L^2.

The definition of Sobolev spaces H^s for s real is based on Parseval's theorem. First recall that every $u\in H^k$ with non-negative integer k satisfies $\partial^\alpha u\in L^2$ for $|\alpha|\leqslant k$, so Parseval's theorem implies $|\omega|^k\hat{u}(\omega)\in L^2$. Second, notice that there exists a positive constant c such that $(1/c)\langle\omega\rangle\leqslant(1+|\omega|)\leqslant c\langle\omega\rangle$, where $\langle\omega\rangle=(1+|\omega|^2)^{1/2}$. Therefore, one arrives at the following definition. The Sobolev space H^s for any $s\in\mathbb{R}$ consists of locally square integrable functions in $\mathbb{R}^n$ such that $\langle\omega\rangle^s\hat{u}\in L^2$. This space is a Hilbert space with the inner product

$$(u,v)_s:=\int_{\mathbb{R}^n}\langle\omega\rangle^{2s}\overline{\hat{u}(\omega)}\hat{v}(\omega)d\omega,$$

and the associated norm is denoted by

$$\|u\|_s^2:=\int_{\mathbb{R}^n}\langle\omega\rangle^{2s}|\hat{u}(\omega)|^2d\omega.$$

One can check that $H^s\subset H^{s'}$ whenever $s'\leqslant s$. Notice that negative indices are allowed. The elements of those spaces are distributions. Furthermore, the Hilbert space H^{-s} is the dual of H^s. Finally, two more spaces are needed later on, $H^{-\infty}:=\cup_{s\in\mathbb{R}}H^s$ and $H^{\infty}:=\cap_{s\in\mathbb{R}}H^s$. These spaces are, with appropriate metrics on them, Fréchet spaces. A closer picture of the kind of element these spaces may contain is given by the following observations. The Sobolev embedding lemma implies that $H^\infty\subset C^\infty$, while the opposite inclusion is not true. Also notice that $\mathcal{S}\subset H^\infty$, and therefore $H^{-\infty}\subset\mathcal{S}'$, so the elements of $H^{-\infty}$ are tempered distributions.

3. Pseudodifferential operators

Let S^m, with $m\in\mathbb{R}$, be the set of complex valued smooth functions $\underline{p}(x,\omega)$ from $\mathbb{R}^n\times\mathbb{R}^n$, such that

$$|\partial_x^\beta\partial_\omega^\alpha\underline{p}(x,\omega)|\leqslant C_\alpha\langle\omega\rangle^{m-|\alpha|},\tag{A1}$$

with C_α a constant depending on the multi-index α, and $\langle\omega\rangle=(1+|\omega|^2)^{1/2}$. This is the space of functions whose elements are associated with operators. It is called the space of symbols, and its elements $\underline{p}(x,\omega)$ symbols. There is no asymptotic behavior needed in the x variable, because Fourier integrals are thought to be carried out in the ω variable. The asymptotic behavior of this variable is related to the order of the associated differential operator, as one can shortly see in the definition of the map that associates functions $\underline{p}(x,\omega)$ with operators $p(x,\partial_x)$. One can check that $S^{m'}\subset S^m$ whenever $m'\leqslant m$. Two more spaces are needed later on, $S^\infty:=\cup_{m\in\mathbb{R}}S^m$ and $S^{-\infty}:=\cap_{m\in\mathbb{R}}S^m$.

Given any $\underline{p}(x,\omega)\in S^m$, the associated operator $p(x,\partial_x):\mathcal{S}\rightarrow\mathcal{S}$ is said to belong to ψ^m and is determined by

$$p(x,\partial_x)(u)=\int_{\mathbb{R}^n}e^{ix\cdot\omega}\underline{p}(x,\omega)\hat{u}(\omega)\bar{d}\omega,\tag{A2}$$

for all $u\in\mathcal{S}$. The constant m is called the order of the operator. It is clear that $u\in\mathcal{S}$ implies $p(x,\partial_x)(u)\in C^\infty$; however, the proof that $p(u)\in\mathcal{S}$ is more involved. One has to show that $p(u)$ and its derivatives decay faster than any polynomial in x. The idea is to multiply Eq. (A2) by x^α and recall the relation $i^{|\alpha|}x^\alpha e^{i\omega\cdot x}=\partial_\omega^\alpha e^{i\omega\cdot x}$. Integration by parts and the inequality (A1) imply that the resulting integral converges and is bounded in x. This gives the decay.

The polynomial symbols $\underline{p}(x,\omega)=\Sigma_{|\alpha|=0}^m a_\alpha(x)(i\omega)^\alpha$ with non-negative integer m correspond to differential operators of order m, $p(x,\partial_x)=\Sigma_{|\alpha|=0}^m a_\alpha(x)\partial_x^\alpha$. An example of a pseudodifferential operator that is not differential is given by the symbol $\underline{p}(\omega)=\chi(\omega)|\omega|^k\sin[\ln(|\omega|)]$, where k is a real constant and $\chi(\omega)$ is a cut function at $|\omega|=1/2$. That is a smooth function that vanishes for $|\omega|\leqslant 1/2$ and is identically 1 for $|\omega|\geqslant 1$. The cut function is needed to have a smooth function at $\omega=0$. This symbol belongs to S^k. The function $\underline{p}(\omega)=\chi(\omega)\ln(|\omega|)$ is not a symbol, because $|\underline{p}(\omega)|\leqslant c_0\langle\omega\rangle^\epsilon$, for every $\epsilon>0$, but $|\partial_\omega\underline{p}(\omega)|\leqslant c_1\langle\omega\rangle^{-1}$, and the change in the decay is bigger than 1, which is the value of $|\alpha|$ in this case. Another useful example to understand the symbol spaces is $\underline{p}(\omega)=\chi(\omega)|\omega|^k\ln(|\omega|)$, with k a real constant. This function is not a symbol for k natural or zero, for the same reason as in the previous example. However, it is a symbol for the remaining cases, belonging to $S^{k+\epsilon}$, for every $\epsilon>0$.

A very useful operator is $\Lambda^s:\mathcal{S}\rightarrow\mathcal{S}$ given by

$$\Lambda^s(u):=\int_{\mathbb{R}^n}e^{i\omega\cdot x}\langle\omega\rangle^s\hat{u}(\omega)\bar{d}\omega,$$

where s is any real constant. This is a pseudodifferential operator that is not differential. Its symbol is $\underline{\Lambda}^s=\langle\omega\rangle^s$, which belongs to S^s, and then one says $\Lambda^s\in\psi^s$. It is usually denoted as $\Lambda^s=(1-\Delta)^{s/2}$. It can be extended to Sobolev spaces, that is, to an operator $\Lambda^s:H^s\rightarrow L^2$. This is done by noticing the bound $\|\Lambda^s(u)\|=\|u\|_s$ for all $u\in\mathcal{S}$, and recalling that $\mathcal{S}$ is a dense subset of L^2. This operator gives a picture of what is meant by an s derivative, for s real. One can also rewrite the definition of H^s, saying that $u\in H^s$ if and only if $\Lambda^s(u)\in L^2$.

Pseudodifferential operators can be extended to operators acting on Sobolev spaces. Given $p\in\psi^m$, it defines an operator $p(x,\partial_x):H^{s+m}\rightarrow H^s$. This is the reason to call m the order of the operator. The main idea of the proof is again to translate the basic estimate (A1) into the symbol to an L^2-type estimate for the operator, and then use the density of $\mathcal{S}$ in L^2. The translation is more complicated for a general pseudodifferential operator than for Λ^s, because symbols can depend on x. Intermediate steps are needed, involving estimates on an integral representation of the symbol, called the kernel of the pseudodifferential operator. Pseudodifferential operators can also be extended to act on distribution spaces $\mathcal{S}'$, the dual of Schwartz spaces $\mathcal{S}$.

An operator $p:H^{-\infty}\rightarrow H^{-\infty}$ is called a smoothing operator if $p(H^{-\infty})\subset C^\infty$. That means $p(u)$ is smooth regardless of u being smooth. One can check that a pseudodifferential operator whose symbol belongs to $S^{-\infty}$ is a smoothing operator. For example, $\underline{p}(\omega)=e^{-|\omega|^2}\in S^{-\infty}$. However, not every smoothing operator is pseudodifferential. For example,

$\underline{p}(\omega)=\rho(\omega)$, with $\rho\in H^s$ for some s and having compact support, is a smoothing operator which is not pseudodifferential unless ρ is smooth. Friedrichs' mollifiers, J_ϵ for $\epsilon\in(0,1]$, are a useful family of smoothing operators, which satisfy $J_\epsilon(u)\to u$ in the L^2 sense, in the limit $\epsilon\to 0$, for each $u\in L^2$.

Consider one more example, the operator $\lambda:\mathcal{S}\to\mathcal{S}$ given by

$$\lambda(u):=\int_{\mathbb{R}^n}e^{i\omega\cdot x}i|\omega|\chi(\omega)\hat{u}(\omega)\bar{d}\omega,$$

where $\chi(\omega)$ is again a cut function at $|\omega|=1/2$. The symbol is $\underline{\lambda}(\omega)=i|\omega|\chi(\omega)$. The cut function χ makes $\underline{\lambda}$ smooth at $\omega=0$. The operator without the cut function is $\ell:\mathcal{S}\to L^2$ given by

$$\ell(u):=\int_{\mathbb{R}^n}e^{i\omega\cdot x}i|\omega|\hat{u}(\omega)\bar{d}\omega.$$

Its symbol $\underline{\ell}(\omega)=i|\omega|$ does not belong to any S^m because it is not smooth at $\omega=0$. Both operators λ,ℓ can be extended to maps $H^1\to L^2$. What is more important, their extensions are essentially the same, because they differ in a smoothing, although not pseudodifferential, operator.

The asymptotic expansion of symbols is maybe the most useful notion related to pseudodifferential calculus. Consider a decreasing sequence $\{m_j\}_{j=1}^{\infty}$, with $\lim_{j\to\infty}m_j=-\infty$. Let $\{p_j\}_{j=1}^{\infty}$ be a sequence of symbols $\underline{p}_j(x,\omega)\in S^{m_j}$. Assume that these symbols are asymptotically homogeneous in ω of degree m_j, that is, they satisfy $\underline{p}_j(x,t\omega)=t^{m_j}\underline{p}_j(x,\omega)$ for $|\omega|\geqslant 1$. Then, a symbol $\underline{p}\in S^{m_1}$ has the asymptotic expansion $\Sigma_j\underline{p}_j$ if and only if

$$\left(\underline{p}-\sum_{j=1}^{k}\underline{p}_j\right)\in S^{m(k+1)},\quad\forall\quad k\geqslant 1,\tag{A3}$$

and it is denoted by $\underline{p}\sim\Sigma_j\underline{p}_j$. The first order term in the expansion, $\underline{p}_1$, is called the principal symbol. Notice that m_j are real constants, not necessarily integers. Every asymptotic expansion defines a symbol, that is, every function of the form $\Sigma_j\underline{p}_j$ belongs to some symbol space S^{m_1}. However, not every symbol $\underline{p}\in S^m$ has an asymptotic expansion. Consider the example $\underline{p}(\omega)=\chi(\omega)|\omega|^{1/2}\ln(|\omega|)$. The set of symbols that admit an asymptotic expansion of the form (A3) is called classical, it is denoted by $S^m_{\rm cl}$, and the corresponding operators are said to belong to $\psi^m_{\rm cl}$. One then has $S^m_{\rm cl}\subset S^m$. Notice that if two symbols $\underline{p}$ and $\underline{q}$ have the same asymptotic expansion $\Sigma_j\underline{p}_j$, then they differ in a pseudodifferential smoothing operator, because

$$\underline{p}-\underline{q}=\left(\underline{p}-\sum_{j=1}^{k}\underline{p}_j\right)-\left(\underline{q}-\sum_{j=1}^{k}\underline{p}_j\right)\in S^{m(k+1)}$$

for all k, and $\lim_{k\to\infty}m_j=-\infty$, so $(\underline{p}-\underline{q})\in S^{-\infty}$. This is the precise meaning for the rough sentence, "what really matters is the asymptotic expansion."

There is in the literature a more general concept of asymptotic expansion. It does not require that the $\underline{p}_j$ to be asymptotically homogeneous. We do not consider this generalization in these notes.

Most of the calculus of pseudodifferential operators consists of performing calculations with the highest order term in the asymptotic expansion and keeping careful track of the lower order terms. The symbol of a product of pseudodifferential operators is not the product of the individual symbols. Moreover, the former is difficult to compute. However, an asymptotic expansion can be explicitly written for classical symbols, and one can check that the principal symbol of the product is equal to the product of the individual principal symbols. More precisely, given $p\in\psi^r_{\rm cl}$ and $q\in\psi^s_{\rm cl}$, then the product is a well defined operator $pq\in\psi^{r+s}_{\rm cl}$ and the asymptotic expansion of its symbol is

$$\underline{pq}\sim\sum_{|\alpha|\geqslant 0}\frac{1}{i^{|\alpha|}\alpha!}[\partial^\alpha_\omega\underline{p}(x,\omega)][\partial^\alpha_x\underline{q}(x,\omega)].$$

Notice that the first term in the asymptotic expansion of a commutator $[p,q]=pq-qp$, that is, its principal symbol, is precisely $1/i$ times the Poisson bracket of their respective symbols, $\{\underline{p},\underline{q}\}=\Sigma_j(\partial_{\omega^j}\underline{p}\,\partial_{x^j}\underline{q}-\partial_{x^j}\underline{p}\,\partial_{\omega^j}\underline{q})$.

Similarly, the symbol of the adjoint pseudodifferential operator is not the adjoint of the original symbol. However, this is true for the principal symbols. The proof is based in an asymptotic expansion of the following equation:

$$\underline{(p^*)}(x,\omega)=\int\int_{\mathbb{R}^n}e^{-i(x-x')\cdot(\omega-\omega')}(\underline{p})^*(x',\omega')\bar{d}x\bar{d}\omega.$$

There are three main generalizations of the theory of pseudodifferential operators present in the literature. First, the operators act on vector valued functions instead of on scalar functions. While this is straightforward, the other generalizations are more involved. Second, the space of symbols is enlarged, first done in [27]. It is denoted as $S^m_{\rho\delta}$, and its elements satisfy $|\partial^\beta_x\partial^\alpha_\omega\underline{p}(x,\omega)|\leqslant C_{\alpha,\beta}\langle\omega\rangle^{m-\rho|\alpha|+\delta|\beta|}$, with $C_{\alpha,\beta}$ a constant depending on the multi-indices α and β. The extra indices have been tuned to balance two opposite tendencies; on the one hand, to preserve some properties of differential operators; on the other hand, to maximize the amount of new objects in the generalization. These symbol spaces contain functions like $\underline{p}(x,\omega)=\langle\omega\rangle^{a(x)}$, which belongs to $S^m_{1,\delta}$, where $\delta>0$ and $m=\max_{x\in\mathbb{R}^n}a(x)$. Third, the domain of the functions $p(x,\omega)$ is changed from $\mathbb{R}^n\times\mathbb{R}^n$ to $\Omega\times\mathbb{R}^n$, with $\Omega\subset\mathbb{R}^n$ any open set. A consequence in the change of the domain is that $p:C^\infty_0(\Omega)\to C^\infty(\Omega)$, so the domain and range of p are not the same, which makes it more difficult to define the product of pseudodifferential operators. These notes are intended to be applied to hyperbolic PDEs on $\mathbb{R}^n$, which are going to be converted to pseudodifferential operators in S^1, so there is no need to consider the last two generalizations.

4. Further reading

There is no main reference followed in these notes; however, a good place to start is [3]. Notice that the notation is not precisely the one in that reference. The introduction is good, and the definitions are clear. The proofs are difficult to follow. More extended proofs can be found in [29], together with some historical remarks. The whole subject is clearly written in [30]. It is not the most general theory of pseudodifferential operators, but it is close to these notes. A slightly different approach can be found in [31], and detailed calculations to find parametrices are given in [32]. The introduction of [33] is very instructive. The first order reduction using Λ is due to Calderón in [26], and a clear summary of this reduction is given in [30].

The field of pseudodifferential operators grew out of a special class of integral operators called singular integral operators. Mikhlin in 1936 and Calderón and Zygmund in the beginning of 1950s carried out the first investigations. The field started to develop really fast after a suggestion by Peter Lax in 1963 [34], who introduced the Fourier transform to represent singular integral operators in a different way. Finally, the work of Kohn and Nirenberg [35] presented the pseudodifferential operators as they are known today, and they proved their main properties. They showed that singular integral operators are the particular case of pseudodifferential operators of order zero. Further enlargements of the theory were due to Hörmander [27,36].

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