The behavior of hyperbolic heat equations' solutions near their parabolic limits

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Standard energy methods are used to study the relation between the solutions of one parameter families of hyperbolic systems of equations describing heat propagation near their parabolic limits, which for these cases are the usual diffusive heat equation. In the linear case it is proven that given any solution to the hyperbolic equations there is always a solution to the diffusion equation which after a short time stays very close to it for all times. The separation between these solutions depends on the square of the ratio between the assumed very short decay time appearing in Cattaneo's relation and the usual characteristic smoothing time (initial data dependent) of the limiting diffusive equation. The techniques used in the linear case can be readily used for nonlinear equations. As an example we consider the theories of heat propagation introduced by Coleman, Fabrizio, and Owen, and prove that near a solution to the limiting diffusive equation there is always a solution to the nonlinear hyperbolic equations for a time which usually is much longer than the decay time of the corresponding Cattaneo relation. An alternative derivation of the heat theories of divergence type, which are consistent with thermodynamic principles, is given as an appendix.

I. INTRODUCTION

It is usually understood that heat propagation in a solid is governed by *Fourier's law*

$$\mathbf{q}=-k(T)\nabla T, \tag{1}$$

where $\mathbf{q}$ is the heat flux, T is the absolute temperature, and k is the thermal conductivity. If we assume that the specific internal energy e is only dependent on the temperature (so that $de=\gamma_0\, dT$ where γ_0 is the specific heat of the body) and that k is constant, then conservation of energy

$$\frac{de}{dt}=-\nabla\cdot\mathbf{q}, \tag{2}$$

combined with Eq. (1) leads us to the well-known heat diffusion equation for the temperature

$$\frac{\partial T}{\partial t}=\kappa\nabla^2 T, \tag{3}$$

where $\kappa=k/\gamma_0$ is the thermal diffusivity.

Equation (3) is a parabolic equation and consequently has the unphysical property that the information propagates at arbitrarily high speed. If, for example, we solve the Cauchy problem for

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Eq. (3) on the real line with initial data of compact support, then the corresponding solution will not have compact support for any $t>0$. The first attempt to solve this problem was carried out by Cattaneo,[1] who suggested that Fourier's law should be replaced by a more general law, now known as *Cattaneo's equation*

$$\tau(T)\,\frac{\partial \mathbf{q}}{\partial t}+\mathbf{q}=-k(T)\nabla T, \tag{4}$$

where τ is a relaxation time which depends on the mechanism of heat transport. [Other equations which modify Fourier law, like the heat flux of Jeffrey's type, have been considered in the literature.[2] However, Jeffrey's type equations together with Eq. (2) do not give rise to hyperbolic systems of equations, and so its study is beyond the strength of the techniques used here.] Equations (2) and (4) cannot in general be combined to obtain a single scalar equation for the temperature. However, we can consider the particular case in which τ and k are constants, and the energy e depends only on T in a linear way; then conservation of energy (2) combined with (4) leads us to the so-called *Telegraph equation* for the temperature

$$\frac{1}{c^2}\,\frac{\partial^2 T}{\partial t^2}+\frac{1}{\kappa}\,\frac{\partial T}{\partial t}-\nabla^2 T=0. \tag{5}$$

Equation (5) is hyperbolic and $c=\sqrt{\kappa/\tau}$ is its characteristic speed. The solutions of Eq. (5) are dissipative wavelike solutions (in the sense that no information at a given time can propagate with speed higher than c). If we now solve the Cauchy problem for Eq. (5) on the real line with initial data of compact support, then the solution will have compact support for all $t\geqslant 0$. This is not a property of Eq. (5) alone, but also of the system (2), (4), which in most cases of interest—that is, choosing an energy e such that the theory is physically meaningful—is a hyperbolic system of equations. This property solves the "paradox of instant propagation" of information, and constitutes a fundamental difference between the Cauchy problems for Eq. (3) and system (2), (4).

There exists another important difference between the Cauchy problems for Eq. (3) and system (2), (4). To obtain a unique solution one has to give, in the first case, only one function as initial data, namely, the temperature at $t=0$. Instead, in the second case one has to give two functions, the temperature and the heat flux at $t=0$. Thus, system (2,4) has many more solutions than system (3). From a physical point of view this means that if one were to prepare an experiment in which the temperature of a body is a relevant variable, then, not only the initial temperature of the body would have to be controlled, but also "extra" initial data, namely, its initial heat flux. This would not be necessary if one knew that the extra data had no appreciable influence on the future temperature of the body or, in other words, if the temperature remained near the temperature predicted by diffusion equation (3). Then the question arises, why if the physically reasonable equations of heat propagation are hyperbolic, the more simple parabolic heat equation is, in most cases, in excellent agreement with experiments? Or why is it that only under very special circumstances, heat waves (hyperbolic phenomena) have been measured?[3]

In this article we will try to answer the questions above for two different theories of heat propagation, a simple linear one and a more complicated nonlinear one. The way to study this problem will be the following. We will think about the system of equations (2), (4) as a monoparametric system of equations—where τ is the parameter that recovers a parabolic "limiting" system of equations when the parameter goes to zero. In this limit, one has to replace Cattaneo's equation by Fourier's law. We will then study the relation between the solution of the parabolic system with some initial data, and the solution of the hyperbolic system with the same data for the temperature and some appropriate initial data for the other component.

Usual energy techniques, as those used in studies of hyperbolic partial differential equations, will be used to treat these problems. The *energy* (not to be confused with the *real* energy e of the physical system), a positive definite functional, will be defined essentially as an appropriate Sobolev norm of the difference between the solutions of the hyperbolic and parabolic systems. This energy will then be bounded as is usually done in the existence of solutions' theorems.

In Sec. II the simplest hyperbolic linear theory of heat propagation will be treated. The equations in this theory are (2) and (4) with constant coefficients and an energy which is linear in T [this system is thus equivalent to the Telegraph equation (5)]. It will be proven that given *any* arbitrary initial data for this hyperbolic system, the corresponding solution converges uniformly, as the parameter of the hyperbolic family goes to zero, to the solution of the limiting equation (3) with the same initial data for the temperature. Furthermore, given a finite value of the parameter, a bound for the difference between both solutions will be given in terms of their initial data and the parameter. In this simple linear case it will also be shown that the contribution of the extra data (mentioned above) to the solution of the hyperbolic system is the sum of two terms, one of which vanishes exponentially as t grows while the other is always small. All these assertions will be shown to hold globally—that is, for all $t \geqslant 0$.

For this linear system some of these results have already been shown. Caffarelli and Virga[4] partially obtained them by comparing the solutions of Eqs. (3) and (5) but using Green's functions to write the general solutions of those equations. Geroch[5] used Fourier transform techniques. The advantage of the energy techniques we use here, resides in the fact that they also work for nonlinear systems of equations.

Many authors have studied the problem of heat propagation and heat waves from different points of view. A broad review containing a bibliography was written by Joseph and Preziosi.[2] It turns out that the simple linear system we treat in Sec. II [and Eq. (5)] is incompatible with thermodynamic principles. Coleman, Fabrizio, and Owen (CFO)[6] have found that Eq. (4) is compatible with thermodynamics when the specific internal energy e is not only a function of the temperature but also of the heat flux

$$e = e_0(T) + a(T)q^2, \quad \text{where} \quad a(T) = \frac{Z(T)}{T} - \frac{Z'(T)}{2}, \tag{6}$$

with $Z(T) = \tau(T)/k(T)$. We will refer to CFO theories as the system of Eqs. (2), (4), and (6). Given three positive functions $k(T)$, $e_0(T)$, and $\tau(T)$ one obtains a particular theory. Morro and Ruggeri[7] have also found a family (closely related but different than the CFO) of nonlinear generalizations of Cattaneo's equations which is also compatible with thermodynamics and has better stability properties than the CFO family. The energy in Morro and Ruggeri theories is a function of only the temperature, and the corresponding system of equations is determined in terms of three functions of the temperature: the thermal conductivity $k(T)$, the specific heat $\gamma_0(T)$, and the temperature pulse speed $U_0(T)$. We will refer to these theories as MR theories.

Following the ideas of Liu,[8] and subsequent works culminating in that of Geroch and Lindblom,[9] we give in the Appendix a derivation of a class of dissipative heat theories of divergence type, that emerge from a set of postulates which are consistent with thermodynamic principles. After making some further assumptions it will be seen that subsets of these theories are, respectively, coincident with the CFO theories and with the Morro–Ruggeri ones.

In Sec. III the nonlinear system of equations of CFO theories will be studied. In this case the limiting system is a nonlinear parabolic equation. It will be proven that given a solution to the limiting equation near enough to equilibrium, that is near enough to the solution T=const and $q=0$, there always exist a solution to the hyperbolic system that is close to it during a finite (nonempty) time interval. Even more, it will be concluded that within this time interval, the difference between both temperatures uniformly converges to zero as the parameter goes to zero. The initial data for the hyperbolic system has to be chosen coincident with the data for the given parabolic solution. (This choice of the initial data is known as the *Initialization Procedure*,[10,11] and it essentially means that the extra data we mentioned above cannot be arbitrary. It has to be coincident with the value at $t=0$ of the corresponding variable in the limiting theory.)

The techniques employed here can be used for many classes of nonlinear systems, in particular they are applicable to the theories of heat propagation recently introduced by Morro and Ruggeri.[7] These theories have a simpler structure than the CFO ones treated here and so allow for greater generality (stronger results) and complete estimates which can then be used in real physical situations. Some progress is being made in this subject and the results will be published elsewhere, for they need a rather more detailed use of the structure of the equations.

II. LINEAR HYPERBOLIC HEAT THEORY

We consider the problem of heat propagation in the cylinder (we only consider a one dimensional problem for simplicity, the proofs for higher dimensional cases are more involved but do not add any substantial understanding) $S^1\times\mathbb{R}^+$ given by, $0\leq x\leq L$ with 0 and L identified, and $t\geq 0$. The linear hyperbolic heat theory we are going to consider in this section emerges from Eqs. (2) and (4) when all the coefficients are constant, plus an energy linear with respect to the temperature. Then the equations are

$$\gamma_0 \frac{\partial T}{\partial t} = -\frac{\partial q}{\partial x}, \quad \tau \frac{\partial q}{\partial t} = -k \frac{\partial T}{\partial x} - q.$$

We remind that γ_0 is the specific heat, k is the thermal conductivity, and τ is a relaxation time. Now, if we divide the first equation by γ_0 and the second by $-k$, and introduce new variables $u=T$ and $v=-q/\gamma_0$, the equations become (subscript t and x denote partial derivatives)

$$u_t = v_x, \tag{7}$$

$$\epsilon^2 v_t = u_x - \frac{1}{\kappa} v, \tag{8}$$

where $\kappa = k/\gamma_0$ is the thermal diffusivity and $\epsilon = 1/c = \sqrt{\tau/\kappa}$ is the reciprocal of the characteristic speed of the system, which will be taken as the parameter of the family. [In the Introduction we said that τ was the parameter of the system. Notice that ϵ^2 is proportional to τ, so that it is an equally good parameter. It is easy to see, eliminating v from the system (7), (8), that the temperature u obeys the Telegraph equation (5).] The equations so written are explicitly seen to be a symmetric hyperbolic[12] system for all $\epsilon > 0$. In the limit $\epsilon \to 0$ the system becomes

$$u_t = v_x, \quad 0 = u_x - \frac{1}{\kappa} v.$$

Eliminating v and calling u^0 the solution of this limiting equation, we obtain

$$\frac{1}{\kappa} u_t^0 - u_{xx}^0 = 0, \tag{9}$$

which is nothing but the usual diffusive heat equation (3) and u^0 is the parabolic temperature.

Now let us think about the Cauchy problem (initial value problem) for both Eq. (9) and system (7), (8). To solve the parabolic equation (9) one has to give only one function as initial data, while to solve the hyperbolic system (7), (8) one has to give two functions. We want to study the relation between the solution (u,v) of the hyperbolic system (7), (8) and the solution u^0 of the parabolic equation (9) when both temperatures (u and u^0) have the same initial datum (to order ϵ^2), and v has arbitrary initial datum. That is to say

$$u|_{t=0} = f(x), \tag{10}$$

$$v|_{t=0}=g(x), \tag{11}$$

$$u^0|_{t=0}=f(x)+\kappa\epsilon^2(g_x(x)-\kappa f_{xx}(x)). \tag{12}$$

The questions we want to answer are the following. What is the relation between (u,v) and u^0 for a finite value of ϵ? Does the hyperbolic solution u tend to the parabolic solution u^0 in the limit $\epsilon\to 0$?; and what happens to the extra initial data g we have to give to solve system (7), (8)? Finally, can we bound the difference $u-u^0$ in terms of ϵ and the initial data? We answer these questions through the theorem we state below.

Theorem 1: *Let $f\in C^{n+2}(S^1)$ and $g\in C^{n+1}(S^1)$ be the initial data for the system (7), (8) and Eq. (9) as shown in Eqs. (10), (11), (12) with $n\geqslant 2$. Then the corresponding solutions (u,v) and u^0 are related as follows:*

$$u=u^0-\kappa\epsilon^2\Delta_x \exp(-t/\kappa\epsilon^2)+u_R, \tag{13}$$

$$v=\kappa u_x^0+\Delta \exp(-t/\kappa\epsilon^2)+v_R, \tag{14}$$

where $\Delta(x)=g(x)-\kappa f_x(x)\in C^{n+1}(S^1)$ and

(i) The Sobolev norms of u_R and v_R can be bounded for all $t\geqslant 0$ in terms of the Sobolev norms of the initial data as follows:

$$\|u_R\|^2_{H^m}+\epsilon^2\|v_R\|^2_{H^m}\leqslant\epsilon^4\kappa^4(\|f_{xx}\|^2_{H^m}+\epsilon^2\tfrac{3}{2}\|\Delta_{xx}\|^2_{H^m}+\epsilon^4\kappa^2\|\Delta_{xxx}\|^2_{H^m}), \tag{15}$$

with $0\leqslant m\leqslant n-2$.

(ii) When $n\geqslant 3$, $u_R=O(\epsilon^2)$, $v_R=O(\epsilon)$ pointwise and $u,v\in C^{n-3}(S^1)$ even in the limit $\epsilon\to 0$. [We say that a function $F(x,t,\epsilon)$ is of order ϵ^p in the H^m sense—and write $F=O(\epsilon^p)$, if $0\neq\lim_{\epsilon\to 0}[\epsilon^{-p}F(x,t,\epsilon)]\in H^m$. $O(1)=O(\epsilon^0)$.]

We can now answer the questions asked prior to the theorem. Theorem 1 explicitly shows that the new temperature u equals the usual temperature u^0 plus terms of order ϵ^2, so that these terms vanish when $\epsilon\to 0$. The second component v [Eq. (14)] behaves as the derivative of the temperature plus two terms; one is of order one for $t=0$ but exponentially decreasing as t grows—this allows the initial data to be arbitrary (no "initialization procedure"[11] is necessary here); while the other term is of order ϵ.

To see the physical implications of this inequality we can take the expression for $m=0$, and after dividing by the whole initial Sobolev energy rewrite it as

$$\frac{E_R}{E_T(0)}\leqslant\frac{\tau^2}{\tau_d^2}, \tag{16}$$

where τ_d is a time constructed from the initial data. For the case where $g(x)=\kappa f_x(x)$, $(\Delta=0)$, we have

$$\tau_d^2=\frac{1}{\kappa^2}\frac{\|f\|^2_{L^2}+\tau\kappa\|f_x\|^2_{L^2}}{\|f_{xx}\|^2_{L^2}}. \tag{17}$$

If we now assume $f(x)\approx f_0\sin((2\pi/l)x)$ we see that

$$\tau_d^2\approx\frac{l^4}{(2\pi)^4\kappa^2} \tag{18}$$

and so

$$\frac{E_R}{E_T(0)} = \frac{\tau^2}{\tau_d^2} \approx \frac{(2\pi)^4 \tau^2 \kappa^2}{l^4}. \tag{19}$$

As an example of everyday material consider a piece of iron at room temperature, then we have $\kappa \approx 2.1 \times 10^{-1}$ cm^2/s, $\epsilon \approx 1/$(speed of sound)$\approx 2.0 \times 10^{-6}$s/cm, and so $\tau = \epsilon^2 \kappa \approx 8.4 \times 10^{-13}$ s. So we see that for perturbations off equilibrium (constant temperature) to behave in a different form than the usual dissipation their length scale has to be $l \leqslant 2.2 \times 10^{-6}$ cm.

For other initial data configurations similar order of magnitude estimates follow. Notice also that while the heat flow and the gradient of temperature approach their equilibrium configurations, $T=$const, $q=0$, at a rate given by the initial data time, τ_d, their difference approaches zero at the much faster rate given by τ. So that the statement of the theorem is not that "*things go to equilibrium*," but rather that "*the failure of the Fourier Law approach a small value in a time much shorter than the time needed for equilibrium to settle.*"

It is well known that the solution (u,v) of a symmetric hyperbolic system like (7), (8) has a certain degree of differentiability (depending on the smoothness of the initial data). What is important about this point in Theorem 1, is that even in the limit $\epsilon \to 0$ the solution preserves certain differentiability; even more, if $n \geqslant 3$ then u uniformly converges to u^0 when $\epsilon \to 0$, as follows from the following inequality:

$$\begin{aligned}
|u-u^0| &= |u_R - \kappa\epsilon^2 \Delta_x \exp(-t/\kappa\epsilon^2)| \leqslant |u_R| + \kappa\epsilon^2 |\Delta_x| \exp(-t/\kappa\epsilon^2) \\
&\leqslant \sqrt{2}[\|u_R\|_{H^1} + \epsilon^2\kappa \|\Delta_x\|_{H^1} \exp(-t/\kappa\epsilon^2)] \leqslant \sqrt{2}[\sqrt{E_R} + \epsilon^2 \kappa \|\Delta_x\|_{H^1} \exp(-t/\kappa\epsilon^2)] \\
&\leqslant \sqrt{2}\epsilon^2\kappa[\kappa(\|f_{xx}\|^2_{H^m} + \epsilon^2 \tfrac{3}{2} \|\Delta_{xx}\|^2_{H^m} + \epsilon^4\kappa^2 \|\Delta_{xxx}\|^2_{H^m})^{1/2} + \|\Delta_x\|_{H^1} \exp(-t/\kappa\epsilon^2)],
\end{aligned}$$

where we have used Sobolev's embedding theorem in the second inequality, and inequality (15) in the last one. The whole factor in square brackets is a regular function of ϵ; thus, the inequality above shows uniform convergence, and gives an upper-bound for the difference between u and u^0 in terms of ϵ and the initial data.

Proof of Theorem 1: We can always write the solution of Eqs. (7), (8) as is done in Eqs. (13), (14). Then we have to prove that statements (i) and (ii) of the theorem hold. To do this we first obtain the equations that u_R and v_R satisfy; this is done by replacing u and v in Eqs. (7), (8) for the expressions given in Eqs. (13) and (14). The resulting equations are

$$u_{Rt} = v_{Rx}, \tag{20}$$

$$\epsilon^2 v_{Rt} = u_{Rx} - \frac{1}{\kappa} v_R + \epsilon^2 \rho, \tag{21}$$

where $\rho = -\kappa(\Delta_{xx} \exp(-t/\kappa\epsilon^2) + u^0_{xt})$. We now define the energy for (u_R, v_R) as

$$E_R = \frac{1}{L} \int_0^L dx[(u_R)^2 + \epsilon^2 (v_R)^2] = \|u_R\|^2_{L^2} + \epsilon^2 \|v_R\|^2_{L^2}. \tag{22}$$

To bound this energy we proceed as follows. Taking the time derivative of Eq. (22)

$$\dot{E}_R = \frac{2}{L} \int_0^L dx[u_R u_{Rt} + \epsilon^2 v_R v_{Rt}] \tag{23}$$

and using Eqs. (20), (21) we get

$$\dot{E}_R = \frac{2}{L}\int_0^L dx\left[u_R v_{Rx} + v_R u_{Rx} - \frac{1}{\kappa}(v_R)^2 + \epsilon^2 \rho v_R\right].$$

The first and second terms in the integral cancel each other under integration by parts, so that

$$\dot{E}_R = \frac{2}{L}\int_0^L dx\left[-\frac{1}{\kappa}(v_R)^2 + \epsilon^2 \rho v_R\right].$$

At this point we could drop the first term and conclude that $\dot{E}_R \leqslant (1/\tau)E_R + \tau \int_0^L dx\, \rho^2$. But we can do better; completing squares we obtain

$$\dot{E}_R = \frac{2}{L}\int_0^L dx\left[-\left(\frac{v_R}{\sqrt{\kappa}} - \epsilon^2 \rho \frac{\sqrt{\kappa}}{2}\right)^2 + \frac{\kappa}{4}\epsilon^4 \rho^2\right] \leqslant \epsilon^4 \frac{\kappa}{2L}\int_0^L dx\, \rho^2 \tag{24}$$

gaining an ϵ^4 factor. Integrating this inequality we obtain

$$E_R(t) \leqslant E_R(0) + \epsilon^4 \frac{\kappa}{2L}\int_0^t dt' \|\rho\|^2_{L^2}(t',\epsilon), \tag{25}$$

with

$$\|\rho\|^2_{L^2}(t,\epsilon) := \epsilon^4 \frac{\kappa}{2L}\int_0^L dx\, \rho^2(x,t,\epsilon).$$

But the choice of initial data implies $u_R(t=0)=0$ and $v_R(t=0) = -\epsilon^2 \kappa \Delta_{xx}$, so that

$$E_R(0) = \epsilon^6 \kappa^4 \|\Delta_{xx}\|^2_{L^2}$$

and the bound for the energy is

$$E_R(t) \leqslant \epsilon^6 \kappa^4 \|\Delta_x\|^2_{L^2} + \epsilon^4 \frac{\kappa}{2L}\int_0^t dt' \|\rho\|^2_{L^2}(t',\epsilon). \tag{26}$$

The quantity ρ is made out of initial data and solutions of the limiting equation, so that $\rho = O(1)$ $\forall t \geqslant 0$, and consequently the expression in brackets is finite even in the limit $\epsilon \to 0$. Then from Eq. (26) and the definition of energy it follows that $\|u_R\|^2_{L^2} = O(\epsilon^4)$ and $\|v_R\|^2_{L^2} = O(\epsilon^2)$. The next step in the proof is to express the term involving ρ of Eq. (26) in terms of the initial data. Recalling the definition of ρ we have

$$\frac{1}{2}\int_0^t dt' \|\rho\|^2_{L^2} = \frac{\kappa^2}{2L}\int_0^t dt' \int_0^L dx(\Delta_{xx}\exp(-t'/\kappa\epsilon^2) + u^0_{xt'})^2$$

$$\leqslant \frac{\kappa^2}{L}\int_0^t dt' \int_0^L dx[(\Delta_{xx})^2 \exp(-2t'/\kappa\epsilon^2) + (u^0_{xt'})^2], \tag{27}$$

where we have used $(a+b)^2 \leqslant 2(a^2+b^2)$. Performing the time integral in the first term in Eq. (27) and throwing away negative terms we obtain

$$\int_0^t dt' \int_0^L dx(\Delta_{xx})^2 \exp(-2t'/\kappa\epsilon^2) \leqslant \frac{\kappa\epsilon^2}{2}\int_0^L dx(\Delta_{xx})^2.$$

Using the limiting equation (9) we obtain for the second term in Eq. (27)

$$\int_0^t dt' \int_0^L dx(u^0_{xt'})^2 \leq \kappa \int_0^t dt' \int_0^L dx\, u^0_{xt'} u^0_{xxx}$$

$$\leq -\kappa \int_0^t dt' \int_0^L dx\, u^0_{xxt'} u^0_{xx}$$

$$\leq -\kappa \int_0^L dx \left[\frac{1}{2} (u^0_{xx})^2 \right]_0^t \leq \kappa \int_0^L dx [(f_{xx})^2 + \epsilon^4 \kappa^2 (\Delta_{xxx})^2].$$

Putting all this into Eq. (26) we obtain

$$\|u_R\|^2_{L^2} + \epsilon^2 \|v_R\|^2_{L^2} \leq \epsilon^4 \kappa^4 (\|f_{xx}\|^2_{L^2} + \epsilon^2 \tfrac{3}{2} \|\Delta_{xx}\|^2_{L^2} + \epsilon^4 \kappa^2 \|\Delta_{xxx}\|^2_{L^2}). \tag{28}$$

We thus have the inequality sought but for $m=0$. To obtain the inequality for arbitrary m notice that Eqs. (20), (21) for $(u_R,\ v_R)$ are linear and consequently the first spatial derivative of these functions will obey the same equations if we change ρ by ρ_x in Eq. (21). Then, using an energy $E'_R = \| u_{Rx} \|^2_{L^2} + \epsilon^2 \| v_{Rx} \|^2_{L^2}$ we obtain exactly the same inequality (28) but with each function differentiated once with respect to x. Now, adding $1/L^2$ times this last inequality plus Eq. (28) we obtain the inequality (15) for the H^1 norm of (u_R, v_R). It is clear that we can continue this procedure increasing one spatial derivative every time. We then obtain the inequality (15) for all the H^m norms of (u_R, v_R). These inequalities will be meaningful only if their right hand sides remain finite; and, because of the differentiability of the initial data, this is true for $0 \leq m \leq n-2$, but may be not true for $m \geq n-1$. This concludes the proof of statement (i).

Finally, comparing the ϵ orders of each term in the inequality (15), we note that statement (ii) is a direct consequence of statement (i) and Sobolev's embedding theorem. ■

III. NONLINEAR EQUATIONS OF HEAT PROPAGATION

As in Sec. II we consider the problem of heat propagation in the cylinder (the restriction to one spatial dimension is only for simplicity) $S^1 \times \mathrm{R}^+$ given by, $0 \leq x \leq L$ with 0 and L identified, and $t \geq 0$. We shall study in this section the nonlinear theories introduced by Coleman, Fabrizio, and Owen;[6] see the Appendix. Equations (A29) and (A30) for the temperature T and heat flux q, when restricted to one spatial dimension and written as a symmetric system, become (the prime denotes the derivative of a function of a single variable)

$$[\gamma_0(T) + a'(T) q^2] T_t = -q_x + 2 \frac{a(T) k(T)}{\tau(T)} q T_x + 2 \frac{a(T)}{\tau(T)} q^2, \tag{29}$$

$$\frac{\tau(T)}{k(T)} q_t = -T_x - \frac{1}{k(T)} q, \tag{30}$$

where—as explained in the Appendix, $\gamma_0(T) > 0$ is the specific heat, $k(T) > 0$ is the thermal conductivity, $\tau(T)$ is a relaxation time, and $a(T) = -T^2(\tau/T^2 k)'/2$. We shall consider the system (29),(30) as a monoparametric family of systems; instead of thinking of τ as that parameter—with the complication that it is a function—we introduce a parameter ϵ such that $\tau(T) = \epsilon^2 \tilde{\tau}(T)$, that is $\tau(T) = O(\epsilon^2)$ while $\tilde{\tau}(T) = O(1)$. Thus $a(T) = \epsilon^2 \tilde{a}(T)$ with $\tilde{a}(T) = -T^2(\tilde{\tau}/T^2 k)'/2$, so that Eqs. (29),(30) can be written as

$$\Gamma_T T_t = -q_x + 2\,\frac{ak}{\tau}\,q T_x + 2\,\frac{a}{\tau}\,q^2, \tag{31}$$

$$\epsilon^2 \Gamma_q q_t = -T_x - \frac{1}{k}\,q, \tag{32}$$

where

$$\Gamma_T(T,q) = \gamma_0 + \epsilon^2 \tilde{a}' q^2, \quad \Gamma_q(T) = \frac{\tilde{\tau}}{k}. \tag{33}$$

The limiting system, which is obtained taking the limit $\epsilon \to 0$, is a parabolic system.[Strictly speaking, one knows that this is the limiting system only in the cases for which the initial data for both systems (hyperbolic and parabolic) are chosen so that the theorem below holds, which implies that $T - T^0$ and $q - q^0$ do not diverge when $\epsilon \to 0$.] Calling (T^0, q^0) the solution of this system, it can be written as

$$\gamma_0(T^0) T^0_t = -q^0_x, \tag{34}$$

$$0 = -T^0_x - \frac{1}{k(T^0)}\,q^0 \tag{35}$$

or equivalently (as q^0 is a dependent variable)

$$\gamma_0(T^0) T^0_t = [k(T^0) T^0_x]_x. \tag{36}$$

Notice that (T=const, $q=0$) is an equilibrium (stationary) solution for both the hyperbolic and the parabolic systems. Also notice that to solve the initial value problem for Eqs. (31), (32) it is necessary to give two functions as initial data, while for Eqs. (34), (35) [or equivalently Eq. (36)] it is necessary to give only one function. We write the initial datum for Eq. (36) as

$$T^0|_{t=0} = \overline{T^0} + f(x), \quad \text{with} \quad \overline{T^0} = \frac{1}{L}\int_0^L dx\; T^0|_{t=0} > 0. \tag{37}$$

Now, given a solution T^0 of Eq. (36) we want to find out whether or not there exists a solution (T,q) of the hyperbolic system that "approximates" $(T^0, -kT^0_x)$ during a finite time interval. It will be shown that the answer is affirmative provided that the parabolic solution T^0 is near enough to equilibrium. The solution (T,q) of the hyperbolic system will be obtained by choosing the following initial data:

$$T|_{t=0} = T^0|_{t=0} = \overline{T^0} + f(x) \quad \text{and} \quad q|_{t=0} = -k\,\frac{df}{dx}. \tag{38}$$

This choice of initial data is known as the *Initialization Procedure* for the system (29), (30) (see the introduction and Refs. 10, 11). In what follows we shall restrict our study to the case where the thermal conductivity k is constant. This will be assumed just to simplify some calculations, though there is a version of the theorem below without this restriction.

Theorem 2: *Let $\gamma_0(T)$ and $\tilde{\tau}(T)$ be smooth and positive functions when $T>0$. Given a solution T^0 of Eq. (36) with initial datum chosen as in Eq. (37) with $\|f\|_{H^5} < C\overline{T^0}$ [How small the constant C is required to be will be stated in the proof of the theorem. See Eqs. (44), (67).], there exists a time interval $[0, t_0], t_0>0$ such that the solution (T,q) of Eqs. (31), (32) with initial data chosen as in Eq. (38) satisfy*

$$T=T^0+T_R \quad \text{and} \quad q=-kT^0_x+q_R, \quad t\in[0, t_0], \tag{39}$$

with $T_R=O(\epsilon^2)$ *and* $q_R=O(\epsilon)$ *in the* H^2 *sense.*

Before giving the proof we want to remark on an important difference between the linear and nonlinear cases. In the former, the H^0 Sobolev norm of any derivative of the difference between the hyperbolic and parabolic solutions could be bounded independently. Whereas in the latter, different order derivatives cannot be bounded independently; one has to choose a precise Sobolev norm order (depending only on the number of spatial dimensions of the problem) to define the energy, so that all the derivatives involved can be bounded together. In the one dimensional case the lowest Sobolev norm that can be bounded is H^2. Thus our energy for this case contains up to second derivatives of the variables.

Proof: We can always write (T,q) as done in Eq. (39); we have to prove that T_R and q_R are, respectively, $O(\epsilon^2)$ and $O(\epsilon)$ in the H^2 sense.

Using Eqs. (31), (32), and (36) we get the equations for (T_R, q_R)

$$\Gamma_T T_{Rt}=-q_{Rx}+2\frac{\tilde{a}}{\Gamma_q}qT_{Rx}+2k\frac{\tilde{a}}{\Gamma_q}qq_R-\epsilon^2\tilde{a}''T^0_t q^2+H, \tag{40}$$

$$\epsilon^2\Gamma_q q_{Rt}=-T_{Rx}-\frac{1}{k}q_R+\epsilon^2 k\Gamma_q T^0_{xt}, \tag{41}$$

where

$$H(T)=-[\gamma_0(T)-\gamma_0(T^0)]T^0_t.$$

The equations for (T_{Rx}, q_{Rx}) and (T_{Rxx}, q_{Rxx}) are obtained taking derivatives of Eqs. (40), (41). The choice of initial data implies that $T_R|_{t=0}=T_{Rx}|_{t=0}=T_{Rxx}|_{t=0}=0$ and $q_R|_{t=0}=q_{Rx}|_{t=0}=q_{Rxx}|_{t=0}=0$.

We now write the solution T^0 as follows:

$$T^0(x,t)=\overline{T^0}+\tilde{T}^0(x,t)$$

so that $\tilde{T}^0$ obeys the nonlinear heat equation

$$\tilde{T}^0_t=\kappa(\tilde{T}^0)\tilde{T}^0_{xx}, \quad \kappa(\tilde{T}^0):=\frac{k}{\gamma_0(\overline{T^0}+\tilde{T}^0)}>0, \tag{42}$$

with initial datum $\tilde{T}|_{t=0}=f(x)$. It is known[13] that the solution $\tilde{T}^0$ of a strongly parabolic equation like Eq. (42) obeys the energy estimate

$$\|\tilde{T}^0\|_{H^s}(t)\leq P(\|\kappa\|_{H^s},\|\tilde{T}^0|_{t=0}\|_{H^s},t)<\infty, \quad t\geq 0, \quad s=3,4,5,\ldots, \tag{43}$$

where P is a regular function of all its arguments that vanishes when $\|\tilde{T}^0|_{t=0}\|_{H^s}\rightarrow 0$. This implies that if we take $s=5$ in Eq. (43) and C small enough in the hypothesis that bounds the fifth norm of f, $\tilde{T}^0$ will obey $\|\tilde{T}^0\|_{H^5}\leq \overline{T^0}/3\sqrt{2}$ during a finite time interval $[0, t_1]$, then $|\tilde{T}^0| \leq \sqrt{2}\|\tilde{T}^0\|_{H^1}\leq \overline{T^0}/3$ and consequently

$$\tfrac{2}{3}\overline{T^0}\leq T^0(x,t)\leq \tfrac{4}{3}\overline{T^0}, \quad t\in[0, t_1]. \tag{44}$$

To be able to built the energy estimate we make some *a priori* assumptions. We take $0<\epsilon\leq\epsilon_0$ (ϵ_0 to be chosen by the end of the proof).
(i) The first *a priori* assumption is that there exists a pair of positive constants K_T and K_q such that $\Gamma_T>K_T$ and $\Gamma_q>K_q$.

This assumption allows us to define the energy functional for (T_R,q_R) as follows:

$$E_R(t,\epsilon)=\frac{1}{\epsilon^n L}\int_0^L \Gamma_T[(T_R)^2+L^2(T_{Rx})^2+L^4(T_{Rxx})^2]dx+\frac{1}{\epsilon^{(n-2)}L}\times\int_0^L \Gamma_q[(q_R)^2+L^2(q_{Rx})^2+L^4(q_{Rxx})^2]dx, \tag{45}$$

where n is non-negative. A direct consequence of assumption (i) is that the H^2 Sobolev norms of T_R and q_R satisfy

$$\|T_R\|_{H^2}^2\leq\frac{\epsilon^n}{K_T}E_R, \tag{46}$$

$$\|q_R\|_{H^2}^2\leq\frac{\epsilon^{(n-2)}}{K_q}E_R. \tag{47}$$

These inequalities and the Sobolev's embedding theorem imply that

$$|T_R|\leq\sqrt{2}\|T_R\|_{H^2}\leq\sqrt{\frac{2}{K_T}}\,\epsilon^{n/2}\sqrt{E_R}, \tag{48}$$

$$|T_{Rx}|\leq\frac{\sqrt{2}}{L}\|LT_{Rx}\|_{H^1}\leq\sqrt{\frac{2}{K_T}}\,\frac{\epsilon^{n/2}}{L}\sqrt{E_R}, \tag{49}$$

$$|q_R|\leq\sqrt{2}\|q_R\|_{H^2}\leq\sqrt{\frac{2}{K_q}}\,\epsilon^{(n-2)/2}\sqrt{E_R}, \tag{50}$$

$$|q_{Rx}|\leq\frac{\sqrt{2}}{L}\|Lq_{Rx}\|_{H^1}\leq\sqrt{\frac{2}{K_q}}\,\frac{\epsilon^{(n-2)/2}}{L}\sqrt{E_R}, \tag{51}$$

which say that while the energy E_R remains small, the variables T_R, q_R, T_{Rx}, and q_{Rx} also remain small.

As our aim is not to find explicitly the best estimate for the energy E_R, but just to show that there exists an estimate regular in ϵ, we now make a second *a priori* assumption that allows us to restrict the variables T_R, T_{Rx}, q_R, and q_{Rx} to some finite intervals, and consequently to bound the functions of these variables by their infinite norms. This eases the building of the energy estimate.
(ii) The second *a priori* assumption is then

$$E_R(t,\epsilon)\leq\frac{K_T}{2\epsilon_0^n}\left(\frac{\overline{T^0}}{3}\right)^2.$$

Assumption (ii) allows us to write inequalities (48)–(51) in terms of $\overline{T^0}$, ϵ, ϵ_0 K_T, and K_q as follows:

$$|T_R|\leq\left(\frac{\epsilon}{\epsilon_0}\right)^{n/2}\frac{\overline{T^0}}{3}, \tag{52}$$

$$|T_{Rx}| \leq \left(\frac{\epsilon}{\epsilon_0}\right)^{n/2} \frac{\overline{T^0}}{3L}, \tag{53}$$

$$|q_R| \leq \sqrt{\frac{K_T}{K_q}} \frac{\epsilon^{(n-2)/2}}{\epsilon_0^{n/2}} \frac{\overline{T^0}}{3}, \tag{54}$$

$$|q_{Rx}| \leq \sqrt{\frac{K_T}{K_q}} \frac{\epsilon^{(n-2)/2}}{\epsilon_0^{n/2}} \frac{\overline{T^0}}{3L}. \tag{55}$$

Finally, notice that Eqs. (44) and (52) imply that $0 < \overline{T^0}/3 \leq T \leq 5\overline{T^0}/3$ and then $1/T \leq 3/\overline{T^0} < \infty$. *Consequently* $\tilde{\tau}(T) > 0$ and so $1/\tilde{\tau}(T)$ is a smooth function within the allowed range of T. This will be used in what follows.

The key step in the proof is to build an energy estimate regular in ϵ. To obtain it we take the time derivative of Eq. (45) to get

$$E_{Rt} = \frac{1}{\epsilon^n L} \int_0^L \Gamma_{Tt}[(T_R)^2 + L^2(T_{Rx})^2 + L^4(T_{Rxx})^2]dx + \frac{1}{\epsilon^{(n-2)}L} \int_0^L \Gamma_{qt}[(q_R)^2 + L^2(q_{Rx})^2$$
$$+ L^4(q_{Rxx})^2]dx + \frac{2}{\epsilon^n L} \int_0^L \Gamma_T[T_R T_{Rt} + L^2 T_{Rx} T_{Rxt} + L^4 T_{Rxx} T_{Rxxt}]dx$$
$$+ \frac{2}{\epsilon^{(n-2)}L} \int_0^L \Gamma_q[q_R q_{Rt} + L^2 q_{Rx} q_{Rxt} + L^4 q_{Rxx} q_{Rxxt}]dx. \tag{56}$$

Now, we use equations [the equations for the spatial derivatives are obtained taking the corresponding derivatives of Eqs. (40), (41), and using (40), (41) back to arrange the equations] for T_R, T_{Rx}, T_{Rxx}, q_R, q_{Rx}, and q_{Rxx} and the definitions of Γ_T and Γ_q. Then, using integrations by parts, Eq. (56) can be written (after some algebra) in the form [It is convenient to remark that the expression given here is highly nonunique. Many of the terms that appear when one develops Eq. (56) can be treated in different ways, thus contributing to different functions in Eq. (57).]

$$E_{Rt}(t) = \frac{1}{\epsilon^n L} \int_0^L [Q_1(T_R)^2 + Q_2 L^2(T_{Rx})^2 + Q_3 L^4(T_{Rxx})^2]dx + \frac{1}{\epsilon^{(n-2)}L} \int_0^L [P_1 T_R + P_2 L T_{Rx}$$
$$+ P_3 L^2 T_{Rxx}]dx + \frac{1}{\epsilon^{(n-2)}L} \int_0^L [Q_4(q_R)^2 + Q_5 L^2(q_{Rx})^2 + Q_6 L^4(q_{Rxx})^2]dx$$
$$+ \frac{1}{\epsilon^{(n-2)}L} \int_0^L [R_1 q_R + R_2 L q_{Rx} + R_3 L^2 q_{Rxx}]dx + \frac{1}{\epsilon^n L} \int_0^L [R_4 q_R + R_5 L q_{Rx} + R_6 L^2 q_{Rxx}]dx$$
$$- \frac{1}{\epsilon^n L} \int_0^L [S_1(q_R)^2 + S_2 L^2(q_{Rx})^2 + S_3 L^4(q_{Rxx})^2]dx, \tag{57}$$

where (functions Q_i, P_i, and R_i will not be given explicitly because some of their expressions are too complicated—and give no insight into the calculation—and none of them are uniquely determined) Q_i for $i=1,...,6$, P_i and R_i for $i=1,2,3$ are smooth functions of the variables T_R, T_{Rx}, q_R and q_{Rx}, and the parameter ϵ. That is, they are bounded while the variables move in the ranges allowed by the *a priori* assumptions and $0 \leq \epsilon \leq \epsilon_0$.

R_i for $i=4,5,6$ are of the form

$$R_i = R_{i1} T_R + R_{i2} T_{Rx} + R_{i3} T_{Rxx},$$

where R_{ij}, $j=1,2,3$ are smooth functions of the variables T_R, T_{Rx}, q_R, and q_{Rx}, and the parameter ϵ, and

$$S_1 = \frac{1}{k}\Bigg[2 + \Delta\Gamma_q L^2 T^0_{xx} - \Delta\Gamma_q L^4 T^0_{xxxx} + 2(\Delta\Gamma_q)^2 L^4 (T^0_{xx})^2 + 2(\Delta\Gamma_q)^2 L^4 T^0_x T^0_{xxx}$$
$$+ \frac{1}{2}\frac{d\Delta\Gamma_q}{dT} L^2 (T^0_x)^2 + 4\frac{d\Delta\Gamma_q}{dT} L^4 (T^0_x)^2 T^0_{xx} + 2\left(\frac{d\Delta\Gamma_q}{dT}\right)^2 L^4 (T^0_x)^4$$
$$+ 6\frac{d\Delta\Gamma_q}{dT}\Delta\Gamma_q L^4 (T^0_x)^2 T^0_{xx} - 4\frac{d\Delta\Gamma_q}{dT} L^4 T^0_x T^0_{xxx} - 3\frac{d\Delta\Gamma_q}{dT} L^4 (T^0_{xx})^2$$
$$+ 2\frac{d^2\Delta\Gamma_q}{dT^2} L^4 \Delta\Gamma_q (T^0_x)^4 + 6\frac{d^2\Delta\Gamma_q}{dT^2} L^4 \Delta\Gamma_q (T^0_x)^2 T^0_{xx} - \frac{d^3\Delta\Gamma_q}{dT^3} L^4 (T^0_x)^4 \Bigg], \quad (58)$$

$$S_2 = \frac{1}{k}\left[2 + \frac{3}{2}\Delta\Gamma_q L^2 T^0_{xx} - 2(\Delta\Gamma_q)^2 L^2 (T^0_x)^2 + \frac{3}{2}\frac{d\Delta\Gamma_q}{dT} L^2 (T^0_x)^2 \right], \quad (59)$$

$$S_3 = \frac{2}{k}, \quad (60)$$

where

$$\Delta\Gamma_q(T) = \frac{\Gamma_q'}{\Gamma_q}. \quad (61)$$

Equation (57) gives an exact expression for the derivative of E_R. Now we bound each term in Eq. (57).

The first term is bounded as follows:

$$\frac{1}{\epsilon^n L}\int_0^L [Q_1 (T_R)^2 + Q_2 L^2 (T_{Rx})^2 + Q_3 L^4 (T_{Rxx})^2] dx \le (|Q_1|_\infty + |Q_2|_\infty + |Q_3|_\infty) \frac{\|T_R\|^2_{H^2}}{\epsilon^n}, \quad (62)$$

where $|Q_i|_\infty < \infty$ can be thought of as the least upper bound of the function Q_i while the variables obey Eqs. (52)–(55) and $\epsilon \le \epsilon_0$.

The second term is bounded as follows:

$$\frac{1}{\epsilon^{(n-2)} L}\int_0^L [P_1 T_R + P_2 L T_{Rx} + P_3 L^2 T_{Rxx}] dx$$
$$= \frac{1}{\epsilon^{(n-2)} L}\int_0^L \left[(\epsilon\sqrt{\tilde t} P_1)\left(\frac{T_R}{\epsilon\sqrt{\tilde t}}\right) + (\epsilon\sqrt{\tilde t} P_2) L \left(\frac{T_{Rx}}{\epsilon\sqrt{\tilde t}}\right) + (\epsilon\sqrt{\tilde t} P_3) L^2 \left(\frac{T_{Rxx}}{\epsilon\sqrt{\tilde t}}\right)\right] dx$$
$$\le \frac{1}{2\epsilon^{(n-2)} L}\int_0^L \epsilon^2 \tilde t [(P_1)^2 + (P_2)^2 + (P_3)^2] dx + \frac{1}{\epsilon^{(n-2)} L}\int_0^L \frac{1}{\epsilon^2 \tilde t}[(T_R)^2$$
$$+ L^2 (T_{Rx})^2 + L^4 (T_{Rxx})^2] dx \le \frac{\epsilon^{(4-n)}}{2}\tilde t (|P_1^2|_\infty + |P_2^2|_\infty + |P_3^2|_\infty) + \frac{1}{2\tilde t}\frac{\|T_R\|^2_{H^2}}{\epsilon^n}, \quad (63)$$

where we have used inequality $ab \leq \frac{1}{2}(a^2+b^2)$. The constant $0<\tilde{t}<\infty$ has units of time, and is chosen to minimize the bound (63).

The third term, in analogy with the first one, is bounded as follows:

$$\frac{1}{\epsilon^{(n-2)}L} \int_0^L [Q_4(q_R)^2+Q_5L^2(q_{Rx})^2+Q_6L^4(q_{Rxx})^2]dx \leq (|Q_4|_\infty+|Q_5|_\infty+|Q_6|_\infty) \frac{\|q_R\|_{H^2}^2}{\epsilon^{(n-2)}} . \tag{64}$$

To bound the fourth, fifth, and sixth terms we proceed as follows. Notice that S_1 and S_2 obey the inequalities

$$\begin{aligned} S_1 \geq \frac{1}{k} \Bigg\{ & 2-|\Delta\Gamma_q|L^2|T^0_{xx}|-|\Delta\Gamma_q|L^4|T^0_{xxxx}|-2|\Delta\Gamma_q|^2L^4|T^0_{xx}|^2-2|\Delta\Gamma_q|^2L^4|T^0_x||T^0_{xxx}| \\ & -\frac{1}{2}\left|\frac{d\Delta\Gamma_q}{dT}\right|L^2|T^0_x|^2-4\left|\frac{d\Delta\Gamma_q}{dT}\right|L^4|T^0_x|^2|T^0_{xx}|-2\left|\frac{d\Delta\Gamma_q}{dT}\right|^2L^4|T^0_x|^4 \\ & -6\left|\frac{d\Delta\Gamma_q}{dT}\right||\Delta\Gamma_q|L^4|T^0_x|^2|T^0_{xx}|-4\left|\frac{d\Delta\Gamma_q}{dT}\right|L^4|T^0_x||T^0_{xxx}|-3\left|\frac{d\Delta\Gamma_q}{dT}\right|L^4|T^0_{xx}|^2 \\ & -2\left|\frac{d^2\Delta\Gamma_q}{dT^2}\right|L^4|\Delta\Gamma_q||T^0_x|^4-6\left|\frac{d^2\Delta\Gamma_q}{dT^2}\right|L^4|\Delta\Gamma_q||T^0_x|^2|T^0_{xx}|-\left|\frac{d^3\Delta\Gamma_q}{dT^3}\right|L^4|T^0_x|^4\Bigg\}, \end{aligned} \tag{65}$$

$$S_2 \geq \frac{1}{k}\left\{2-\frac{3}{2}|\Delta\Gamma_q|L^2|T^0_{xx}|-2|\Delta\Gamma_q|^2L^2|T^0_x|^2-\frac{3}{2}\left|\frac{d\Delta\Gamma_q}{dT}\right|L^2|T^0_x|^2\right\}. \tag{66}$$

Notice that these inequalities involve up to fourth derivatives of T^0. We now require that the constant C to be small enough so that Eq. (43) and Sobolev's embedding theorem guarantee that

$$\begin{aligned} |\Delta\Gamma_q|_\infty \left|L^i \frac{d^iT^0}{dx^i}\right| &\leq \frac{1}{20}, \quad i\leq 4, \\ \left|\frac{d\Delta\Gamma_q}{dT}\right|_\infty \left|L^i \frac{d^iT^0}{dx^i}\right|\left|L^j \frac{d^jT^0}{dx^j}\right| &\leq \frac{1}{20}, \quad i+j\leq 4, \\ \left|\frac{d^2\Delta\Gamma_q}{dT^2}\right|_\infty \left|L^i \frac{d^iT^0}{dx^i}\right|\left|L^j \frac{d^jT^0}{dx^j}\right|\left|L^l \frac{d^lT^0}{dx^l}\right| &\leq \frac{1}{20}, \quad i+j+l\leq 4, \\ \left|\frac{d^3\Delta\Gamma_q}{dT^3}\right|_\infty \left|L \frac{dT^0}{dx}\right|^4 &\leq \frac{1}{20} \end{aligned} \tag{67}$$

during the time interval $[0, t_1]$. These conditions together with Eqs. (65), (66) imply that

$$S_1 \geq \frac{1}{k} > 0 \tag{68}$$

and

$$S_2 \geq \frac{1}{k} > 0. \tag{69}$$

We also have

$$S_3=\frac{2}{k}>0. \tag{70}$$

All this tells us that the sixth line in Eq. (57) is negative definite. [Notice that in the case we are considering (k=const), $\Delta\Gamma_q=\tilde{\tau}'/\tilde{\tau}=\tau'/\tau$. Thus, the smaller the spatial variations of τ, the weaker the condition on the initial data $f(x)$ has to be. In particular, if Γ_q=const, it is not necessary that the parabolic solution T^0 be near equilibrium.] We can then, by completing squares, bound the fourth, fifth, and sixth terms in Eq. (57) as follows:

$$\begin{aligned}
&\frac{1}{\epsilon^{(n-2)}L}\int_0^L[R_1q_R+R_2Lq_{Rx}+R_3L^2q_{Rxx}]dx+\frac{1}{\epsilon^nL}\int_0^L[R_4q_R+R_5Lq_{Rx}+R_6L^2q_{Rxx}]dx\\
&\quad-\frac{1}{\epsilon^nL}\int_0^L[S_1(q_R)^2+S_2L^2(q_{Rx})^2+S_3L^4(q_{Rxx})^2]dx\\
&\leqslant\frac{1}{\epsilon^nL}\int_0^L\bigg[(\epsilon^2R_1+R_4)q_R+(\epsilon^2R_2+R_5)Lq_{Rx}+(\epsilon^2R_3+R_6)L^2q_{Rxx}\\
&\quad-\frac{1}{k}(q_R)^2-\frac{L^2}{k}(q_{Rx})^2-\frac{2L^4}{k}(q_{Rxx})^2\bigg]dx\\
&\leqslant\frac{1}{\epsilon^nL}\int_0^L\frac{k}{4}\left[(\epsilon^2R_1+R_4)^2+(\epsilon^2R_2+R_5)^2+\frac{1}{2}(\epsilon^2R_3+R_6)^2\right]dx\\
&\leqslant\frac{\epsilon^{(4-n)}}{L}\int_0^L\frac{1}{2}\left[(R_1)^2+(R_2)^2+\frac{1}{2}(R_3)^2\right]+\frac{1}{\epsilon^nL}\int_0^L\frac{1}{2}\left[(R_4)^2+(R_5)^2+\frac{1}{2}(R_6)^2\right]dx\\
&\leqslant\frac{\epsilon^{(4-n)}}{2}\left[|R_1^2|_\infty+|R_2^2|_\infty+\frac{1}{2}|R_3^2|_\infty\right]+\frac{1}{\epsilon^nL}\int_0^L\frac{3}{2}\bigg\{\left[(R_{41})^2+(R_{51})^2+\frac{1}{2}(R_{61})^2\right](T_R)^2\\
&\quad+L^2\left[(R_{42})^2+(R_{52})^2+\frac{1}{2}(R_{62})^2\right](T_{Rx})^2+L^4\left[(R_{43})^2+(R_{53})^2+\frac{1}{2}(R_{63})^2\right](T_{Rxx})^2\bigg\}\\
&\leqslant\frac{\epsilon^{(4-n)}}{2}\left[|R_1^2|_\infty+|R_2^2|_\infty+\frac{1}{2}|R_3^2|_\infty\right]+\frac{3}{2}\bigg[|R_{41}^2|_\infty+|R_{51}^2|_\infty+\frac{1}{2}|R_{61}^2|_\infty+|R_{42}^2|_\infty+|R_{52}^2|_\infty\\
&\quad+\frac{1}{2}|R_{62}^2|_\infty+|R_{43}^2|_\infty+|R_{53}^2|_\infty+\frac{1}{2}|R_{63}^2|_\infty\bigg]\frac{\|T_R\|_{H^2}^2}{\epsilon^n},
\end{aligned}\tag{71}$$

where we have used Eqs. (68), (69), (70) to get the first inequality; we have completed squares and thrown away the negative terms to get the second inequality, and have done some algebra to get the rest of Eq. (71).

Now, using all the bounds (62), (63), (64), and (71) together with (46) and (47) we get

$$E_{Rt}(t)\leqslant\epsilon^{(4-n)}P(t,\epsilon)+Q(t,\epsilon)E_R, \tag{72}$$

where $P(t,\epsilon)$ and $Q(t,\epsilon)$ are $O(1)$ functions given by

$$P(t,\epsilon)=\tfrac{1}{2}[\tilde{t}(|P_1^2|_\infty+|P_2^2|_\infty+|P_3^2|_\infty)+|R_1^2|_\infty+|R_2^2|_\infty+|R_3^2|_\infty],$$

$$Q(t,\epsilon)=|Q_1|_\infty+|Q_2|_\infty+|Q_3|_\infty+|Q_4|_\infty+|Q_5|_\infty+|Q_6|_\infty+\frac{1}{2\tilde{t}}+\frac{3}{2}(|R^2_{41}|_\infty+|R^2_{51}|_\infty+\tfrac{1}{2}|R^2_{61}|_\infty$$
$$+|R^2_{42}|_\infty+|R^2_{52}|_\infty+\tfrac{1}{2}|R^2_{62}|_\infty+|R^2_{43}|_\infty+|R^2_{53}|_\infty+\tfrac{1}{2}|R^2_{63}|_\infty).$$

It is now obvious that the highest possible value for n, so that the right hand side of Eq. (72) is order one, is $n=4$. We thus conclude that the energy $E_R(t,\epsilon)$ is bounded by the solution of an ordinary differential equation of the form

$$y_t(t,\epsilon)=F(y,t,\epsilon), \tag{73}$$

with F a regular function of all variables and with initial data $y|_{t=0}=E_R(t=0,\epsilon)=0$. Thus there is a finite time interval for which a solution to this equation exists and is a smooth function of t and ϵ.

Now we turn our attention to the *a priori* assumptions (i) and (ii). At $t=0$, on the one hand, all the functions are evaluated in $T=f(x)\geqslant 2\overline{T^0}/3$ and $q^0=-kf_x$ so that the positivity of γ_0 ensures that there exists a positive constant K_T such that $\gamma_0>2K_T$, and consequently there exists $\epsilon>0$ small enough such that $\Gamma_T>K_T$ [see Eq. (33)]. On the other hand, existence of K_q is ensured just because $T=f>0$ and $\Gamma_q(f)>0$. All this says that at $t=0$ both *a priori* assumptions hold. Thus, since the solution to Eq. (73) is smooth for a finite interval, there will be another smaller finite interval, $[0, t_0]$ $(t_0\leqslant t_1)$, in which the *a priori* assumptions will be valid and so a finite interval in which the energy is a smooth function of ϵ. More precisely, there exist $\epsilon_0>0$ such that E_R is a smooth function of (t,ϵ) in $[0, t_0]\times[0, \epsilon_0]$; and thus the statement of the theorem follows from Eqs. (46), (47). ■

A direct consequence of this theorem and the Sobolev's embedding theorem is that under the hypothesis of Theorem 2, the absolute hyperbolic temperature T uniformly converges to the parabolic temperature T^0 as $\epsilon\rightarrow 0$, for $t\in[0, t_0]$.

It is interesting from a practical point of view to know how long the ϵ smoothness of the energy lasts for a finite (small) value of ϵ, that is, to have an idea of the order of magnitude of t_0. For if t_0 were of the order of magnitude of the relaxation time τ, the theorem, though true, would not be interesting. We can give a simple argument to see that this is not the case. Given a finite (small) value of ϵ, the energy E_R will remain small and smooth in ϵ during a finite interval $[0,t_0(\epsilon))$. [The time $t_0>0$ previously mentioned is such that $[0, t_0]\subset\cap_{\epsilon\in[0,\epsilon_0]}[0, t_0(\epsilon))$.] At $t_0(\epsilon)$, E_R would grow so much that some of the *a priori* assumption made in the proof would cease to hold, being then that E_R would not be bounded anymore by the solution of Eq. (73). However while the energy remains small, the bound obtained from Eq. (73) holds; and as F is smooth in ϵ, we can approximate $t_0(\epsilon)$ by its value at $\epsilon=0$. But in this limit F is just an expression involving the nonlinearities evaluated at the parabolic solutions and these solutions themselves, so the time interval $[0, t_0(0)]$ will not have any relation with—and therefore will have a different magnitude than—the presumably much shorter relaxation time τ (see Sec. II). Thus the expectation, which depends on the particular choice of the model—that is, the choice of the functions γ_0 and τ, is that t_0 is indeed relevant and is of the order of the cooling time τ_d, a function of the initial data and the nonlinearities. [In fact, it can be shown that as the energy of the system goes to zero, $P(t,\epsilon)$ can be taken smaller and smaller, so that the existence time can be made larger and larger, and in principle the time t_0 can be much larger than the decay time. However, the restriction obtained that way on the initial data could be too strong on physical grounds.]

IV. CONCLUSIONS

The behavior of the solutions of hyperbolic systems of equations for heat propagation compared with the solutions of the corresponding parabolic limiting systems have been studied. The hyperbolic systems of equations have been treated as monoparametric families, such that the (linear and nonlinear) heat equation is recovered when the parameter goes to zero.

In Sec. II the hyperbolic linear system for temperature T and heat flux q given by the conservation of energy (2) with constant specific heat, and Cattaneo's equation (4) with constant coefficients has been studied.

It has been proven that given any arbitrary initial data for the hyperbolic system, the solution T^0 of the parabolic equation with the same initial data (to order ϵ^2) for the temperature, remains near the hyperbolic one for all $t\geqslant 0$. More precisely, an $O(\epsilon^2)$ bound for the H^n $(n\geqslant 0)$ Sobolev norm of the difference between the hyperbolic and the parabolic temperatures has been found for all $t\geqslant 0$, in terms of ϵ and the initial data. Using this, an explicit $O(\epsilon^2)$, pointwise, bound for the absolute difference between both temperatures has been found. In the same way, a pointwise bound for the absolute difference between the spatial derivatives of any order of both temperatures can be obtained. As regards the difference between the hyperbolic and parabolic heat flux, it has been proven that its H^n Sobolev norm is $O(\epsilon)$ for all $t>0$, even if that difference is an arbitrary smooth function at $t=0$. This is so, because $O(1)$ terms exponentially vanishes as t grows with an $O(\epsilon^2)$ time constant.

In Sec. III, a family of nonlinear hyperbolic systems of equations has been studied, namely, CFO theories. It has been proven that given a solution T^0 of the nonlinear heat equation near enough to the equilibrium solution $T^0=\text{const}$, $q^0=0$, there exists a solution of the hyperbolic system that remains near the parabolic one. The initial data for the hyperbolic system was chosen to be initialized—if a slight generalization could be done, we could have taken initialized data plus terms of order $O(\epsilon)$ which would have essentially not modified the proof. More precisely, it has been shown that there exists a bound of $O(\epsilon^2)$ for the H^2 Sobolev norm of the difference between the hyperbolic and the parabolic temperatures. This implies that an $O(\epsilon^2)$ pointwise bound for this difference can be found.

The results obtained for the linear case could hardly be made stronger, but the same cannot be said for the results in the nonlinear case. There are several important differences between them.

(i) The bounds obtained in the linear case are valid for all $t\geqslant 0$, while in the nonlinear case they hold only for a finite time interval. We believe that this result can be considerably improved, to obtain results like "for a given data the time of existence goes like $1/\epsilon$."

(ii) Any given solution of the linear heat equation can always be approximated by (many) solutions of a hyperbolic linear system. We do not know if this is so in the nonlinear case, because we have shown only the sufficiency of the condition "T^0 is near enough to equilibrium." It would be interesting to test, at least numerically, whether this condition can be relaxed for the approximation.

(iii) Every solution of the linear hyperbolic system is near a solution of the linear heat equation. Though we could not prove the same for the nonlinear case yet, because of the restriction imposed on the initial data for the heat flux, we think our results can probably be improved in this respect, at least for some nonlinear cases for which the initialization condition could be relaxed. Efforts are being done in this sense.

(iv) In the linear case the bounds were explicitly worked out, while in the nonlinear case only the existence of bounds was shown. Though the bounds in this last case could have been obtained explicitly in terms of the initial data and the parabolic solution, both the complexity of the expressions and the fact that the time intervals of validity would have been only implicitly defined, discouraged us to carry out this calculation. However, we are working with this aim for the physically more interesting nonlinear theories of Morro and Ruggeri. Results will be presented elsewhere.

The energy techniques we have used to prove the theorems can also be applied in a variety of physical problems that involve hyperbolic partial differential equations. In particular we have been working with similar problems to those treated here but for certain "hyperbolizations" of Navier–Stoke's equations. Some of these results will be given in further works.

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APPENDIX: DISSIPATIVE HEAT THEORIES OF DIVERGENCE TYPE

In this appendix we give a general formulation of the dissipative theories of divergence type for heat propagation in a medium at rest. This subject and some related ones were studied by several authors.[8,6,7,9] The present study will be carried out using the basic ideas of extended thermodynamics. Our derivation is almost equivalent to that of Morro and Ruggeri, though a little more general and including some new results. It will be seen at the end of the appendix that under some assumptions these theories become either (see introduction) CFO theories or MR theories.

We require the theories to obey the following postulates:

(i) The dynamical variables of the theories are a scalar field and a vector field (the dynamical variables could, for instance, be chosen as the physical temperature field T and the heat flux field q^a), and all the physical fields are pointwise functions of them and the metric tensor h_{ab}. (We assume that the properties of the medium are described by a symmetric positive-definite metric tensor h^{ab}. Throughout this appendix the indices $a,b,c,...$ will run over spatial coordinates. The corresponding spatial derivatives will be denoted by $D_a, D_b, D_c,...$, while a dot over a variable will indicate time derivative. Rising and lowering of indices are carried out with the metric tensor h_{ab} and its inverse h^{ab}, that is, $\xi_a = h_{ab}\xi^b$, $\xi^a = h^{ab}\xi_b$ where $h_{ab}h^{bc} = \delta_a^c$.)

(ii) Apart from the conservation of energy

$$\dot{e} + D_a q^a = 0 \tag{A1}$$

there will be another divergence type equation

$$\dot{A}^a + D_b H^{ba} = I^a, \tag{A2}$$

where, as assumed in (i), the five tensors e, q^a, A^a, H^{ba}, and I^a are functions of the dynamical variables.

(iii) There exists an entropy density s and an entropy flux s^a obeying, as a consequence of Eqs. (A1), (A2), the entropy law

$$\dot{s} + D_a s^a = \sigma, \tag{A3}$$

where σ is a non-negative function of the dynamical variables.

Let us analyze the consequences of these postulates. In order that Eq. (A3) be a consequence of Eqs. (A1), (A2) and that σ be just a function of the dynamical fields, $\dot{s} + D_a s^a$ must be a linear combination of the left hand sides of Eqs. (A1), (A2). We denote the coefficients in this linear combination as $-\xi$ and $-\xi_a$. Thus

$$\dot{s} + D_a s^a = -\xi(\dot{e} + D_a q^a) - \xi_a(A^a + D_b H^{ba})$$

so that Eqs. (A1), (A2) imply $\sigma = -\xi_a I^a \geqslant 0$. From now on we will think of $\{\xi, \xi_a\}$ as a set of dynamical variables.

Consider now the tensors χ and χ^a defined as

$$\chi = s + \xi e + \xi_a A^a, \tag{A4}$$

$$\chi^a = s^a + \xi q^a + \xi_b H^{ab} \tag{A5}$$

and notice that χ has units of entropy density, χ^a has units of entropy flux, and ξ has units of inverse temperature. Now, notice that

$$\dot{\chi} + D_a \chi^a = \frac{\partial \chi}{\partial \xi} \dot{\xi} + \frac{\partial \chi}{\partial \xi_a} \dot{\xi}_a + \frac{\partial \chi^a}{\partial \xi} D_a \xi + \frac{\partial \chi^a}{\partial \xi_b} D_a \xi_b$$

and using Eqs. (A1)–(A3) and the definition of σ

$$\dot{\chi} + D_a \chi^a = \dot{\xi} e + q^a D_a \xi + \dot{\xi}_a A^a + H^{ab} D_a \xi_b .$$

Since these last two equations are supposed to be valid for all pairs $\{\xi, \xi_a\}$, it must be true that

$$e = \frac{\partial \chi}{\partial \xi}, \quad q^a = \frac{\partial \chi^a}{\partial \xi}, \quad A^a = \frac{\partial \chi}{\partial \xi_a}, \quad \text{and} \quad H^{ab} = \frac{\partial \chi^a}{\partial \xi_b}. \tag{A6}$$

As all the physical fields are pointwise functions of $\{\xi, \xi_a\}$, the energy density e—as any other scalar in the theory—must be a function of ξ and $\mu = h_{ab} \xi^a \xi^b / 2$. Both vector fields A^a and q^a must be proportional to ξ^a, and consequently proportional to each other, that is

$$A^a = \lambda(\xi, \mu) q^a. \tag{A7}$$

Finally, since the only tensors present are ξ^a and the three metric h_{ab}, the tensor H^{ab} can only have the form

$$H^{ab} = \gamma(\xi, \mu) h^{ab} + \eta(\xi, \mu) \xi^a \xi^b \tag{A8}$$

so that H^{ab} is necessarily a symmetric tensor. Equation (A6) and (A7) imply that

$$A^a = \frac{\partial \chi}{\partial \xi_a} = \lambda q^a = \lambda \frac{\partial \chi^a}{\partial \xi}$$

but $\chi = \chi(\xi, \mu)$ so that

$$q^a = \frac{1}{\lambda} \frac{\partial \chi}{\partial \mu} \xi^a. \tag{A9}$$

Now, Eq. (A8) implies

$$\frac{\partial H^{ab}}{\partial \xi} = \frac{\partial \gamma}{\partial \xi} h^{ab} + \frac{\partial \eta}{\partial \xi} \xi^a \xi^b, \tag{A10}$$

while Eqs. (A6) and (A9) imply

$$\frac{\partial H^{ab}}{\partial \xi} = \frac{\partial^2 \chi^a}{\partial \xi \partial \xi_b} = \frac{\partial}{\partial \xi_b} \left(\frac{1}{\lambda} \frac{\partial \chi}{\partial \mu} \xi^a \right) = \frac{1}{\lambda} \frac{\partial \chi}{\partial \mu} h^{ab} + \frac{\partial}{\partial \mu} \left(\frac{1}{\lambda} \frac{\partial \chi}{\partial \mu} \right) \xi^a \xi^b. \tag{A11}$$

Comparing Eqs. (A10) and (A11) we conclude that

$$\frac{\partial \eta}{\partial \xi} = \frac{\partial}{\partial \mu} \left(\frac{1}{\lambda} \frac{\partial \chi}{\partial \mu} \right) \quad \text{and} \quad \frac{\partial \gamma}{\partial \xi} = \frac{1}{\lambda} \frac{\partial \chi}{\partial \mu} \tag{A12}$$

so that λ and χ uniquely determine η and γ, and therefore H^{ab}.

As a direct consequence of Eqs. (A4) and (A6) we get

$$ds = -\xi\, de - \xi_a\, dA^a. \tag{A13}$$

This equation suggests to us to identify $-1/\xi$ with the absolute temperature T of the system. Assuming this, the transformation between the dynamical variables $\{\xi,\xi_a\}$ and the physical variables $\{T,q^a\}$ is given by

$$T = -\frac{1}{\xi}, \quad q^a = \frac{1}{\lambda}\frac{\partial\chi}{\partial\mu}\,\xi^a. \tag{A14}$$

Now, s^a can be written as

$$s^a = \chi^a - \xi\,\frac{\partial\gamma}{\partial\xi}\,\xi^a - \gamma\xi^a - 2\mu\eta\xi^a. \tag{A15}$$

Differentiating with respect to ξ and using Eq. (A12) we get

$$\frac{\partial s^a}{\partial\xi} = -\frac{\partial}{\partial\xi}\left(\xi\,\frac{\partial\gamma}{\partial\xi} + 2\mu\,\frac{\partial\gamma}{\partial\mu}\right)\xi^a$$

and integrating we get

$$s^a = -\left(\xi\,\frac{\partial\gamma}{\partial\xi} + 2\mu\,\frac{\partial\gamma}{\partial\mu}\right)\xi^a + f(\mu)\xi^a. \tag{A16}$$

We now differentiate both Eqs. (A15) and (A16) with respect to ξ^b and compare the results to get the identities

$$f(\mu) = 0 \quad \text{and} \quad \eta = \frac{\partial\gamma}{\partial\mu}. \tag{A17}$$

Using this, Eq. (A16) becomes

$$s^a = -\xi q^a - 2\mu\,\frac{\partial\gamma/\partial\mu}{\partial\gamma/\partial\xi}\,q^a. \tag{A18}$$

These theories describe heat flux in a medium at rest (there is no matter movement); consequently, the entropy flux and the heat flux have to obey

$$s^a = \frac{q^a}{T}$$

and as we have chosen $\xi = -1/T$, Eq. (A18) implies $\partial\gamma/\partial\mu = 0$. Some consequences can be drawn from this fact. First, Eq. (A17) tell us that $\eta = 0$ and consequently the tensor H^{ab} is proportional to the metric tensor h^{ab}, which simplifies the structure of Eq. (A2) for all these theories. $\partial\gamma/\partial\mu = 0$ also implies $\partial^2\gamma/\partial\mu\partial\xi = 0$ and so—using Eq. (A12)—we get

$$\frac{\partial\chi}{\partial\mu} = \lambda(\xi,\mu)\beta(\xi)$$

for some function $\beta(\xi)$; and integrating we get

$$\chi(\xi,\xi^a)=\beta(\xi)\int_0^{\mu} d\tilde{\mu}\lambda(\xi,\tilde{\mu})+\chi_0(\xi). \tag{A19}$$

Thus, any particular theory is obtained giving the functions χ_0, λ, and β, and the vector I^a as functions of ξ and ξ^a. From these tensor fields, Eqs. (A1) and (A2) can explicitly be written in terms of the physical fields. Notice that the entropy density is given by

$$s=\chi-\xi\frac{\partial\chi}{\partial\xi}-2\mu\frac{\partial\chi}{\partial\mu}. \tag{A20}$$

In what follows, we will restrict our study to a subclass of these theories whose "generating function" χ is quadratic in q^a.

Quadratic theories. The simplest possible choice for the function $\chi(\xi,\mu)$ is to take it linear in μ, that is, quadratic in q^a. These theories are obtained if the function λ is only dependent on ξ. Assuming this, the generating function χ becomes

$$\chi=\chi_0(\xi)+\lambda(\xi)\beta(\xi)\mu. \tag{A21}$$

Using this χ we obtain

$$e=\frac{d\chi_0}{d\xi}+\frac{d(\lambda\beta)}{d\xi}\mu, \quad q^a=\beta(\xi)\xi^a$$

so that conservation of energy (A1) becomes

$$\left(\frac{d^2\chi_0}{d\xi^2}+\frac{d^2(\lambda\beta)}{d\xi^2}\mu\right)\dot{\xi}+\frac{d(\lambda\beta)}{d\xi}\xi_a\dot{\xi}^a+D_a(\beta\xi^a)=0. \tag{A22}$$

Now, as $A^a=\lambda\beta\xi^a$ and $H^{ab}=\gamma h^{ab}$ (with $\partial\gamma/\partial\xi=\beta$), Eq. (A2) becomes

$$\lambda\beta\dot{\xi}^a+\frac{d(\lambda\beta)}{d\xi}\xi^a\dot{\xi}+\beta h^{ab}D_b\xi-I^a=0. \tag{A23}$$

Finally, the entropy density becomes

$$s=\chi_0-\mu\lambda\beta-\xi\frac{d\chi_0}{d\xi}-\mu\xi\frac{d(\lambda\beta)}{d\xi}.$$

We will now show how both CFO and MR theories arise making particular choices for χ_0, λ, β, and I^a.

(i) CFO theories[6]

The theories of Coleman, Fabrizio, and Owen arise from these quadratic theories if one chooses λ=const, so that Eqs. (A22) and (A23) reduce to

$$\left(\frac{\partial^2\chi_0}{\partial\xi^2}+\frac{\lambda}{2}\frac{\partial^2\beta}{\partial\xi^2}\xi_a\xi^a\right)\dot{\xi}+\lambda\frac{\partial\beta}{\partial\xi}\xi_a\dot{\xi}^a+D_a(\beta\xi^a)=0, \tag{A24}$$

$$\lambda\beta\dot{\xi}^a+\lambda\frac{\partial\beta}{\partial\xi}\xi^a\dot{\xi}+\beta h^{ab}D_b\xi-I^a=0. \tag{A25}$$

Now, suppose we are given with the thermal conductivity $k(T)$, the energy for zero heat flux $e_0(T)$, and the relaxation time $\tau(T)$, which are the functions needed to fix a particular CFO theory (see introduction). We now define χ_0, β, and I^a in terms of these three functions as follows:

$$\chi_0(\xi)=\int_{-\infty}^{\xi} e_0(-1/\zeta)d\zeta, \quad \beta(\xi)=\lambda\,\frac{(-1/\xi)}{\xi^2\tau(-1/\xi)}, \quad \text{and} \quad I^a=-\frac{\lambda^2}{\xi^2}\frac{k(-1/\xi)}{\tau^2(-1/\xi)}\,\xi^a, \tag{A26}$$

where we have assumed that the energy density vanishes at zero temperature. From these definitions we get, after some algebra, the physical fields which can be written in terms of T and q^a as follows (using the notation of Ref. 6):

$$e=e_0(T)+a(T)q_aq^a, \tag{A27}$$

$$s=s_0(T)+b(T)q_aq^a, \tag{A28}$$

where

$$a(T)=\frac{Z(T)}{T}-\frac{Z'(T)}{2}, \quad b(T)=\frac{Z(T)}{2T^2}-\frac{Z'(T)}{2T}, \quad s_0(T)=\int_0^T \frac{e_0(r)}{r^2}\,dr+\frac{e_0(T)}{T},$$

with $Z(T)=\tau(T)/k(T)$ and where a prime denotes differentiation with respect to T. Notice that $s_0'(T) = e_0'(T)/T$ as expected. Definitions (A26) transform Eqs. (A24) and (A25) into

$$[\gamma_0(T)+a'(T)q_aq^a]\dot{T}+2a(T)q_a\dot{q}^a+D_aq^a=0, \tag{A29}$$

$$\tau(T)\dot{q}^a+k(T)h^{ab}D_bT+q^a=0, \tag{A30}$$

where $\gamma_0(T) = e_0'(T)$ is the specific heat. The entropy source σ satisfies the non-negativity condition

$$\sigma=-I^a\xi_a=\frac{q_aq^a}{T^2k}\geqslant 0.$$

Equations (A27), (A28), (A29), and (A30) constitute CFO theories.

(ii) MR theories[7]

The theories of Morro and Ruggeri arise from the quadratic theories given above if one chooses $\lambda\beta=1$. In this case Eqs. (A22) and (A23) become

$$\frac{d^2\chi_0}{d\xi^2}\,\dot{\xi}+D_a\left(\frac{\xi^a}{\lambda}\right)=0, \tag{A31}$$

$$\lambda\dot{q}^a+\frac{d\lambda}{d\xi}\,\dot{\xi}\,\frac{\xi^a}{\lambda}+\frac{1}{\lambda}\,h^{ab}D_b\xi-I^a=0. \tag{A32}$$

Now, suppose we are given with the thermal conductivity $k(T)$, the specific heat $\gamma_0(T)$, and the heat pulse speed $U_0(T)$ which are the functions needed to determine a particular MR theory (see introduction). We now define χ_0, λ, and I^a as follows:

$$\chi_0(\xi)=\int_{-\infty}^{\xi} d\eta \int_{-\infty}^{\eta} \gamma_0(-1/\zeta)d\zeta, \quad \lambda(\xi)=\frac{\xi}{U_0(-1/\xi)\sqrt{\gamma_0(-1/\xi)}}, \quad \text{and} \quad I^a=-\frac{\xi^2}{\lambda^2}\,\xi^a. \tag{A33}$$

So that the physical fields can be written as

$$e(T)=e_0(T)=\int_0^T \gamma_0(r)dr, \tag{A34}$$

$$s(T)=\int_0^T \frac{dv}{v^2}\int_0^v \gamma_0(r)dr+\frac{1}{T}\int_0^T \gamma_0(r)dr \tag{A35}$$

such that $s'(T) = e_0'(T)/T$. Equations (A31) and (A32) become

$$\gamma_0(T)\dot{T}+D_a q^a=0 \tag{A36}$$

and

$$\left(\frac{kT}{U_0\sqrt{\gamma_0}}\right)\frac{d}{dt}\left(\frac{q^a}{U_0 T\sqrt{\gamma_0}}\right)+kh^{ab}D_bT+q^a=0, \tag{A37}$$

which are the equations in the Morro–Ruggeri theories. Finally, notice that σ satisfies the non-negativity condition

$$\sigma=-\xi_a I^a=\frac{q_a q^a}{T^2 k}>0.$$

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