Any of the following exercises are fair game for your final oral exam. I suggest that you write up your solutions neatly in your own handwriting to consult during that exam!

Additional Reminder: You have an option of doing a project instead of doing the homework... so keep that in mind (and check the syllabus) if that sounds more appealing.

Problems Related to Randomized Numerical Linear Algebra

14. Do homework exercise 3.1.5 on page 84 of the notes (`https://math.msu.edu/~iwenmark/Notes_Fall2020_Iwen_Classes.pdf`).

15. Let $\times$ denote the Cartesian product of two sets. Suppose $M \in \mathbb{C}^{m \times N/2}$ is an ϵ-JL map of $T \cup S \subset \mathbb{C}^{N/2}$ into $\mathbb{C}^m$. Show that $g : \mathbb{C}^N \to \mathbb{C}^{2m}$ defined by $g(\mathbf{x}) := g\left((\mathbf{x}_1, \mathbf{x}_2)\right) = (M\mathbf{x}_1, M\mathbf{x}_2)$ is an ϵ-JL map of $(S \times S) \cup (S \times T) \cup (T \times S) \cup (T \times T) \subset \mathbb{C}^N$ into $\mathbb{C}^{2m}$.

16. Let $M_1 \in \mathbb{C}^{\tilde{m} \times N}$ be an ϵ-JL map of $S \subset \mathbb{C}^N$ into $\mathbb{C}^{\tilde{m}}$, and let $M_2 \in \mathbb{C}^{m \times \tilde{m}}$ be an ϵ-JL map of $M_1 S := \left\{ M_1 \mathbf{x} \mid \mathbf{x} \in S \right\} \subset \mathbb{C}^{\tilde{m}}$ into $\mathbb{C}^m$. Show that $M_2 M_1 \in \mathbb{C}^{m \times N}$ is a 3ϵ-JL map of $S \subset \mathbb{C}^N$ into $\mathbb{C}^m$.

17. Let $A \in \mathbb{C}^{n \times p}$, and suppose that $M \in \mathbb{C}^{m \times n}$ is an ϵ-JL map of the p-columns of A into $\mathbb{C}^m$. Prove that $\left| \|MA\|_F^2 - \|A\|_F^2 \right| \leq \epsilon \|A\|_F^2$.

18. Let $A \in \mathbb{C}^{n \times p}$, and suppose that $M \in \mathbb{C}^{m \times n}$ is an ϵ-JL map of the column span of A into $\mathbb{C}^m$. Prove that $\left| \|MA\mathbf{y}\|_F^2 - \|A\mathbf{y}\|_F^2 \right| \leq \epsilon \|A\mathbf{y}\|_F^2$ holds for all $\mathbf{y} \in \mathbb{C}^p$.

19. Do homework exercise 3.2.1 on page 88 of the notes (`https://math.msu.edu/~iwenmark/Notes_Fall2020_Iwen_Classes.pdf`).

20. Do homework exercise 3.2.2 on page 89 of the notes (`https://math.msu.edu/~iwenmark/Notes_Fall2020_Iwen_Classes.pdf`).

21. Do homework exercise 4.4.3 on page 157 of the notes (`https://math.msu.edu/~iwenmark/Notes_Fall2020_Iwen_Classes.pdf`).

22. Let $A \in \mathbb{C}^{p \times q}$ with rank $\tilde{r} \leq \min\{p, q\}$ have the full SVD $A = U \Sigma V^*$ where $U \in \mathbb{C}^{p \times p}$ is unitary, $V \in \mathbb{C}^{q \times q}$ is unitary, and $\Sigma \in [0, \infty)^{p \times q}$ is diagonal with its diagonal entries satisfying $\Sigma_{j,j} =: \sigma_j(A) \geq \Sigma_{j+1,j+1} =: \sigma_{j+1}(A) \geq 0$ for all $j \in [\min\{p, q\} - 1]$. Choose $r \in [p]$ and let $U_r \in \mathbb{C}^{p \times r}$ be U with it's last $(p - r)$ columns removed, $V_r \in \mathbb{C}^{q \times r}$ be V with it's last $(q - \min\{r, q\})$ columns removed (or, if $r \geq q$, we set all entries of V_r to be 0), and let $\Sigma_r \in [0, \infty)^{r \times r}$ be a diagonal matrix with
$$(\Sigma_r)_{k,j} = \begin{cases} \Sigma_{k,j} \text{ for all } k \in [r], j \in [\min\{r, q\}] \\ \Sigma_{k,j} \text{ for all } k \in [r], j \notin [\min\{r, q\}] \end{cases}.$$
Set $A_r := U_r \Sigma_r V_r^*$ and $A_{\setminus r} := A - A_r$.
    (a) Show that $A_r = U_r U_r^* A = A V_r V_r^*$.
    (b) Show that $A_{\setminus r} A_r^* = A_r A_{\setminus r}^* = 0$, and that $A_{\setminus r}^* A_r = A_r^* A_{\setminus r} = 0$.
    (c) Show that $A_{\setminus r} V_{\min\{r, \tilde{r}\}} = 0$.

23. Let $A \in \mathbb{C}^{p \times q}$ and suppose that $M \in \mathbb{C}^{m \times p}$ is an ϵ-JL map of the column span of A into $\mathbb{C}^m$. Prove that
$$\left| \operatorname{tr}\left(B^*(A^*A - A^*M^*MA)B\right) \right| = \left| \|AB\|_{\mathrm{F}}^2 - \|MAB\|_{\mathrm{F}}^2 \right| \leq \epsilon \|AB\|_{\mathrm{F}}^2 = \epsilon \|B^*A^*\|_{\mathrm{F}}^2$$
holds for all $B \in \mathbb{C}^{q \times n}$.