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Initial data for fluid bodies in general relativity

Sergio Dain*
Albert-Einstein-Institut, am Mühlenberg 1, D-14476, Golm, Germany

Gabriel Nagy†
Albert-Einstein-Institut, am Mühlenberg 1, D-14476, Golm, Germany
and Département de Mathématiques Faculté des Sciences, Tours, France

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We show that there exist asymptotically flat almost-smooth initial data for the Einstein–perfect-fluid equation that represent an isolated liquid-type body. By liquid-type body we mean that the fluid energy density has compact support and takes a strictly positive constant value at its boundary. By almost-smooth we mean that all initial data fields are smooth everywhere on the initial hypersurface except at the body boundary, where tangential derivatives of any order are continuous at that boundary.

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I. INTRODUCTION

A description by an initial value formulation in general relativity of a self-gravitating ideal body in a general situation is still missing. By an ideal body we mean a perfect fluid where the thermodynamical variables and the fluid velocity have spatially compact support. Examples are, within some approximation, a star, a neutron star, or a fluid planet. By a general situation we mean first a body without symmetry, because spherically symmetric bodies are already described, whether static [1] or in radial motion [2], and second a situation including nearly static objects.

The lack of this type of description is remarkable. Stars are common objects in the universe, and a perfect fluid is the simplest matter model for them. General relativity is the currently accepted theory of gravitation to describe stars, as well as planets, white dwarfs, and neutron stars. An initial value formulation will be a useful tool to predict the time evolution of such objects as predicted by Einstein's equation, without any approximation other than the choice of the matter model.

The main difficulty is to find a solution in a neighborhood of the timelike hypersurface corresponding to the fluid-vacuum interface, where the Einstein-Euler equation transforms into Einstein's vacuum equation. It is known how to describe regions not including this interface, since an initial value formulation for Einstein's vacuum equation was first given in [3], and for the Einstein-Euler equation with nonvanishing energy density everywhere, in [4]. The problem at the interface is inherent in the fluid equations and it is also present in a Newtonian description. A summary of known results on free boundary problems is given in Sec. 2.5 in [5], in the context of Newton's theory as well as of general relativity.

A first step to set up an initial value formulation for the Einstein-Euler equation in a neighborhood of the fluid-vacuum interface requires finding, from the complete system of equations, a symmetric hyperbolic system that remains symmetric hyperbolic even at that interface. A first system of this type was found in [6] for a certain class of fluid state functions. However, spherically symmetric static situations cannot be described by these state functions. The reason is that every solution found in that reference satisfies the fluid particles at the body boundary following a timelike geodesic. But this is not the case for a spherically symmetric static star, as the following argument, already given in [6], shows. Consider a static spherically symmetric stellar model. The fluid 4-velocity must be proportional to the timelike, hypersurface orthogonal Killing vector. The proportionality factor finds that the 4-velocity is a unitary vector field. In the vacuum region the space-time must coincide with Schwarzschild's. At the star boundary the timelike Killing vector field must coincide with the timelike Killing vector in Schwarzschild's space-time. However, the 4-velocity obtained with this Killing vector field is not geodesic. Therefore, spherically symmetric stellar models are not included among the solutions given in [6] and so one does not expect that nearly static stellar models can be described with these solutions.

A second system of the type mentioned above was found, for a general class of state functions, in [7]. However, the initial value formulation that describes nearly static objects is still missing. Fix some smooth state function such that both the energy density and the sound velocity vanish at pressure zero, and then consider the system given in [7] for that fluid. Assume that there exists a smooth solution that describes such a fluid body. Then, one can check that the fluid particles at the boundary of the body follow timelike geodesics. In other words, smooth perfect fluid solutions of the system given in [7] cannot describe nearly static situations.

Therefore, a first attempt to describe nearly static perfect fluid bodies by an initial value formulation would be that the fluid state function satisfies the following condition: Neither the fluid energy density nor the sound velocity must vanish when the pressure vanishes. Because at the boundary of the body the pressure vanishes, we are then requiring that the border of the body have nonzero energy density. We call them ''liquid-type'' bodies, and ''liquid-type'' state functions. Therefore, it is natural to study what are the appropriate initial data for a liquid-type body. It turns out that the

*Email address: dain@aei-potsdam.mpg.de
†Email address: nagy@aei-potsdam.mpg.de

answer to this question was not known; it is subtle, and is the subject of this work.

Fix once and for all a simple perfect fluid, that is a perfect fluid with one-dimensional manifold of fluid states; for example, one with a state function of the form $p(\rho)$, where p is the fluid pressure and ρ the fluid comoving energy density. Assume that this state function is smooth and of liquid-type. By liquid-type ideal body initial data we mean a three-dimensional initial hypersurface, its first and second fundamental forms, and the fluid initial 3-velocity and comoving energy density. The first two fields must be asymptotically flat, the last two must have compact support, the support of the fluid velocity must be included in the support of the energy density, and all of them must be a solution of the constraint equations, and satisfy some energy condition. In addition, there exists an extra constraint on the initial fluid fields: The fluid comoving energy density must be *strictly positive and constant* at the border of its support. Constant because the simple perfect fluid state function implies that there exists only one single value of the fluid comoving energy density such that the pressure vanishes, and this is the value of the energy density at the border of the body. Notice that only the energy density as measured by a fluid comoving observer must be constant. This extra condition only arises for liquid-type fluids, because it is trivially satisfied in the case where the fluid comoving energy density vanishes at the border of the body. [See Eqs. (5),(6).] Also notice that this data must have a C^1 first fundamental form, that is, with at least one continuous derivative everywhere; if not, Dirac's delta appears in the fluid energy density.

Initial data of this type was not known in the literature. The only result on solutions of the constraint equations with discontinuous matter sources and C^1 first fundamental form [8] does not guarantee that the fluid comoving energy density be constant at the border of the body. Here is why. The solutions with discontinuous matter sources are found by the usual conformal rescaling that also rescales the matter sources. Then, the initial physical energy density [the energy density as measured by an observer at rest with the initial surface, function $\tilde{\mu}$ in Eq. (5)] is the product of the initial unphysical energy density (free data) times the conformal factor at power minus eight. We do not know any procedure to choose the free initial data such that the solutions given in [8] guarantee both that the fluid comoving energy density [function ρ in Eq. (5)] be constant at the border of the body and the conformal factor be C^1, simultaneously. We should mention that there are also found in [8] solutions of the constraints without rescaling the energy density, but, in this case, only *continuous* energy densities are considered; that is, they vanish at the border of the body.

Here, we conformally rescale all the fields except the initial physical energy density, which is now free data, given positive and constant at the boundary of the body. We impose that the fluid 3-velocity vanishes at the body boundary. These two conditions imply that the fluid comoving energy density is positive and constant at the body boundary. [See Eq. (7).] The subtle part now is to solve, with the initial physical energy density as free data, the equation for the conformal factor. This is a semilinear elliptic equation, with the nonlinear term given by a discontinuous function (the initial physical energy density) times the unknown (the conformal factor) at the power plus five. In other words, a nondecreasing function of the unknown, times a discontinuous given function (in contrast with [8] where this function is continuous). We introduce a compact manifold, and we prove existence of a C^1 conformal factor, based on Schauder's fixed-point theorem. The proof follows the ideas given in an Appendix of [9], and in [10]. There is only one (technical) requirement on the initial physical energy density: Its L^2-norm, computed with the unphysical metric, must not exceed some given upper bound. We show in the appendix that, although this condition excludes possible initial data, it is mild enough to include interesting physical situations, such as neutron stars.

We also give a statement on the regularity of these data. They cannot be smooth, because the liquid-type energy density is, by definition, a discontinuous function of the space variables. How regular can it be? The smoothest liquid-type body is almost-smooth; that is, smooth everywhere except at the body boundary, where tangential derivatives (appropriately defined) of any order are continuous. In other words, we prove the following: If the unphysical metric is a smooth field on the initial hypersurface, and the fluid free initial data are smooth up to the body boundary with every derivative tangential to the boundary continuous through that boundary, then the same holds for all the initial data fields. A crucial requirement to prove this statement is that the conformal factor be C^1 on a neighborhood of the body boundary. (See Sec. III C.)

Summarizing, we prove that *almost-smooth initial data representing liquid-type simple perfect fluid bodies can be obtained as solutions of the constraint equations, in the case where the fluid 3-velocity vanishes at the body-boundary, through a suitable modification of the usual conformal rescaling techniques.*

Finally, some technical remarks: (i) We rescale the initial fluid momentum density but not the initial fluid energy density, in order to solve the constraint equations. Therefore, we have to choose the rescaled momentum density small enough, in the sense given in Sec. IV A, in order to have physical data satisfying the dominant energy condition. (ii) We impose that the fluid 3-velocity vanish at the body boundary. There is no physical justification for this assumption, it is made because it is the only way we know, with the rescaling of the initial data field that we have chosen, to guarantee that the fluid comoving physical energy density be constant at the body boundary. [See Eq. (7).] We give an interpretation of this condition in Sec. II B. (iii) Besides the conformal rescaling to solve the constraint equations we perform a conformal compactification in order to solve elliptic equations on some unphysical compact manifold. Asymptotic decay properties of fields in the physical initial hypersurface are translated into differentiability properties of these fields at a particular point in the unphysical compact manifold. These differentiability properties at the point at infinity are completely independent of the differentiability of the fields near the body boundary.

In Sec. II we introduce the main definitions we need in order to present the principal result, theorems 1 and 2. We also give a proof based on results obtained in Secs. III and IV. In Secs. III A and III B we give the main existence proofs for the semilinear and linear elliptic equation associated with the Hamiltonian and momentum constraint, respectively. In Sec. III C we prove the regularity statements, theorems 7 and 8. In Sec. IV A we explain why this discussion on an energy condition appears, and we give a simple condition on the free data such that the physical initial data satisfies the dominant energy condition. The constraint equations are naturally written in terms of the initial fluid energy density and the initial fluid momentum density. Equations (5),(6) relate them to the fluid pressure, comoving energy density, and 3-velocity. In Sec. IV B, we prove that these equations are invertible. In Sec. V, we comment on the initial value formulation for liquid-type ideal bodies. In theorem 3 we need to assume that the fluid initial energy density satisfies an inequality [Eq. (24)] involving both the unphysical manifold and the unphysical rescaled metric. In the Appendix, we study this inequality. In Sec. 1, we show (lemma 4) that a similar (but weaker) inequality holds for every initial data. By an explicit example, in Sec. 2, we prove that the inequality required for the existence theorems is in fact a restriction on the allowed initial data. This example also suggests that this restriction is mild, in the sense that interesting physical systems, like neutron stars, satisfy it.

II. DEFINITIONS AND MAIN RESULT

A. Liquid-type ideal body data

We first introduce what we mean by initial data for a liquid-type ideal body. Afterwards, we split the concept of almost-smooth into two pieces. Given a field and an open bounded set Ω on some manifold, we introduce the concept of an Ω-piecewise smooth and Ω-tangentially smooth field. Finally, in the next subsection, we present our main result.

Consider an initial data set for Einstein's equation with matter. That is, consider a 3-dimensional, smooth, connected manifold $\tilde{M}$, a positive definite metric, $\tilde{q}_{ab}$, and a symmetric tensor field, $\tilde{p}^{ab}$, on $\tilde{M}$, together with a vector field, $\tilde{j}^a$, and a positive scalar function, $\tilde{\mu}$, subject to the condition $\tilde{j}_a\tilde{j}^a \leqslant \tilde{\mu}^2$, and solution on $\tilde{M}$ of

$$\tilde{D}_a\tilde{p}^{ab} - \tilde{D}^b\tilde{p}_a{}^a = -\kappa\tilde{j}^b, \tag{1}$$

$$\tilde{R} + (\tilde{p}_a{}^a)^2 - \tilde{p}_{ab}\tilde{p}^{ab} = 2\kappa\tilde{\mu}, \tag{2}$$

where $\tilde{D}_a$ and $\tilde{R}$ are the Levi-Civita connection and the Ricci scalar associated with $\tilde{q}_{ab}$, and $\kappa = 8\pi$. Indices on tensors with "tilde" are raised and lowered with $\tilde{q}^{ab}$ and $\tilde{q}_{ab}$, respectively, where $\tilde{q}_{ac}\tilde{q}^{cb} = \delta_a{}^b$. Latin letters a, b, c, represent abstract indices. The fields solving Eqs. (1),(2) have a meaning as part of a 4-dimensional space-time solution of Einstein's equation with matter sources. The manifold $\tilde{M}$ represents a three-dimensional spacelike hypersurface such that $\tilde{q}_{ab}$ and $\tilde{p}^{ab}$ are their first and second fundamental forms. This hypersurface will be a maximal slice if and only if $\tilde{p}_a{}^a = 0$. The fields $\tilde{\mu}$ and $\tilde{j}^a$ represent the normal-normal and the (negative) normal-parallel components to $\tilde{M}$ of the stress-energy tensor. The dominant energy condition on the stress-energy tensor in the space-time implies $\tilde{j}_a\tilde{j}^a \leqslant \tilde{\mu}^2$ on $\tilde{M}$. This condition is the reason why one does not, in general, pick *any* $\tilde{q}_{ab}$ and $\tilde{p}^{ab}$ and then *define* $\tilde{j}^a$ and $\tilde{\mu}$ by Eqs. (1),(2). Because if one does that, the resulting fields form an initial data set iff the energy condition is satisfied by these $\tilde{j}^a$ and $\tilde{\mu}$. It is an open question whether there exists a procedure to find appropriate $\tilde{q}_{ab}$ and $\tilde{p}^{ab}$ besides the conformal rescaling one.

The initial data set is asymptotically flat if the complement of a compact set in $\tilde{M}$ can be mapped by a coordinate system $\tilde{x}^j$ diffeomorphically onto the complement of a closed ball in $\mathbb{R}^3$ such that we have in these coordinates

$$\tilde{q}_{ij} = (1 + 2m/\tilde{r})\delta_{ij} + O(\tilde{r}^{-2}), \tag{3}$$

$$\tilde{p}^{ij} = O(\tilde{r}^{-2}), \tag{4}$$

as $\tilde{r} := \sqrt{\delta_{ij}\tilde{x}^i\tilde{x}^j} \to \infty$, where m is a constant that represents the ADM mass of the data. Latin letters i, j, k, denote coordinates indices and take values 1, 2, 3, while $\delta_{ij} = \mathrm{diag}(1,1,1)$.

Fix as the matter source a simple perfect fluid. That is, first, introduce on $\tilde{M}$ a non-negative scalar field ρ, interpreted as the fluid comoving energy density, a vector field $\tilde{v}^a$, interpreted as the fluid initial 3-velocity, and fix a function $p(\rho)$, the state function, interpreted as the fluid pressure as a function of the comoving energy density. Second, introduce on $\tilde{M}$ the equations

$$\tilde{\mu} = \frac{\rho + p\tilde{v}^2}{1 - \tilde{v}^2}, \tag{5}$$

$$\tilde{j}^b = \frac{(\rho + p)\tilde{v}^a}{1 - \tilde{v}^2}, \tag{6}$$

with $\tilde{v}^2 = \tilde{v}_a\tilde{v}^a < 1$. In the space-time solution of Einstein's equation with matter sources, the normal-normal and the (negative) normal-parallel components to $\tilde{M}$ of the usual perfect fluid stress-energy tensor are precisely the left hand side of Eqs. (5),(6), respectively. If u^a denotes the unit fluid 4-velocity, and n^a the unit normal to the initial hypersurface, then $u^a = (n^a + \tilde{v}^a)/\sqrt{1 - \tilde{v}^2}$. For this matter model the dominant energy condition is equivalent to $\rho \geqslant p$. In addition one can prove that $\rho \geqslant p$ implies $\tilde{j}_a\tilde{j}^a < \tilde{\mu}^2$. The sketch of the proof is the following: from Eqs. (5),(6) define $f(\rho,\tilde{v}) := \sqrt{\tilde{j}_a\tilde{j}^a}/\tilde{\mu}$. Notice that $f(\rho,0) = 0$, and $f(\rho,1) = 1$. One can prove that $\rho \geqslant p$ implies $\partial f/\partial\tilde{v} > 0$, for $0 \leqslant \tilde{v} < 1$; then, it follows that $f < 1$ for $0 \leqslant \tilde{v} < 1$.

Consider a liquid-type ideal body data set. That is, an asymptotically flat initial data with a simple perfect fluid whose state function is of liquid-type, and both the fluid 3-velocity and the comoving energy density have the same compact support, $\bar{\Omega}\subset\tilde{M}$. By a liquid-type state function we mean a non-negative, nondecreasing, smooth function $p(\rho)$ that vanishes at $\rho_0>0$. As an example consider a big water drop (in order to neglect surface tension effects), or a fluid planet, or a neutron star. The support of the energy density represents the place occupied by the body. The value of the comoving energy density at the border is determined by the function of state as the value where the pressure vanishes. (Otherwise, the acceleration of fluid particles lying on this border becomes infinite.) Therefore, a liquid-type ideal body satisfies $\rho|_{\partial\Omega}=\rho_0$. Equations (5),(6) translate this condition for the comoving energy density into a constraint on the fields $\tilde{\mu}$ and $\tilde{j}^a$ at $\partial\Omega$, where they are no longer free but they must satisfy

$$[\tilde{\mu}(1-\tilde{j}_a\tilde{j}^a/\tilde{\mu}^2)]|_{\partial\Omega}=\rho_0. \tag{7}$$

As a summary, we state the following.

Definition 1. A liquid-type ideal body initial data set consists of fields $\tilde{q}_{ab}$, $\tilde{p}^{ab}$, $\tilde{v}^a$, *and* ρ *on* $\tilde{M}$, *and a state function* $p(\rho)$, *such that (i)* $\tilde{q}_{ab}$ *is a Riemannian metric,* $\tilde{p}^{ab}$ *is a symmetric tensor, and both are asymptotically flat; (ii)* $p(\rho)$ *is liquid-type, and vanishes at* $\rho_0>0$; *(iii) supp(*$\tilde{v}^a$*)*$\subset$ *supp(*ρ*)*$=\bar{\Omega}$ *compact; (iv)* $\rho|_{\partial\Omega}=\rho_0$; *(v) these fields are solutions of Eqs. (1), (2) and (5), (6) on* $\tilde{M}$.

Given an open set $\Omega'\subset\mathbb{R}^3$, we denote by $C^s(\Omega')$ and $C^{s,\alpha}(\Omega')$ the spaces of s-times continuously and Hölder continuously differentiable functions, respectively, with $s\geqslant 0$ integer, and $0<\alpha<1$. We use the notation $C^\alpha(\Omega')=C^{0,\alpha}(\Omega')$. We also denote by $L^p(\Omega')$, $W^{s,p}(\Omega')$, and by $L^p_{\mathrm{loc}}(\Omega')$, $W^{s,p}_{\mathrm{loc}}(\Omega')$ the Lebesgue and Sobolev spaces, and the local Lebesgue and local Sobolev spaces, respectively, where $1<p<\infty$. We follow the definitions given in [11,12], and the generalizations for smooth manifolds, M', given in [13]. Finally, we say that a tensor field on $\Omega'\subset M'$ belongs to one of the functional spaces mentioned above, if all its components, in some smooth atlas of M', belong to such a space.

It is convenient to split the concept of an almost-smooth field, presented in the Introduction, into the following two definitions. The first one is an Ω-piecewise smooth field. Consider a smooth manifold M', a tensor field u on that manifold, and a open set $\Omega\subset M'$, with compact closure. We say that u is Ω-piecewise smooth, if $u\in C^\infty(\bar{\Omega})\cap C^\infty(M'\backslash\Omega)$. Note that this definition involves conditions on the field both in Ω and its complement. An example of an Ω-piecewise smooth but not smooth function is any f such that $0<f\in C^\infty(\bar{\Omega})$ and $f=0$ on $M'\backslash\bar{\Omega}$. The fluid energy density of a liquid-type body is such a function.

The second concept is an Ω-tangentially smooth field. Let q_{ab} be a smooth, positive definite, metric on M'. Assume that $\partial\Omega$ is a smooth submanifold of codimension one. Let $\hat{n}^a$ be a normal vector to $\partial\Omega$ with respect to q_{ab}. Consider a Gaussian normal foliation in a neighborhood of $\partial\Omega$, that is, a foliation orthogonal to the geodesics tangent to $\hat{n}^a$ at every point of $\partial\Omega$. Define $\hat{n}^a$ outside $\partial\Omega$ to be tangent to these geodesics. Let $V^a_{\partial\Omega}$ be any smooth tangent vector field to this foliation, i.e., any smooth vector field such that $V^a_{\partial\Omega}\hat{n}_a=0$. We say that an Ω-piecewise smooth field u is Ω-tangentially smooth if for all $k\geqslant 1$ the tangential derivatives $V^{(k)}_{\partial\Omega}(u)$ are continuous, where $V^{(1)}_{\partial\Omega}(u):=V^a_{\partial\Omega}D_a u$, and $V^{(k)}_{\partial\Omega}(u):=V^a_{\partial\Omega}D_a[V^{(k-1)}_{\partial\Omega}(u)]$, for $k\geqslant 1$. For example, choose the field to be the energy density of a liquid-type ideal body, and Ω the interior of its support. A necessary condition for this field to be Ω-tangentially smooth is to be constant at $\partial\Omega$.

B. Main result

The strategy is, first, to find fields $\tilde{q}_{ab}$, $\tilde{p}^{ab}$, $\tilde{j}^a$, and $\tilde{\mu}$, the solution of Eqs. (1),(2) with the desired properties. Conformal rescaling techniques are used in this part. We also introduce a compact manifold where equations associated with Eqs. (1),(2) are solved with boundary conditions chosen in such a way that the decompactification of these solutions gives asymptotically flat initial data. Then we prove that under specific assumptions on the state function, Eqs. (5),(6) can be inverted for all $\rho\geqslant\rho_0$ and for $\tilde{v}^a$ with $0\leqslant\tilde{v}<1$.

Fix a 3-dimensional, orientable, connected, compact, smooth manifold. Fix $i\in M$, and $\tilde{M}:=M\backslash\{i\}$. The choice $M=S^3$, and so $\tilde{M}=\mathbb{R}^3$, describes, for example, ordinary stars. A restriction of this type in the topology, however, plays no role in what follows. Let $h_{ab}\in C^\infty(M)$ be a Riemannian metric on M. Let x^i be its associated Riemann normal coordinate system at i, and r the geodesic distance. Let $\hbar_{ab}\in C^\infty(M)$ be a symmetric tensor such that

$$x^i\hbar_{ij}=0. \tag{8}$$

Latin indices i,j,k denote tensor components on coordinates x^i. Let $q_{ab}\in C^\infty(\tilde{M})$ be a Riemannian metric on M with scalar curvature R. Let B_ϵ be an open ball of geodesic radius ϵ centered at i. Assume that there exists $\epsilon>0$, such that the metric q_{ab} on B_ϵ has the form

$$q_{ij}=h_{ij}+r^3\hbar_{ij} \tag{9}$$

in the coordinates x^i. Since we have assumed Eq. (8), these coordinates are also normal coordinates of the metric q_{ab}. The motivation for Eq. (9) is given in the remarks below theorem 1.

Fix a non-negative scalar field $\tilde{\mu}$ and a vector field j^a on M with $\mathrm{supp}(j^a)\subset\mathrm{supp}(\tilde{\mu})=\bar{\Omega}$, where Ω is some open set with compact closure $\bar{\Omega}\subset\tilde{M}$, such that its boundary $\partial\Omega$ is a smooth submanifold of codimension one. Introduce on $\tilde{M}$ the fields θ and p^{ab}, with $p_a{}^a=0$, solutions of

$$D_a p^{ab}=-\kappa j^b, \tag{10}$$

$$L_q(\theta)=-\frac{p_{ab}p^{ab}}{8\theta^7}-\frac{\kappa}{4}\tilde{\mu}\theta^5, \tag{11}$$

where $L_q(\theta) := q^{ab}D_aD_b\theta - R\theta/8$, and D_a is the Levi-Civita connection associated to q_{ab}. Indices of "non-tilde" tensors are raised and lowered with q^{ab} and q_{ab}, respectively, where $q_{ac}q^{cb} = \delta_a{}^b$. Fix the boundary condition

$$p^{ij} = O(r^{-4}), \tag{12}$$

$$\lim_{r\to 0} r\theta = 1. \tag{13}$$

The main part of this work is to prove the existence of a solution to Eqs. (10)–(13), and then to show that if the source functions $\tilde{\mu}$ and j^a are Ω-piecewise and tangentially smooth, then so are the solutions p^{ab} and θ. Once these fields θ and p^{ab} are known the initial data set is given by the following conformal rescaling:

$$\tilde{q}_{ab} = \theta^4 q_{ab}, \quad \tilde{p}^{ab} = \theta^{-10}p^{ab}, \quad \tilde{j}^a = \theta^{-10}j^a. \tag{14}$$

Notice that we do not rescale the energy density $\tilde{\mu}$, but we do rescale the momentum density $\tilde{j}^a$. In this way we achieve both that $\tilde{\mu}$ be free data, and that the momentum constraint decouples the Hamiltonian constraint, respectively. One can check that if θ and p^{ab} are solutions of Eqs. (10),(11) then the rescaled fields in Eqs. (14) satisfy Eqs. (1),(2). One can also check that the boundary conditions (12),(13) on θ and p^{ab} imply that the rescaled initial data is asymptotically flat. (See [9,10].)

Let γ be the Green function of the operator L_q given in Eq. (11), which is defined in Eqs. (20),(21). Our first main theorem is concerned with the momentum constraint (10), (12).

Theorem 1. Fix M, $\tilde{M}$, Ω, *and* q_{ab} *as above. Let* s^{ab} *be a symmetric trace-free tensor in* $W^{1,q}(M)\cap C^\infty(\tilde{M})$, $q>3$. *Let* $\overline{p}^{ab}$ *be given by Eq. (37). Assume that*

(i) $\mathrm{supp}(j^a)\subset\overline{\Omega}\subset\tilde{M}$.

(ii) $j^a\in L^q(M)$ *and it is* Ω*-piecewise and* Ω*-tangentially smooth.*

(iii) Condition (42) is satisfied.

Then there exists a unique tensor p^{ab} *given by the Eq. (41) solution of Eqs. (10), (12). Moreover,* p^{ab} *is* Ω*-piecewise and* Ω*-tangentially smooth and satisfies*

$$p_{ab}p^{ab}/\gamma^7 \in L^2(M). \tag{15}$$

The existence part of this theorem is essentially the standard York splitting (cf. [14]) adapted to our setting. It is given in theorem 5, under a weaker hypothesis. s^{ab} is free data related to the arbitrary amount of gravitational radiation that can be added to the system keeping the matter sources fixed. $\overline{p}^{ab}$ contains the linear and angular momentum of the initial data, it can also be prescribed freely unless there are conformal symmetries. In this case it has to satisfy condition (iii), which is the corresponding Fredholm condition (see the remark after theorem 5 for a physical interpretation). The regularity part of the theorem is proved in Sec. III C.

We have chosen the unphysical metric, q_{ab}, smooth on $M\setminus\{i\}$. This is a reasonable physical assumption. However to also impose smoothness at i it is too restrictive. In this case initial data for stationary space-times are ruled out (see [15]). The differentiability at i of the unphysical metric is related with decay at infinity of the associated physical metric $\tilde{q}_{ab}$, imposing smoothness at i means a restriction in the fall off which is, in particular, incompatible with the stationary solutions. In order to include these data, we have made the assumption (9). Although the functions h_{ij} and $\hbar_{ij}$ are smooth, the metric belongs to $C^{2,\alpha}(B_\epsilon)$ but it does not belong to $C^3(B_\epsilon)$. The data for stationary space-times have precisely this form (see [15]). In order to prove the theorem one certainly does not need smoothness of h_{ij} and $\hbar_{ij}$, only a finite number of derivatives. But the important point is that in order to prove the last part, Eq. (15), it is not enough to require that, for example, $q_{ab}\in C^{2,\alpha}(M)$. Equation (15) is essential in order to prove our second theorem. We prove Eq. (15) in theorem 6.

Proof. The metric given by Eq. (9) satisfies that $q_{ab}\in W^{4,p}(M)$, $p>3/2$. By assumption j^a and s^{ab} belong to $W^{1,q}(M)$. Therefore assumption (iii) and theorem 5 imply that there exists $p^{ab}\in W^{1,q'}(M)$, $1<q'<3/2$, given by Eq. (41) which solves Eqs. (10), (12). The hypothesis on the metric given in Eq. (9) and theorem 6 imply Eq. (15). Assumptions (i), (ii), and theorems 7 and 8 imply that p^{ab} is Ω-piecewise smooth and Ω-tangentially smooth. □

In order to write the next theorem we need to define some constants. Set $C_p = \|p_{ab}p^{ab}/(8\gamma^7)\|_{L^2(M)}$, $\gamma_+ = \max_{\overline{\Omega}}\gamma$, $\gamma_- = \min_{\overline{\Omega}}(\gamma)$, and $j_+ = \max_{\overline{\Omega}}\sqrt{j_a j^a}$. Let K, k, be the positive constants defined in Sec. III A. They essentially depend on the metric q_{ab} and the manifold M. Finally, let $\epsilon_0>0$ be the solution of the following equation:

$$\epsilon_0 j_+\gamma_-^{-8} = \frac{K}{|\Omega|^{1/2}(\gamma_+ + k\epsilon_0^2 C_p)^4}, \tag{16}$$

where $|\Omega|$ is the volume of Ω with respect to the metric q_{ab}. There always exists a unique positive solution to Eq. (16), since for $\epsilon_0>0$ the right-hand side of Eq. (16) is a positive, decreasing function of ϵ_0 which goes to zero at infinity.

Theorem 2. Assume that the hypothesis of theorem 1 holds. Let p^{ab} *be the tensor field given in that theorem. Assume that* $R>0$. *Fix a smooth liquid-type state function,* $p(\rho)$, *with zero-pressure energy density,* $\rho_0>0$, *compatible with condition (iii). Assume that* $0<\partial p/\partial\rho<1$. *Let* $\tilde{\mu}$ *be such that*

(i) $supp(\tilde{\mu}) = \overline{\Omega}$. $\tilde{\mu}|_{\partial\Omega} = \rho_0$, *and* $j^a|_{\partial\Omega} = 0$.

(ii) $\tilde{\mu}$ *is* Ω*-piecewise and* Ω*-tangentially smooth.*

(iii) For $\epsilon<\epsilon_0$, $\tilde{\mu}$ *satisfies*

$$\epsilon j_+\gamma_-^{-8} < \rho_0 \leq \tilde{\mu} \leq \frac{K}{|\Omega|^{1/2}(\gamma_+ + k\epsilon^2 C_p)^4}. \tag{17}$$

Then there exists a positive solution $\theta\in C^{1,\alpha}(\tilde{M})$ *of Eqs. (11), (13) with sources given by* $\tilde{\mu}$ *and* ϵp^{ab}, *where* $0\leq\epsilon<\epsilon_0$ *and* $0<\alpha<1$.

Moreover, the initial data computed with $\tilde{q}_{ab}=\theta^4 q_{ab}$, $\tilde{p}^{ab}=\theta^{-10}\epsilon p^{ab}$, $\tilde{j}^a=\theta^{-10}\epsilon j^a$, *and* $\tilde{\mu}$ *is of liquid-type, as stated in definition 1. They are* Ω*-piecewise smooth, and* Ω*-tangentially smooth,* $\tilde{q}_{ab}\in C^{1,\alpha}(\tilde{M})$, $\tilde{p}^{ab}\in C^{\alpha}(\tilde{M})$. *The fluid 3-velocity,* $\tilde{v}^a$, *vanishes at* $\partial\Omega$.

We use a nontypical conformal rescaling given in Eq. (14). The positive outcome is that, in this way, $\tilde{\mu}$ is essentially free data, and so we can choose it constant at $\partial\Omega$. A negative outcome is that this $\tilde{\mu}$ must satisfy the bound (17). The upper bound in Eq. (17) is related with the existence of the solution, given in theorem 3. In the Appendix we give arguments to show that this bound is only technical, that is, there exist solutions which do not satisfy it. However, the example presented there suggests that this bound will be satisfied for every realistic star. The lower bound in Eq. (17) is related to the energy condition. It is a sufficient condition for the dominant energy condition to hold, see Sec. IV A.

A second negative outcome is that, in order to satisfy the liquid-type constraint (7), we impose $j^a|_{\partial\Omega}=0$; this implies $\tilde{v}^a|_{\partial\Omega}=0$. In order to understand the implications of this condition on the motion of the fluid, assume that we have a simple, liquid type, fluid solution of Einstein-Euler's equation. That is, a 4-dimensional Lorentzian metric g_{ab} and a unit timelike vector field u^a, representing the fluid 4-velocity, solutions of Einstein-Euler's equation. The boundary $\mathcal{B}$ of the fluid is the 3-dimensional, timelike, hypersurface where $p=0$. Since we have a simple fluid, this implies that ρ is constant on $\mathcal{B}$; hence the vector defined by $N^a=g^{ab}\nabla_b\rho$ is normal to $\mathcal{B}$, where ∇_b is the covariant derivative with respect to g_{ab}. By assumption N^a is not zero on $\mathcal{B}$. Fix an arbitrary spacelike foliation, with normal vector n^a; let $\tilde{M}$ be a member of this foliation. Defining $\partial\Omega=\tilde{M}\cap\mathcal{B}$, we will assume that both $\mathcal{B}$ and $\partial\Omega$ are smooth submanifolds. The 3-velocity $\tilde{v}^a$, defined by $u^a=(n^a+\tilde{v}^a)/\sqrt{1-\tilde{v}^2}$, will vanish at $\partial\Omega$ if and only if the following equations hold

$$N_a n^a|_{\partial\Omega}=0, \tag{18}$$

$$N_a \omega^a|_{\partial\Omega}=0, \tag{19}$$

where $\omega^a=\epsilon^{abcd}u_b\nabla_c u_d$ is the twist of u^a (ϵ_{abcd} is the volume element of g_{ab} and the indexes are moved with g_{ab}). Equation (18) is a condition on the foliation: the slice $\tilde{M}$ has to be tangent to N^a. Equation (19) is a condition on u^a, independent of the foliation: the normal component, with respect to the fluid boundary, of the twist of u^a must vanish on $\partial\Omega$. Equation (19) is a consequence of Frobenius's theorem (see for example [21]) and the fact that $\partial\Omega$ is a smooth submanifold and N^a is hypersurface orthogonal. Note that ω^a itself can be different from zero at $\partial\Omega$. Condition (19) is not time-propagated by u^a. This condition is imposed only on the initial slice, not in the subsequent evolution. Although it is a restriction, it is not clear if it is a strong restriction or not.

Another outcome of this particular conformal rescaling is a lack of uniqueness of solutions to Eqs. (1)–(6) in terms of the free data.

We do not require that Ω be connected. A nonconnected domain can describe several compact bodies.

This is not the most general result one can obtain with these methods. One can also find solutions which are not piecewise smooth, but with some finite differentiability in the interior of the support of ρ. One can even obtain solutions where the support of ρ itself has some finite differentiability. The obtainment of such more general data from the techniques used to get our result does not present a substantial difficulty but only a greater level of technical complication, which could obscure the main ideas necessary to find these types of data.

Proof. The upper bound on $\tilde{\mu}$ given by Eq. (17) and theorem 3 implies that there exists a strictly positive solution $\theta=\gamma+\vartheta\in C^{1,\alpha}(\tilde{M})$ of Eqs. (11), (13). Hypothesis (i), (ii) and theorems 7 and 8 imply that θ is Ω-piecewise smooth and Ω-tangentially smooth.

Let $\tilde{q}_{ab}$, $\tilde{p}^{ab}$, $\tilde{j}^a$ be as stated in theorem 2. Those fields are also Ω-piecewise smooth and Ω-tangentially smooth, and they satisfy $\tilde{q}_{ab}\in C^{1,\alpha}(\tilde{M})$, $\tilde{p}^{ab}\in C^{\alpha}(\tilde{M})$. The lower bound in Eq. (17) and lemma 3 implies that the dominant energy condition is satisfied, that is $\tilde{j}_a\tilde{j}^a<\tilde{\mu}^2$. Assumption (i) implies that the liquid-type constraint (7) is trivially satisfied. Finally, theorem 9 implies that Eqs. (5),(6) are invertible. The state function $p(\rho)$ is a smooth function of ρ, so Eqs. (5),(6) imply that the fields $\tilde{v}^a$ and ρ are Ω-piecewise smooth and Ω-tangentially smooth. Equation (6) and assumption (i) imply that $\tilde{v}^a|_{\partial\Omega}=0$. □

III. EXISTENCE AND REGULARITY

A. Hamiltonian constraint

Consider Eqs. (11), (13). To obtain a solution θ we first transform this problem on $\tilde{M}$ with a singular boundary condition at $i\in M$, into a regular problem on M for another function. The metric q_{ab} has strictly positive scalar of curvature R and the assumption given in Eq. (9) implies that $q_{ab}\in W^{4,p}(M)$, $p>3/2$. Therefore, lemma 3.2 and corollary 3.3 in [10], imply that there exist a unique, positive solution $\gamma\in C^{1,\alpha}(\tilde{M})$ of the equation

$$L_q(\gamma)=-4\pi\delta_i, \tag{20}$$

where δ_i is Dirac's delta distribution with support at i. It is also true that $\gamma^{-1}\in C^{\alpha}(M)$ and

$$\lim_{r\to 0} r\gamma=1. \tag{21}$$

We introduce the function $\vartheta=\theta-\gamma$. Then, Eq. (11) for θ on $\tilde{M}$ becomes the following equation for ϑ on M,

$$L_q(\vartheta)=-\frac{p_{ab}p^{ab}}{8(\gamma+\vartheta)^7}-\frac{\kappa}{4}\tilde{\mu}(\gamma+\vartheta)^5. \tag{22}$$

Before stating the theorem concerning existence of solutions to Eq. (22), we need some notation. Given any function

$g \in W^{2,2}(M)$ and the operator L_q, we introduce k to be the constant such that $|g|_{C^0(M)} \leq k\|L_q(g)\|_{L^2(M)}$. This constant can be written as $k = c_s c_L$, where the Sobolev coefficient c_s is the constant such that $|g|_{C^0(M)} \leq c_s\|g\|_{W^{2,2}(M)}$, while c_L is the constant of the elliptic estimate $\|g\|_{W^{2,2}(M)} \leq c_L\|L_q(g)\|_{L^2(M)}$. (See [11,13].) We introduce, as well, the constants $C_p := \|p_{ab}p^{ab}/(8\gamma^7)\|_{L^2(M)}$, and $\gamma_+ := \sup_{\bar{\Omega}}\gamma$. Therefore,

$$p_{ab}p^{ab}/\gamma^7 \in L^2(M) \tag{23}$$

is equivalent to the condition $C_p < \infty$.

Theorem 3. (Existence) Let M and $\tilde{M}$ be as in Sec. II B. Let q_{ab} be a Riemannian metric on M, such that $q_{ab} \in W^{4,p}(M)$, $p > 3/2$, and $R > 0$. Assume that p^{ab} satisfies that $C_p < \infty$. Let $\tilde{\mu}$ be a positive function of compact support in $\Omega \subset \tilde{M}$, such that

$$\|\tilde{\mu}\|_{L^2(\Omega)} \leq \frac{K}{(\gamma_+ + kC_p)^4}, \tag{24}$$

where $K = 4^5/(5^5\kappa k)$. Then there exists a non-negative solution $\vartheta \in W^{2,2}(M)$ of Eq. (22). The solution is strictly positive unless both p^{ab} and $\tilde{\mu}$ are zero. Moreover, it satisfies $\vartheta \leq (\gamma_+ + 5kC_p)/4$.

Remark. The proof is based on Schauder's fixed-point theorem (see for example [16]): Let $B \subset X$ be a nonempty, closed, convex set in a Banach space X, and $F: B \to B$ be a continuous mapping. If $F(B)$ is precompact, then F has a fixed point. The construction of the functional F is similar to the one made in [10] for theorem 3.4. The only difference is the choice of the set B, and the main work is to prove that for this choice we have $F(B) \subset B$.

Proof. Consider $X = C^0(M)$, which is a Banach space under the supremum norm. Given a constant $c > 0$, define $B_c = \{u \in X : 0 \leq u \leq c\}$. One can check that B_c is convex and closed. Define a nonlinear operator $F: B_c \to X$, by setting

$$F := L_q^{-1} \circ f$$

where the $f: B_c \to L^2(M)$ is the continuous map given by

$$f(u) := -\frac{p_{ab}p^{ab}}{8(\gamma+u)^7} - \frac{\kappa}{4}\tilde{\mu}(\gamma+u)^5. \tag{25}$$

Under the assumptions $q_{ab} \in W^{4,p}(M)$, $p > 3/2$, and $R > 0$ it has been proved in [10] that the nonlinear map F is continuous and $F(B_c)$ is precompact. The only difference between the map F and the analogous map T defined in [10] is the second term in the right-hand side of Eq. (25). This term is continuous. Note that γ is singular at i but we assume that $\tilde{\mu}$ has support in Ω and the point i is *not* included in Ω.

We only have to choose the constant "c" such that $F(B_c) \subset B_c$. The rest of the proof shows how to find "c." In what follows we will use lemma 3.1 of [10] many times.

Introduce the functions $\varphi_c \in L^2(M)$ and $\phi_c \in W^{2,2}(M)$ as follows:

$$\varphi_c := -\frac{p_{ab}p^{ab}}{8\gamma^7} - \frac{\kappa}{4}\tilde{\mu}(\gamma+c)^5$$

$$\phi_c := L_q^{-1}(\varphi_c).$$

Then, for all $u \in B_c$ we have that $f(u) - \varphi_c \geq 0$. This is equivalent to $L_q(F(u) - \phi_c) \geq 0$, and then $F(u) \leq \phi_c$. We now choose the best constant "c" such that $\phi_c \leq c$. This is done as follows. Given the bound

$$\begin{aligned}\phi_c &\leq k\|L_q(\phi_c)\|_{L^2(M)}, \\ &\leq k\left[C_p + \frac{\kappa}{4}(\gamma_+ + c)^5\|\tilde{\mu}\|_{L^2(\Omega)}\right]\end{aligned} \tag{26}$$

we impose that the right hand side of Eq. (26) be less or equal to c. We then obtain

$$\|\tilde{\mu}\|_{L^2(\Omega)} \leq \frac{4}{\kappa k}\frac{c - kC_p}{(\gamma_+ + c)^5}. \tag{27}$$

This inequality has to be valid for some c, in particular for its maximum value given by $c_0 = (\gamma_+ + 5kC_p)/4$. Equation (27) evaluated at c_0 gives Eq. (24). Therefore, choosing $B = B_{c_0}$, condition (24) implies $F(B) \subset B$. Finally, Schauder's fixed-point theorem implies that F has a fixed point in B. This fixed point is the solution ϑ. □

We now show that, under slightly stronger assumptions on the source functions $\tilde{\mu}$ and p^{ab}, the function θ belongs to $C^{1,\alpha}(\tilde{M})$. (This differentiability is important for theorem 8.)

Theorem 4. [$C^{1,\alpha}(\tilde{M})$-regularity] Assume the hypothesis on theorem 3 holds, and let $\theta = \gamma + \vartheta$, with $\vartheta \in W^{2,2}(M)$ the solution of Eq. (22). In addition, assume that $\tilde{\mu} \in L^q(\Omega)$, with $q > 3$, and that $p_{ab}p^{ab}/\gamma^7 \in L^q_{\mathrm{loc}}(\tilde{M})$.

Then, $\theta \in W^{2,q}_{\mathrm{loc}}(\tilde{M}) \subset C^{1,\alpha}(\tilde{M})$ is a solution of Eqs. (11), (13).

Proof. From the hypothesis on $\tilde{\mu}$ and p^{ab}, we have $f(\vartheta) \in L^q_{\mathrm{loc}}(\tilde{M})$. Elliptic regularity implies $\vartheta \in W^{2,q}(\tilde{M})$. (See [12].) Sobolev embedding and $q > 3$ imply $\vartheta \in C^{1,\alpha}(\tilde{M})$. Therefore, $\gamma \in C^{1,\alpha}(\tilde{M})$ implies that $\theta \in C^{1,\alpha}(\tilde{M})$. Equation (21) implies that θ satisfies the boundary condition (13). □

B. Momentum constraint

Consider Eqs. (10), (12). The main idea is, as in Sec. III A, to transform these equations on $\tilde{M}$ with a singular boundary condition into an equation on M for a regular variable. Solutions of this regular equation can be found by the transverse, traceless decomposition of symmetric tensors. See [17] for a transverse decomposition, and [14] for a transverse, traceless decomposition. See also [18], and references therein.

All of this procedure is performed, however, not in Eq. (10) itself, but in a properly conformal rescaled version of that equation. The new rescaled metric is chosen such that its

Ricci tensor vanishes at i. (The restriction that the unphysical metric, q_{ab}, have strictly positive Ricci scalar on M is not needed in this subsection.) The positive outcome of this new rescaling is that it is not hard to prove that solutions p^{ab} with nonvanishing total linear momentum are included.

The plan of this subsection is, first, to introduce some notation; second, to set up the procedure to prove existence of solutions p^{ab} in a weak sense (theorem 5); and third, to prove that, under a slightly stronger assumption on the matter source, the solution satisfies Eq. (23) (theorem 6).

We start with the new conformal rescaling. Let M, $\tilde{M}$, q_{ab}, x^i, r, and B_ϵ as in Sec. II B. Let χ be a cut function, that is a smooth function with support in $B_{2\epsilon}$ and such that $\chi=1$ in B_ϵ. Fix on M the metric $\hat{q}_{ab}$ given by

$$\hat{q}_{ab}=\omega_0^4 q_{ab}, \tag{28}$$

where the conformal factor ω_0 has the form

$$\omega_0=e^{\chi f_0}, \quad f_0=\frac{1}{2}x^j x^k L_{jk}(i), \tag{29}$$

and we have evaluated at i the tensor field

$$L_{ab}:=R_{ab}-\frac{1}{4}R q_{ab}, \tag{30}$$

with R_{ab} the Ricci tensor of q_{ab}. Therefore, $\hat{q}_{ab}=q_{ab}$ on $M\backslash B_{2\epsilon}$, and they differ only on $B_{2\epsilon}$. One can check that $\hat{R}_{abc}{}^d(i)=0$, that is the Riemann tensor of $\hat{q}_{ab}$ evaluated at i vanishes. [An explicit computation shows $\hat{R}_{ab}(i)=0$. Since $\hat{q}_{ab}$ is a 3-dimensional metric, $\hat{R}_{abc}{}^d(i)=0$.] This property implies that in its associated Riemann normal coordinate system at i, $\hat{x}^j$, the metric $\hat{q}_{ab}$ has the form

$$\hat{q}_{ij}=\delta_{ij}+O(\hat{r}^3), \quad \hat{\Gamma}^{j}_{i}{}_{k}=O(\hat{r}^2), \tag{31}$$

where $\hat{r}$ is the geodesic distance from i measured by $\hat{q}_{ab}$. This is the reason for doing the new conformal rescaling.

We complete the rescaling introducing the fields $\hat{p}^{ab}$ and $\hat{j}^a$ as

$$\hat{p}^{ab}=\omega_0^{-10}p^{ab}, \quad \hat{j}^a=\omega_0^{-10}j^a.$$

Therefore, Eqs. (10),(12) transform into

$$\hat{D}_a\hat{p}^{ab}=-\kappa\hat{j}^b, \quad \hat{p}^{ij}=O(\hat{r}^{-4}), \tag{32}$$

where $\hat{D}_a$ is the metric connection associated to $\hat{q}_{ab}$. Latin indices on "hatted" quantities represent components in the coordinate system $\hat{x}^i$.

We now start the procedure to transform Eq. (32) on $\tilde{M}$ with a singular boundary condition into an equation on M for a regular variable. The singular behavior at i of a solution θ of Eqs. (11), (13) was captured by the Green function γ. In the case of Eq. (32), the role analogous to γ is played by a tensor $\overline{p^{ab}}$. The construction of this tensor field that follows is detailed in [10], Secs. 4.1–4.2, but we briefly sketch it here.

Consider the manifold $(M,\hat{q}_{ab})$. Let $B_{2\hat{\epsilon}}\subset U$ be a ball of $\hat{q}_{ab}$-geodesic radius $2\hat{\epsilon}$ centered at i, and $\hat{\chi}$ the associated cut function, that is, a smooth function that vanishes on $M\backslash B_{2\hat{\epsilon}}$ and $\hat{\chi}=1$ in $B_{\hat{\epsilon}}$. Let $\hat{n}_a=\hat{D}_a\hat{r}$. Introduce on $B_{2\hat{\epsilon}}\backslash\{i\}$ the tensor fields [35]

$$\phi^{ab}_{(1)}=\frac{3}{2\hat{r}^2}[2Q^{(a}\hat{n}^{b)}-(\delta^{ab}-\hat{n}^a\hat{n}^b)\hat{n}_c Q^c], \tag{33}$$

$$\phi^{ab}_{(2)}=\frac{A}{\hat{r}^3}(\delta^{ab}-3\hat{n}^a\hat{n}^b), \tag{34}$$

$$\phi^{ab}_{(3)}=\frac{6}{\hat{r}^3}\hat{n}^{(a}\epsilon^{b)cd}J_c\hat{n}_d, \tag{35}$$

$$\phi^{ab}_{(4)}=-\frac{3}{2\hat{r}^4}[2P^{(a}\hat{n}^{b)}+(\delta^{ab}-5\hat{n}^a\hat{n}^b)\hat{n}_c P^c], \tag{36}$$

where A is constant, and P^a, J^a, and Q^a are constants in the coordinate system $\hat{x}^i$. Here $\hat{n}^b=\hat{n}_a\delta^{ab}$, and in Riemann normal coordinates, $\hat{n}^j=\hat{x}^j/\hat{r}$. These tensors are transverse and traceless with respect to the flat metric. Let $\overline{p^{ab}_{(k)}}:=\hat{\chi}(\phi^{ab}_{(k)}-\hat{q}^{ab}\hat{q}_{cd}\phi^{cd}_{(k)}/3)$. Finally, introduce $\overline{p^{ab}}$ as follows:

$$\overline{p^{ab}}:=\sum_{k=1}^{4}\overline{p^{ab}_{(k)}}. \tag{37}$$

By construction, the tensor $\overline{p^{ab}}$ depends on 10 parameters, is smooth on $\tilde{M}$, vanishes on $M\backslash B_{2\hat{\epsilon}}$, is symmetric and $\hat{q}_{ab}$-traceless, and satisfies $\overline{p^{ij}}=O(\hat{r}^{-4})$, as $\hat{r}\to 0$. It also satisfies

$$\hat{D}_a\overline{p^{ab}}=O(\hat{r}^{-2}) \quad \text{as} \quad \hat{r}\to 0. \tag{38}$$

The last equation is obtained as follows: write $\hat{D}_a\overline{p^{ab}}$ explicitly, and then note that first, the tensor fields $\phi^{ab}_{(k)}$ are divergent and trace free with respect to the flat metric, and second, that in the coordinates $\hat{x}^k$ the metric connection coefficients satisfy Eq. (31).

We finally recall some needed properties of conformal Killing vector fields. Consequently, this paragraph is applicable to both q_{ab} and $\hat{q}_{ab}$. We point out the differentiability of the various fields, for later purposes. Fix a manifold (M,q_{ab}), with $q_{ab}\in W^{4,p}(M)$, with $p>3/2$. A conformal Killing vector field, ξ^a, is defined by $(\mathcal{L}_q\xi)^{ab}:=2[D^{(a}\xi^{b)}-q^{ab}D_c\xi^c/3]=0$, where $\mathcal{L}_q$ is the conformal Killing operator associated to the metric q_{ab}. There are at most ten conformal Killing vector fields for a 3-dimensional metric. Given a vector field $\omega^a\in L^{p'}(M)$, with $p'>1$, we say that it is orthogonal to ξ^a if

$$\int_M \xi_a \omega^a dV = 0, \tag{39}$$

where the volume element is computed with the unphysical metric q_{ab}. Notice that the differentiability assumption on the metric implies that $\xi^a \in C^{2,\alpha}(M)$. This, in turn, with the Hölder inequality, implies that the integral above is well defined. We also introduce the conformal Killing data at *i*, that is,

$$k_a = \frac{1}{6} D_a D_b \xi^b(i), \quad S^a = \epsilon^{abc} D_b \xi_c(i), \quad q^a = \xi^a(i),$$

$$a = \frac{1}{3} D_a \xi^a(i). \tag{40}$$

Since *M* is connected, the integrability conditions for conformal Killing fields (cf. [19]) entail that these ten "conformal Killing data at *i*" determine the field ξ^a uniquely on *M*.

We have the following existence theorem, which is a generalization of theorem 16 proved in [10].

Theorem 5. (Existence) Let M, and $\tilde{M}$ *be as in Sec. II B. Assume* $q_{ab} \in W^{4,p}(M)$, $p > 3/2$. *Let* $\overline{p^{ab}}$ *be defined by Eq. (37), and* $\hat{q}_{ab}$ *as in Eq. (28). Let* $s^{ab} \in W^{1,p'}(M)$ *be a symmetric traceless tensor, and* $j^a \in L^{p'}(M)$, *with* $p' > 1$.

(i) If the metric q_{ab} *admits no conformal Killing vectors on M, then there exists a unique vector field* $w^a \in W^{2,q}(M)$, *with* $q = p'$ *if* $p' < 3/2$ *and* $1 < q < 3/2$ *if* $p' \geq 3/2$ *such that the tensor field*

$$p^{ab} = \omega_0^{10} [\overline{p^{ab}} + s^{ab} + (\mathcal{L}_{\hat{q}} w)^{ab}] \tag{41}$$

satisfies Eqs. (10), (12).

(ii) If the metric q_{ab} *admits a conformal Killing vector* ξ^a *on M, corresponding to the conformal Killing data given in Eq. (40), then a vector field* w^a *as specified above exists if and only if the following condition holds,*

$$P^a k_a + J_a S^a + Aa + [P^b L^a_b(i) + Q^a] q_a = \kappa \int_M j_a \xi^a dV, \tag{42}$$

where the constants P^a, J_a, *A, and* Q^a *characterize the tensor* $\overline{p^{ab}}$ *as in Eqs. (33)–(37).*

Proof. Because of Eq. (38) we can consider $\hat{D}_a(\overline{p^{ab}} + s^{ab})$ as a function in $L^q(M)$, $1 < q < 3/2$. The equation $D_a p^{ab} = -\kappa j^b$ is equivalent to

$$\hat{D}_a[\overline{p^{ab}} + s^{ab} + (\mathcal{L}_{\hat{q}} w)^{ab}] = -\kappa \omega_0^{-10} j^b, \tag{43}$$

which can be written as

$$(\mathbf{L}_{\hat{q}} w)^b = -\kappa \omega_0^{-10} j^b - \hat{D}_a(\overline{p^{ab}} + s^{ab}), \tag{44}$$

where $(\mathbf{L}_{\hat{q}} w)^a := \hat{D}_b(\mathcal{L}_{\hat{q}} w)^{ab}$ is an elliptic operator. Its kernel consists of all conformal Killing vectors, ξ^a, of $\hat{q}_{ab}$, and so, of q_{ab}. Following [10], one can prove that the right hand side of Eq. (44) is orthogonal [in the sense given in Eq. (39)] to every conformal Killing vector field, ξ^a, if and only if Eq. (42) holds. Therefore, the assumptions on the metric, q_{ab}, and the Fredholm alternative for this operator imply there exists a unique solution $w^a \in W^{2,q}(M)$. [For the smooth metric this is a standard result, for metric in the Sobolev space $W^{4,p}(M)$ see [20].] □

The quantities P^a and J^a in tensor $\overline{p^{ab}}$ represent the total linear and angular momentum of the data. These quantities can be prescribed freely in case (i), so they are not related with the matter sources j^a. The interpretation is that gravitational waves can carry an arbitrary amount of linear and angular momentum. In the case that the unphysical metric has conformal symmetries these quantities are restricted by condition (42). In order to understand this condition, consider the case where only one Killing vector ξ^a exists, and it is a rotation. That is, only S^a is different from zero. We can always choose S^a to be a unit vector. (This vector is parallel to the axis of the rotational symmetry.) Construct the following initial data: first, choose any J^a pointing in the same direction as S^a, and second, choose the other part of the free data preserving the symmetry. Then, all the fields in the initial data set have this symmetry, and therefore the whole space-time obtained from this initial data set will also have a Killing vector ξ^a, suitably extended outside the initial hypersurface. Condition (42) reduces to

$$J = \kappa \int_M j_a \xi^a dV, \tag{45}$$

where $J = \sqrt{J_a J^a}$. Equation (45) is just the standard Komar integral. (See for example [21].) This is consistent with the interpretation that axially symmetric gravitational waves do not carry angular momentum.

Notice that, with the assumptions we have made, we do not even know if w^a is a continuous vector field. We start with the final part of this subsection, namely, to show that under a slightly stronger assumption on the differentiability of j^a on *M*, and on the metric q_{ab} at *i*, the tensor p^{ab} given by Eq. (41) satisfies Eq. (23). We have assumed that $q_{ab} \in W^{4,p}(M)$. We now impose on the metric an extra condition given in Eq. (9). Then, we have the following result:

Theorem 6. (Regularity on $\tilde{M}$*) Assume that the hypothesis in theorem 5 holds. Assume that the metric satisfies Eq. (9). If* $j^a \in L^q(M)$, $s^{ab} \in W^{1,q}(M)$, *where* $q > 3$, *then,* $w^a \in C^{1,\alpha}(\tilde{M})$ *and the tensor* p^{ab} *satisfies* $p_{ab} p^{ab} / \gamma^7 \in L^2(M)$.

That $w^a \in C^{1,\alpha}(\tilde{M})$ is deduced from standard elliptic regularity theorems. The second part is more difficult. The problem is that $(\mathcal{L}_{\hat{q}} w)^a$ is not continuous at *i*, and so conditions that involve products of tensors are difficult to prove. Since the origin of the discontinuity in $(\mathcal{L}_{\hat{q}} w)^a$ is the singular behavior of $\overline{p^{ab}}$, which we know explicitly, we proceed as follows. We split w^a into a regular part at *i* (called ω^a in the proof) plus some divergent terms. These divergent terms are explicitly computed in terms of $\overline{p^{ab}}$ by an integration procedure based on Meyers' result [22]. Finally we show that this ω^a satisfies a linear elliptic equation with a source in $L^q(B_\epsilon)$

with $q>3$. Therefore it is $C^{1,\alpha}$ at i. Once this splitting near i on w^a is established, condition (23) is proved by explicit computation.

Proof. The source $j^a \in L^q(M)$, while $\overline{p^{ab}}$, being smooth on $\tilde{M}$, belongs to $L^q_{\mathrm{loc}}(\tilde{M})$. Therefore, standard elliptic regularity theorems given in [12] imply $w^a \in W^{2,q}_{\mathrm{loc}}(M) \subset C^{1,\alpha}(\tilde{M})$.

We now begin the proof of the second part of the theorem. We work always with the rescaled metric $\hat{q}_{ab}$ and its corresponding covariant derivative $\hat{D}_a$. We explicitly compute the divergent terms of w^a at i. These terms are appropriate Meyers' potentials of the divergent terms present on $\hat{D}_i \overline{p^{ij}_{(k)}}$. (See lemma 1 below.)

Let $B_{\hat{\epsilon}} \subset M$ be an open ball centered at i of geodesic radius $\hat{\epsilon}>0$. Let us choose $\hat{\epsilon}$ small enough such that the metric has the form (31) in Riemann normal coordinates at i, and the cut function $\hat{\chi}$ is identically equal to 1 in $B_{\hat{\epsilon}}$. An explicit computation shows

$$\hat{D}_i \overline{p^{ij}} = \sum_{k=2}^{3} \frac{\overset{\circ}{p}{}^{j}_{(k)}}{\hat{r}^{k-1}} + \varphi^j,$$

with $\varphi^i = O(1)$ and smooth on $B_{\hat{\epsilon}} \setminus \{i\}$, and continuous at i. The $\overset{\circ}{p}{}^{i}_{(k)}$ are functions of $\hat{n}_a$. We adopt the convention that a small circle over a quantity means that this quantity depends smoothly on $\hat{n}_a$, and does not depend on $\hat{r}$.

Let $V^i_{(k)}$ denote the Meyers potentials of $\overset{\circ}{p}{}^{j}_{(k)}/\hat{r}^{(k-1)}$, for each $k=2,3$, that is, vector fields $V^i_{(k)} = (\Sigma_{l=0}^{2} [\ln(\hat{r})]^l \overset{\circ}{v}{}^{i}_{(kl)})/\hat{r}^{(k-3)}$ defined on $B_{\hat{\epsilon}}$, with $\overset{\circ}{v}{}^{i}_{(kl)}$ appropriate functions of $\hat{n}_a$ that can be explicitly computed in terms of $\overset{\circ}{p}{}^{i}_{(k)}$, and satisfying

$$(\mathbf{L}_\delta V_{(k)})^i = \frac{\overset{\circ}{p}{}^{i}_{(k)}}{\hat{r}^{(k-1)}}.$$

So, here is our decomposition of the vector field w^i, on $B_{\hat{\epsilon}}$,

$$w^i = \sum_{k=2}^{3} V^i_{(k)} + \omega^i.$$

The rest of the proof shows that ω^i is indeed differentiable at i.

Thus ω^i satisfies

$$(\mathbf{L}_{\hat{q}}\omega)^i = -\kappa \omega_0^{-10} j^i - \sum_{k=2}^{3} (\tilde{\mathbf{L}}_{\hat{q}} V_{(k)})^i - \varphi^i - \hat{D}_j s^{ij},$$

where $(\tilde{\mathbf{L}}_{\hat{q}} V_{(k)})^i := (\mathbf{L}_{\hat{q}} V_{(k)})^i - (\mathbf{L}_\delta V_{(k)})^i$. One can check that $(\tilde{\mathbf{L}}_{\hat{q}} V_{(k)})^i = O(\hat{r}^{-(k-3)})$. Therefore the terms $(\tilde{\mathbf{L}}_{\hat{q}} V_{(k)})^i$ with $k=2,3$, belong to $L^q(M)$ with $q>3$. Standard elliptic regularity implies that $\omega^i \in W^{2,q}(M) \subset C^{1,\alpha}(M)$.

We have proved that the solution $w^i \in C^{1,\alpha}(\tilde{M})$ has the following expression in $B_{\hat{\epsilon}}$:

$$w^i = \sum_{k=2}^{3} \frac{1}{\hat{r}^{k-3}} \left(\sum_{l=0}^{2} [\ln(\hat{r})]^l \overset{\circ}{v}{}^{i}_{(kl)} \right) + \omega^i.$$

Notice that the conformal factor ω_0 is smooth on M, then an explicit computation implies that p^{ab} given by Eq. (41) satisfies Eq. (23). □

We present here the generalization of Meyers' result, used in the proof of theorem 6.

Lemma 1. (*Meyers' potential for* $\mathbf{L}_\delta$) *Consider the manifold* $(\mathbb{R}^3, \delta_{ab})$, *with* δ_{ab} *the flat metric*, ∂_a *the metric connection, and let* $(\mathbf{L}_\delta V)^a = \partial_b \partial^b V^a + \partial^a \partial_b V^b/3$. *Consider the equation*

$$(\mathbf{L}_\delta V)^a = r^{k-2} \sum_{l=0}^{\ell} [\ln(r)]^l \overset{\circ}{p}{}^{a}_{(l)}(n) \tag{46}$$

where $\ell \geq 0$ *is a fixed integer, r is the geodesic distance from an arbitrary point* $p \in \mathbb{R}^3$, *and* $\overset{\circ}{p}{}^{a}_{(l)}(n)$ *is a* $C^{K,\alpha}(\mathbb{R}^3)$ *function of* $n_a = \partial_a r$, *with* $K \geq 0$.

Then, there exists $C^{K+2,\alpha}(\mathbb{R}^3)$ *functions* $\overset{\circ}{V}{}^{a}_{(l)}(n)$, *with* $l = 0, \ldots, l+2$, *such that*

$$V^a = r^k \sum_{l=0}^{\ell+2} [\ln(r)]^l \overset{\circ}{V}{}^{a}_{(l)}(n) \tag{47}$$

is a solution of Eq. (46).

Proof. We look for solutions of Eq. (46) of the form

$$V^a = v^a - \frac{1}{4} \partial^a \lambda$$

with

$$\partial_b \partial^b v^a = r^{k-2} \sum_{l=0}^{\ell} [\ln(r)]^l \overset{\circ}{p}{}^{a}_{(l)}(n)$$

$$\partial_a \partial^a \lambda = \partial_a v^a.$$

Lemma 4 in [22] implies that there exists $C^{K+2,\alpha}(\mathbb{R}^3)$ functions $\overset{\circ}{v}{}^{a}_{(l)}$, with $l = 0, \ldots, \ell+1$, such that

$$v^a = r^k \sum_{l=0}^{\ell+1} [\ln(r)]^l \overset{\circ}{v}{}^{a}_{(l)}(n)$$

satisfies the first equation above. Then, one can explicitly compute $\partial_a v^a$, and again lemma 4 in [22] implies that there exists $C^{K+3,\alpha}(\mathbb{R}^3)$ functions $\overset{\circ}{\lambda}_{(l)}$, with $l = 0, \ldots, \ell+2$, such that

$$\lambda = r^{k+1} \sum_{l=0}^{\ell+2} [\ln(r)]^l \overset{\circ}{\lambda}_{(l)}(n)$$

is a solution of the second equation above. Therefore, an explicit computation gives Eq. (47). □

C. Local regularity

Consider a solution θ, p^{ab} of Eqs. (10)–(13). Assume now that the free data s^{ab}, $\tilde{\mu}$ and j^a are Ω-piecewise and tangentially smooth. In the first part of this subsection we then prove that the fields θ and p^{ab} are Ω-piecewise smooth (theorem 7). This proof is based on standard elliptic regularity theorems. In the second part of this subsection we prove that these fields are also Ω-tangentially smooth. This result is split into two parts; first for linear elliptic systems (lemma 2), and then for Eqs. (10)–(13) (theorem 8).

Theorem 7. (Ω*-piecewise smooth*) *Let both q_{ab} and s^{ab} be in $C^\infty(\tilde{M})$. Let θ and p^{ab} be solutions of Eqs. (10)–(13) given by theorem 4 and theorem 5. If the source functions $\tilde{\mu}$ and j^a are Ω-piecewise smooth then so are θ and p^{ab}.*

Proof. By the assumption $q_{ab} \in C^\infty(\tilde{M})$ we have that the two elliptic operators L_q and $\mathbf{L}_q$ have smooth coefficients in $\tilde{M}$. Applying the standard interior elliptic regularity to the domains $\bar{\Omega}$ and $\tilde{M}\backslash\Omega$ we obtain that if j^a is Ω-piecewise smooth then w^a is also Ω-piecewise smooth. Because $\bar{p}^{ab} \in C^\infty(\tilde{M})$, and by assumption $s^{ab} \in C^\infty(\tilde{M})$, then p^{ab} is Ω-piecewise smooth.

In the case of θ we note first that by the elliptic regularity γ is smooth in $\tilde{M}$. Consider now Eq. (22) for ϑ. Denote by $f(x,\vartheta)$ the right-hand side of this equation. By the assumption on $\tilde{\mu}$ and the previous argument regarding w^a, we have that the function $f(x,\vartheta)$ satisfies the following property: if ϑ belongs to $C^{s,\alpha}(\Omega)$ [or to $C^{s,\alpha}(\tilde{M}\backslash\Omega)$], then the composition $f(x,\vartheta(x))$ defines a function that belongs to $C^{s,\alpha}(\Omega)$ [or to $C^{s,\alpha}(\tilde{M}\backslash\Omega)$, respectively]. By theorem 4 we know that the solution $\vartheta \in C^{1,\alpha}(\tilde{M})$. [The argument works also with $\vartheta \in C^{\alpha}(\tilde{M})$.] Then we make an iteration, applying the elliptic regularity for the domains $\bar{\Omega}$ and $\tilde{M}\backslash\Omega$ in each step, to obtain that ϑ is Ω-piecewise smooth. Therefore, so is θ. $\square$

Let $\Omega \subset \Omega'$, and V^a be a smooth vector field on Ω'. Let u be any tensor field on M. Denote $V^{(0)}(u) := u$, $V^{(1)}(u) := V^a D_a u$, and $V^{(k)}(u) := V^a D_a[V^{(k-1)}(u)]$, for $k \geq 1$. In order to prove tangential regularity we prove first the following lemma.

Lemma 2. Let L be a linear elliptic operator of second order on some open, bounded set $\Omega' \subset \tilde{M}$ with smooth coefficients. Let V^a be a smooth vector field on an Ω's, with $\bar{\Omega} \subset \Omega'$. Let $u \in W^{2,p}(\Omega')$, with $p>1$, be a tensor field on Ω' solution of the elliptic equation $L(u)=f$. Let $k \geq 0$, be an integer.

(i) If $V^{(k)}(f) \in L^p(\Omega')$, then $V^{(k)}(u) \in W^{2,p}(\Omega')$.

(ii) If $V^{(k)}(f) \in C^{\alpha}(\Omega')$, then $V^{(k)}(u) \in C^{2,\alpha}(\Omega')$.

Proof. The proof is by induction on k. Consider part (i) of the lemma. The case $k=0$ is the standard interior elliptic regularity. See [12] for second order elliptic equations and [23–25] for systems. Assume now that (i) is true for $k-1$. Consider now the following identity

$$V^{(k)}(L(u)) = \sum_{l=0}^{k} \binom{k}{l} \, {}^{(l)}[V,L](V^{(k-l)}(u)), \tag{48}$$

where $(k_l) = k!/[l!(k-l)!]$ and we have introduced the notation

$$\begin{aligned} {}^{(0)}[V,L](u) &:= L(u) \\ {}^{(1)}[V,L](u) &:= [V,L](u) \\ &= V(L(u)) - L(V(u)) \\ {}^{(l+1)}[V,L](u) &:= [V, {}^{(l)}[V,L]](u). \end{aligned}$$

Notice that, for all $l \geq 0$, the operator ${}^{(l)}[V,L]$ is a second order operator with smooth coefficients on $\tilde{M}$. Assume now that $V^{(k)}(f) \in L^p(\Omega')$, and $V^{(l)}(u) \in W^{2,p}(\Omega')$, for all $0 \leq l \leq k-1$. If we write the identity (48) as

$$L(V^{(k)}(u)) = V^{(k)}(f) - \sum_{l=1}^{k} \binom{k}{l} \, {}^{(l)}[V,L](V^{(k-l)}(u)) \tag{49}$$

then all the terms in the right-hand side belong to $L^p(\Omega')$. Then the elliptic regularity theorems imply that $V^{(k)}(u) \in W^{2,p}(\Omega')$, $p>1$. The case (ii) is similar. $\square$

Theorem 8. (Ω*-tangentially smooth*) *Assume the hypothesis on theorem 7. If $\tilde{\mu}$ and j^a are Ω-tangentially smooth then so are the fields θ and p^{ab} which solve Eqs. (10)–(13).*

Proof. Fix Ω', to be any open set in $\tilde{M}$ such that $\bar{\Omega} \subset \Omega'$. Let V^a be the tangent vector field $V^a_{\partial\Omega}$ defined in Sec. II A. Since Eq. (44) is linear, lemma 2 implies that w^a is Ω-tangentially smooth.

Equation (22) is semilinear. However, there exists a solution $\vartheta \in W^{2,q}(\Omega')$ for $q>3$. Therefore, $\vartheta \in C^{1,\alpha}(\Omega')$. This is the subtle step. Because $\vartheta \in C^{1,\alpha}(\Omega')$, it implies that $V(f[x,\vartheta(x)]) \in L^q(\Omega')$, for $q>3$. The reason is that when we compute $V(f)$, terms appear of the form "function in $L^p(\Omega')$" times "$V(\vartheta)$." If ϑ was only continuous, then these terms would not be, in general, in $L^p(\Omega')$. Then lemma 2 implies that $V(\vartheta) \in W^{2,q}(\Omega') \subset C^{1,\alpha}(\Omega')$. Thus $V^{(2)}(\vartheta) \in C^{\alpha}(\Omega')$. Then $V^{(2)}(f[x,\vartheta(x)])/ \in L^q(\Omega')$ and we obtain $V^{(2)}(\vartheta) \in C^{1,\alpha}(\Omega')$. Iterating this argument, the conclusion follows. $\square$

IV. FURTHER REQUIREMENTS

A. Energy condition

In order to understand the origin of this discussion on energy conditions it is useful to compare the usual procedure to find solutions of the constraint equations with matter sources. In that procedure, one rescales both the energy density $\tilde{\mu}$ as well the momentum current density $\tilde{j}^a$. The rescaled $\tilde{j}^a = \theta^{-10} j^a$ is fixed from the requirement that the momentum constraint be independent of θ, while the rescaled $\tilde{\mu} = \theta^{-8}\mu$ is chosen such that $\tilde{j}/\tilde{\mu} = j/\mu$, where we have introduced $\tilde{j} := \sqrt{\tilde{q}_{ab}\tilde{j}^a\tilde{j}^b}$, and $j := \sqrt{q_{ab} j^a j^b}$. Therefore, if the energy condition is satisfied by the rescaled fields j^a and μ, with respect to the unphysical metric, then the physical fields $\tilde{j}^a$ and $\tilde{\mu}$ do satisfy the energy condition. That is the usual procedure. Here we cannot rescale the energy density, because we need the extra condition that the physical energy density $\tilde{\mu}$ be constant at the border of its support.

We show here that the lower bound for $\tilde{\mu}$ given in (iii) in theorem 2 is sufficient to guarantee that the physical matter fields satisfy the dominant energy condition. The idea is this: Initial data with $j^a=0$ and $\tilde{\mu}$ satisfy trivially the energy condition. Therefore the same applies for arbitrary small j. Condition (iii) is just a rough bound on the smallness on j that also guarantees the energy condition. Let $\gamma_- = \min_{\bar{\Omega}}(\gamma)$, with γ the Green function solution of Eqs. (20),(21). Then we have the following:

Lemma 3. Let M, $\tilde{M}$, q_{ab}, $\tilde{\mu}$, and p^{ab} be as in theorem 4. Let θ be the corresponding solution of Eqs. (11), (13). Let $\tilde{\mu}$ and j^a have support in Ω and in $C^0(\bar{\Omega})$. If $j<\rho_0(\gamma_-)^8$ then the fields $\tilde{q}_{ab}=\theta^4 q_{ab}$, $\tilde{j}^a=\theta^{-10}j^a$, and $\tilde{\mu}$ satisfy $\tilde{j}<\tilde{\mu}$.

Proof. Since ϑ is positive (see theorem 3), $\theta\geqslant\gamma$ on $\tilde{M}$. Then we have

$$\tilde{j}=\theta^{-8}j\leqslant(\gamma_-)^{-8}j<\rho_0\leqslant\tilde{\mu}.$$

□

B. Inversion of Eqs. (5),(6)

We show here that Eqs. (5),(6) are invertible; that is, given the functions $\tilde{\mu}$, $\tilde{j}^a$ there exists unique functions ρ, $\tilde{v}^a$ satisfying these equations. In other words, the fluid 4-momentum density as seen by an arbitrary observer determines the fluid comoving 4-momentum density. It turns out that the proof is not obvious and we did not find it in the literature.

The main difficulty is that the map defined by Eqs. (5),(6) is nonlinear. Furthermore, it contains an unknown function, the state function, subject to minimally restrictive properties. Since $\tilde{v}^a$ and $\tilde{j}^a$ are parallel, these equations reduce to

$$\tilde{\mu}=\frac{\rho+p\tilde{v}^2}{1-\tilde{v}^2} \tag{50}$$

$$\tilde{j}=\frac{(\rho+p)\tilde{v}}{1-\tilde{v}^2}, \tag{51}$$

with $\tilde{j}=\sqrt{\tilde{j}_a\tilde{j}^a}$ and $\tilde{v}=\sqrt{\tilde{v}_a\tilde{v}^a}$. We define the map Φ between subsets of $\mathbb{R}^2$ as

$$\Phi(\rho,\tilde{v})=\left(\frac{\rho+p\tilde{v}^2}{1-\tilde{v}^2},\frac{(\rho+p)\tilde{v}}{1-\tilde{v}^2}\right). \tag{52}$$

Equations (50),(51) can be rewritten as $(\tilde{\mu},\tilde{j})=\Phi(\rho,\tilde{v})$. Given a positive constant ρ_0 we define the following two subsets of $\mathbb{R}^2$

$$D:=\{(\rho,\tilde{v})\in\mathbb{R}^2:\rho_0\leqslant\rho,\quad 0\leqslant\tilde{v}<1\} \tag{53}$$

$$I:=\{(\tilde{\mu},\tilde{j})\in\mathbb{R}^2:\tilde{\mu}_0(\tilde{j})\leqslant\tilde{\mu},\quad 0\leqslant\tilde{j}\}, \tag{54}$$

where $\tilde{\mu}_0(\tilde{j}):=\rho_0/2+\sqrt{\rho_0^2/4+\tilde{j}^2}$. Our result is:

Theorem 9. Let $p(\rho)$ be a C^1 state function such that (i) $p(\rho)\geqslant 0$ for $\rho\geqslant\rho_0>0$; (ii) $p(\rho_0)=0$; (iii) $0<\partial p/\partial\rho<1$. Then, the map $\Phi:D\to I$ is a diffeomorphism.

Proof. First we prove that Φ is bijective.

Surjectivity: The essential tool is Brouwer's fixed-point theorem. (See for example [16].) Equations (50),(51) are equivalent to

$$\rho=\tilde{\mu}-\tilde{j}\tilde{v} \tag{55}$$

$$\tilde{v}=\frac{\tilde{j}}{\tilde{\mu}}\,\frac{\rho+p\tilde{v}^2}{\rho+p}. \tag{56}$$

Fix a point $(\tilde{\mu},\tilde{j})\in I$. Consider the map F given by

$$F(x,y):=\left[(\tilde{\mu}-\tilde{j}y),\left(\frac{\tilde{j}}{\tilde{\mu}}\,\frac{x+p(x)y^2}{x+p(x)}\right)\right].$$

Introduce the compact convex set $C:=[\rho_0,\tilde{\mu}]\times[0,1]\subset\mathbb{R}^2$.

We claim that $F:C\to C$. We write $F(x,y)=[F_1(x,y),F_2(x,y)]$. Then, by definition of I, $\tilde{j}/\tilde{\mu}<1$ and so, for all $(x,y)\in D$, we have $0\leqslant F_2(x,y)<1$. We now show that, for all $(x,y)\in D$, $\rho_0\leqslant F_1(x,y)\leqslant\tilde{\mu}$. The assumption $0\leqslant y\leqslant 1$ implies $\tilde{\mu}-\tilde{j}\leqslant\tilde{\mu}-\tilde{j}y\leqslant\tilde{\mu}$. But $\rho_0\leqslant\tilde{\mu}_0(\tilde{j})-\tilde{j}\leqslant\tilde{\mu}-\tilde{j}$. Therefore $\rho_0\leqslant\tilde{\mu}-\tilde{j}y\leqslant\tilde{\mu}$.

The map F is also continuous. Therefore, by Brouwer's fixed-point theorem, there exists a fixed point $F(x,y)=(x,y)$.

Notice that $\tilde{j}<\tilde{\mu}_0(\tilde{j})\leqslant\tilde{\mu}$ implies that there exists no fixed point of the form $(x,1)$. Therefore, we conclude that, given a point $(\tilde{\mu},\tilde{j})\in I$, there exists a point $(\rho,\tilde{v})\in D$ which solves Eqs. (50),(51).

Injectivity: Consider Eqs. (50),(51), written as

$$\tilde{\mu}=\rho+\tilde{j}\tilde{v}$$

$$\tilde{j}=\frac{(\rho+p)\tilde{v}}{1-\tilde{v}^2}.$$

Assume that there exist two points $(\rho_1,\tilde{v}_1)$ and $(\rho_2,\tilde{v}_2)$ which are solutions of these above equations for the same value of $(\tilde{\mu},\tilde{j})$. If $\tilde{v}_1=0$ then the second equation above implies $\tilde{j}=0$, and so $\tilde{v}_2=0$, which in turn implies $\rho_1=\rho_2$. If $\tilde{v}_1=\tilde{v}_2$ then the first equation below implies $\rho_1=\rho_2$.

Assume now that $\tilde{v}_1\neq 0$, $\tilde{v}_2\neq 0$, and $\tilde{v}_1\neq\tilde{v}_2$. Then

$$(\rho_2-\rho_1)+\tilde{j}(\tilde{v}_2-\tilde{v}_1)=0$$

$$(\rho_2-\rho_1)(1-\nu^2)=\tilde{j}\left[\frac{1-(\tilde{v}_2)^2}{\tilde{v}_2}-\frac{1-(\tilde{v}_1)^2}{\tilde{v}_1}\right],$$

where $\nu^2=(\partial p/\partial\rho)|_{\rho'}$, with $\rho'\in[\rho_1,\rho_2]$ and we have used the mean value theorem for $p(\rho)$. Then the above equations and the assumptions on $\tilde{v}_1$ and $\tilde{v}_2$ imply

$$\nu^2\tilde{v}_1\tilde{v}_2=1.$$

But by assumption $\nu^2<1$, so that we have a contradiction. Therefore, injectivity follows.

It remains to be proven that Φ and Φ^{-1} are differentiable. By direct computation and the assumptions on $p(\rho)$ one can check that the derivative map of Φ is invertible at each point of D. Then, by the inverse function theorem, Φ^{-1} is also differentiable. □

Notice that the proof fails if $\rho_0=0$ because the derivative of Φ is not invertible at this point. This will be the case for an equation of state of the form $p=a\rho^\gamma$, where a and γ are constants. On the other hand, in this work we are interested in equations of state of liquid-type, i.e., such that the pressure vanishes for a positive value of the density at the border of the fluid. For example $p=a[(\rho/\rho_0)^\gamma-1]$. For suitable constants a and ρ_0 this equation describes water. (See [26].)

V. DISCUSSION

The principal interest in the initial data given here is to use them to set up an initial value formulation. This formulation should be able to describe isolated, nearly static fluid bodies. That is why we have concentrated on finding liquid-type data and, inside this class, the smoothest possible data, i.e., the simplest to evolve. We have shown here that these data are not simple to obtain.

The discontinuity of the fluid energy density at the boundary of its support and extra constraints at that boundary [see Eq. (7)] were the main difficulties. One main idea was to not rescale the fluid energy density and so, being free data, trivially solve the extra constraint Eq. (7) (while also requiring that the fluid 3-velocity vanish at the body boundary). This unconventional rescaling of the fluid fields introduces difficulties in the task of finding solutions to the Hamiltonian constraint. These difficulties were solved in theorem 3. Smoothest liquid-type data are almost-smooth, i.e., smooth except in the normal direction to the body boundary. The main step in establishing this result is lemma 2. The rest is standard elliptic regularity.

We have shown that at the body boundary, the first fundamental form is only in $W^{2,q}_{\mathrm{loc}}(\tilde{M})$, $q>3$. This differentiability is below the threshold required by the known existence theorems on symmetric hyperbolic equations to prove existence of solutions associated with such data. Theorems in Sec. 5.1 in [27] require initial data in $W^{s,2}(\Omega')$, with $s>5/2$, where $\Omega'\subset\mathbb{R}^3$ is open and bounded. (See [28] for a related improvement of this result and also [29] for a discussion on the possible future development of the subject.) Imbedding theorems imply that $W^{s,2}(\Omega')\subset W^{2,p}(\Omega')$, with $p>3$, but not the other way around. Therefore, data in $W^{2,p}(\Omega')$ is not enough for the known theorems to guarantee existence of solutions.

We guess two possible ways to set up an initial value formulation for these liquid-type bodies. The first one is to study in detail the Einstein-Euler system given in [7], with the hope that particular features of this system allow enough decrease in the differentiability threshold on the initial data to include the data given here. A second way is to set up two initial boundary value formulations, one for the interior of the body and one for the exterior, and then match both, in an appropriate way, at the boundary of the body (see [30] and [31]). It is far from being clear if either of these alternatives works.

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APPENDIX

We discuss here the bound on the physical energy density given by Eq. (24) in theorem 3. In the first subsection, we show an inequality that is true for all maximal, asymptotically flat initial data with matter sources. This inequality is similar to Eq. (24) in the sense that it relates the same quantities, but only the $L^1(\Omega)$ norm of the energy density appears. In the second subsection we show that Eq. (24) is in fact a restriction, that is, there exist solutions of the constraint equations which do not satisfy it. Nevertheless we give arguments to show that physical systems like neutron stars do satisfy the bound (24).

1. An inequality

Consider the following result.

Lemma 4. Let M, $\tilde{M}$, and q_{ab} be as in Sec. II B. Let p^{ab} and θ be any solution of Eqs. (10)–(13) with $p_a{}^a=0$. Fix a point $p\in\tilde{M}$, and denote by B_r an open ball centered at p, of geodesic radius r. Then, for sufficiently small r, we have

$$\|\tilde{\mu}\|_{L^1(B_r)}\leq\frac{4^6}{5^5}\,\frac{2\pi r}{\kappa(\gamma_-)^4} \tag{A1}$$

where $\gamma_-:=\inf_{\partial B_r}\gamma$, where γ is defined in Eq. (20).

Proof. Consider any solution θ of Eq. (11) on $\tilde{M}$. Then,

$$\frac{\kappa}{4}\tilde{\mu}\leq-\frac{L_q(\theta)}{\theta^5}. \tag{A2}$$

Introduce ϑ as in Sec. III A, that is, $\theta=\gamma+\vartheta$. We parametrize all possible solutions, instead of by ϑ, by a function $\sigma:=L_q(\vartheta)$. Denoting by $L_q^{-1}(\sigma)(x)=-(1/4\pi)\int_M\sigma(y)\gamma(x-y)dV(y)$, then the inequality above translates into

$$\frac{\kappa}{4}\tilde{\mu} \leqslant -\frac{\sigma}{[\gamma + L_q^{-1}(\sigma)]^5}.$$

The Green function has the form $\gamma(x-y)=1/|x-y|+g$, where $g\geqslant 0$ on B_r. (See [32,33].) The inequality $\gamma(x-y) \geqslant 1/(2r)$, that is true for all $x,y\in B_r$, implies that $L_q^{-1}(\sigma) \geqslant \|\sigma\|_{L^1(B_r)}/(8\pi r)$, and this, in turn, implies

$$\frac{\kappa}{4}\|\tilde{\mu}\|_{L^1(B_r)} \leqslant \frac{\|\sigma\|_{L^1(B_r)}}{[\gamma_- + \|\sigma\|_{L^1(B_r)}/(8\pi r)]^5}.$$

The last step of the proof is to maximize the right-hand side of the inequality above with respect to all possible functions σ. The maximum value is taken for $\|\sigma\|_{L^1(B_r)}=2\pi r\gamma_-$, and the inequality above gives Eq. (A1). □

2. Static spherical body

In this subsection we explicitly construct an initial data set for a static, spherically symmetric, liquid-type body. We match, in appropriate coordinates and in a C^1 way, a 3-sphere endowed with its standard metric, with a 3-dimensional Schwarzschild slice. The reason for redoing this known construction (see [34]) is twofold. First, this example, for suitable choices of the parameters, violates the bound (24). Second, we want to answer the following question: What kind of physical systems satisfy the bound (24)? We show that the answer turns out to be (at least for this example) stars with radius $R\geqslant 1.08R_s$, where $R_s=2m$ is the Schwarzschild radius and m is the total mass. Note that this bound is below to $R\geqslant \frac{9}{8}R_s$, which is the necessary condition for hydrostatic equilibrium in general relativity (see for example [21]); then this bound is expected to be satisfied for every star near equilibrium.

Let $M=S^3$, the conformal metric q^0_{ab} be the standard metric, unit radius, of S^3, the point i be the North Pole of S^3, and the domain Ω be a ball centered at the South Pole of S^3.

Let δ_{ab} be the flat metric, and r be the corresponding spherical radius. Consider the following initial data set:

$$\tilde{q}_{ab}=\hat{\theta}^4\delta_{ab}, \quad \tilde{p}^{ab}=0. \tag{A3}$$

The conformal factor $\hat{\theta}$ is given by

$$\hat{\theta}=\begin{cases} 1+\dfrac{m}{2r} & \text{if } r\geqslant r_0, \\ a^{1/4}\left(\dfrac{2r_1}{r_1^2+r^2}\right)^{1/2} & \text{if } r\leqslant r_0, \end{cases} \tag{A4}$$

where the positive constants a, r_0, r_1, and m, satisfy the following relations:

$$r_1^2=\frac{2r_0^3}{m}, \quad a=\frac{(r_0+m/2)^6}{2mr_0^3}. \tag{A5}$$

Using Eq. (A5) one can check that $\hat{\theta}$, and hence $\tilde{q}_{ab}$, is a C^1 function in $\mathbb{R}^3$. There are two free parameters; for example we can take m and r_0, m being the total mass of the data. The metric $\tilde{q}_{ab}$ for $r\geqslant r_0$ is the Schwarzschild metric in isotropic coordinates and for $r\leqslant r_0$ is the standard metric on S^3 of radius $a^{1/2}$. The Ricci scalar of the metric $\tilde{q}_{ab}$ is given by

$$\tilde{R}=\begin{cases} 0 & \text{if } r>r_0, \\ \dfrac{6}{a} & \text{if } r\leqslant r_0, \end{cases} \tag{A6}$$

and the physical energy density is

$$\tilde{\mu}=\begin{cases} 0 & \text{if } r>r_0, \\ \dfrac{6mr_0^3}{\kappa(r_0+m/2)^6} & \text{if } r\leqslant r_0. \end{cases} \tag{A7}$$

We see that the energy density $\tilde{\mu}$ has support in a closed ball of radius r_0. In order to make contact with the assumptions in theorem 3 we write this initial data as follows. Using the well known relation

$$\gamma^4 q^0_{ab}=\delta_{ab}, \quad \gamma=\left(\frac{r_1^2+r^2}{2r_1}\right)^{1/2}, \tag{A8}$$

we obtain

$$\tilde{q}_{ab}=\theta^4 q^0_{ab}, \quad \theta=\hat{\theta}\gamma.$$

For convenience, we have chosen a different normalization for the Green function γ than Eqs. (20) and (21), in order to fix q^0_{ab} to be exactly the unit radius standard metric of S^3. This difference in the normalization will, of course, play no role in what follows.

We want to prove that for some choices of the free parameters r_0 and m, these initial data violate the bound 24. In order to do that we calculate explicitly the right- and left-hand side of Eq. (24). Since $\tilde{\mu}$ is constant we have

$$||\tilde{\mu}||_{L^2(\Omega)}=\tilde{\mu}[\mathrm{Vol}_{q^0}(\Omega)]^{1/2}, \tag{A9}$$

where $\mathrm{Vol}_{q^0}(\Omega)$ denotes the volume with respect to the metric q^0_{ab}. From Eq. (A8) we have

$$\gamma_+=\left(\frac{r_1^2+r_0^2}{2r_1}\right)^{1/2}. \tag{A10}$$

Since $C_p=0$ in this example, using Eqs. (A9) and (A10) we obtain that inequality 24 is equivalent to

$$[\mathrm{Vol}_{q^0}(\Omega)]^{1/2}\leqslant\beta\left(1+\frac{m}{2r_0}\right)^4, \tag{A11}$$

where $\beta=4^5/(3\times 5^5 k)\approx 0.77$ (the constant k, which depends only on S^3 and q^0_{ab}, can be calculated explicitly for this case $k=\sqrt{2}$). Note that $\mathrm{Vol}_{q^0}(\Omega)$ depends only on the dimensionless parameter $m_0/(2r_0)$. For $m_0/(2r_0)=0$ we have that $\mathrm{Vol}_{q^0}(\Omega)=\mathrm{Vol}_{q^0}(S^3)=2\pi^2>\beta^2$, then there exist values of $m_0/(2r_0)$ such that the bound 24 is not satisfied. We use that $\mathrm{Vol}_{q^0}(\Omega) \leqslant \mathrm{Vol}_{q^0}(S^3)$ for arbitrary $m_0/(2r_0)$, to obtain a sufficient condition in order to satisfy Eq. (A11)

$$m/(2r_0) \geqslant 1.75. \tag{A12}$$

Since the exterior metric is the Schwarzschild metric with mass m we can write this condition in terms of the physical radial area coordinate R to obtain

$$R \geqslant 2.16m. \tag{A13}$$

[1] A. D. Rendall and B. G. Schmidt, Class. Quantum Grav. **8**, 985 (1991).

[2] S. Kind and J. Ehlers, Class. Quantum Grav. **10**, 2123 (1993).

[3] Y. Bruhat, Acta Math. **88**, 141 (1952).

[4] Y. Choquet-Bruhat, Bull. Soc. Math. France **86**, 155 (1958).

[5] A. D. Rendall, Living Rev. Relativ. **3** (2000).

[6] A. Rendall, J. Math. Phys. **33**, 1047 (1992).

[7] H. Friedrich, Phys. Rev. D **57**, 2317 (1998).

[8] Y. Choquet-Bruhat, J. Isenberg, and J. W. York, Jr., Phys. Rev. D **61**, 084034 (2000).

[9] R. Beig and N. O. Murchadha, Commun. Math. Phys. **176**, 723 (1996).

[10] S. Dain and H. Friedrich, Commun. Math. Phys. **222**, 569 (2001).

[11] R. A. Adams, *Sobolev Spaces* (Academic Press, New York, 1975).

[12] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order* (Springer-Verlag, Berlin, 1983).

[13] T. Aubin, *Nonlinear Analysis on Manifolds. Monge-Ampère Equation* (Springer-Verlag, New York, 1982).

[14] J. W. York, Jr., J. Math. Phys. **14**, 456 (1973).

[15] S. Dain, Classic Quantum Grav. **18**, 4329 (2001).

[16] E. Zeidler, *Nonlinear Functional Analysis and its Applications I* (Springer, New York, 1993).

[17] M. Berger and D. Ebin, J. Diff. Geom. **3**, 379 (1969).

[18] Y. Choquet-Bruhat and J. W. York, Jr., in *General Relativity and Gravitation*, edited by A. Held (Plenum, New York, 1980), Vol. 1, pp. 99–172.

[19] K. Yano, *The Theory of Lie Derivatives and Its Applications* (North-Holland, Amsterdam, 1957).

[20] M. Cantor, Bull. Am. Math. Soc. **5**, 235 (1981).

[21] R. M. Wald, *General Relativity* (The University of Chicago Press, Chicago, 1984).

[22] N. Meyers, J. Math. Mech. **12**, 247 (1963).

[23] S. Agmon, A. Douglis, and L. Niremberg, Commun. Pure Appl. Math. **17**, 35 (1964).

[24] A. Douglis and L. Niremberg, Commun. Pure Appl. Math. **8**, 503 (1955).

[25] C. B. Morrey, Jr., *Multiple Integrals in the Calculus of Variations* (Springer-Verlag, Berlin, 1966).

[26] R. Courant and K. Friedrichs, *Supersonic Flow and Shock Waves* (Springer-Verlag, Berlin, 1976).

[27] M. E. Taylor, in *Progress in Mathematics* (Birkhäuser, Boston-Basel-Berlin, 1991), Vol. 100, second printing 1993.

[28] S. Klainerman and I. Rodnianski, ‘‘Improved local well posedness for quasilinear wave equations in dimension three’’ (2001), URL http://www.math.princeton.edu/~seri/homepage/seri.htm

[29] S. Klainerman, ‘‘PDE as a unified subject’’ (2001), URL http://www.math.princeton.edu/~seri/homepage/seri.htm

[30] H. Friedrich and G. Nagy, Commun. Math. Phys. **201**, 619 (1999).

[31] J. M. Stewart, Class. Quantum Grav. **15**, 2865 (1998).

[32] J. M. Lee and T. H. Parker, Bull. Am. Math. Soc. **17**, 37 (1987).

[33] R. Schoen and S. T. Yau, *Lecture on Differential Geometry* (International Press, Boston, 1994).

[34] N. O. Murchadha, Phys. Rev. Lett. **57**, 2466 (1986).

[35] In the particular case where $\hat{q}_{ab} = \delta_{ab}$, that is the flat metric, these tensor fields arise as appropriate derivatives of the tensor $\gamma_a{}^b := (7\delta_a{}^b + \hat{n}_a \hat{n}^b)/(8\hat{r})$ which satisfies $(\mathbf{L}_\delta \gamma_b)^a = (-4\pi\delta_i)\delta_b{}^a$. The operator $\mathbf{L}_\delta$ is defined after Eq. (44).