Any of the following exercises are fair game for your final oral exam. I suggest that you write up your solutions neatly in your own handwriting to consult during that exam!

Additional Reminder: You have an option of doing a project instead of doing the homework... so keep that in mind (and check the syllabus) if that sounds more appealing.

Gaussian Widths, Covering Numbers, and Linear Systems with Solution Constraints

32. Let $T \subset \mathbb{R}^N$ be bounded, and denote the ϵ-covering number of T by $C_\epsilon(T)$. Prove that the gaussian width of T satisfies
$$\omega(T) \geq c\epsilon\sqrt{\log C_\epsilon(T)},$$
where $c > 0$ is an absolute constant. HINT: Use Theorem 5.2.4 on page 183 in the notes (`https://math.msu.edu/~iwenmark/Notes_Fall2020_Iwen_Classes.pdf`).

33. Use Sudakov's Minorization for Gaussian Widths (Theorem 5.2.6 in the notes (`https://math.msu.edu/~iwenmark/Notes_Fall2020_Iwen_Classes.pdf`) to show that the ϵ-covering number of $B^N_{\ell^2} := \left\{\mathbf{x} \in \mathbb{R}^N \mid \|\mathbf{x}\|_2 \leq 1\right\}$ is less than $\mathrm{e}^{\frac{N}{c^2\epsilon^2}}$ for an absolute constant $c > 0$. Is this a better bound than the covering number bound found in Corollary 3.2.7 for any values of ϵ?

34. Use Sudakov's Minorization for Gaussian Widths (Theorem 5.2.6 in the notes (`https://math.msu.edu/~iwenmark/Notes_Fall2020_Iwen_Classes.pdf`) to upper bound the ϵ-covering number of $B^N_{\ell^1} := \left\{\mathbf{x} \in \mathbb{R}^N \mid \|\mathbf{x}\|_1 \leq 1\right\}$.

35. Let $T \subset \mathbb{R}^N$ be finite. Prove that $\omega(T) \geq \frac{0.265}{\sqrt{2}} \left(\min_{\mathbf{x}\neq\mathbf{y}\in T} \|\mathbf{x}-\mathbf{y}\|_2\right) \sqrt{\ln(|T|)}$.

36. Do homework exercise 5.2.3 on page 193 of the notes (`https://math.msu.edu/~iwenmark/Notes_Fall2020_Iwen_Classes.pdf`).

37. Let $T \subset \mathbb{R}^N$ be bounded, and assume that $\omega(T_0) \leq C$ holds for all finite $T_0 \subseteq T$. Prove that $\omega(T) \leq C$ then also holds.

38. Let $f : \mathbb{R} \to \mathbb{R}^+$ be a nondecreasing function such that $\omega(T) \leq \int_0^{\sup\{\|\mathbf{x}\|_2 \ \mid\ \mathbf{x}\in T\}} f\left(C_\epsilon(T)\right)\, d\epsilon$ holds for all bounded $T \subset \mathbb{R}^N$, where $C_\epsilon(T)$ denotes the ϵ-covering number of T. Prove that
$$\mathbb{E}\left[\sup_{\mathbf{x}\in T} |\langle \mathbf{g}, \mathbf{x}\rangle|\right] \leq \int_0^{\sup\{\|\mathbf{x}\|_2 \ \mid\ \mathbf{x}\in T\}} f\left(2C_\epsilon(T)\right)\, d\epsilon$$
then also holds for all bounded $T \subset \mathbb{R}^N$.

39. Let $A \in \mathbb{R}^{m\times N}$, and $T \subset \mathbb{R}^N$ be finite with $10 \geq \mathrm{diam}(T) \geq \min_{\mathbf{x}\neq\mathbf{y}\in T} \|\mathbf{x}-\mathbf{y}\|_2 \geq 0.1$. Let $\mathrm{conv}(T) \subset \mathbb{R}^N$ denote the convex hull of T. Suppose you want to solve $A\mathbf{x} = \mathbf{b}$ subject to the constraint that $\mathbf{b} \in A\left(\mathrm{conv}(T)\right) \subset \mathbb{R}^m$. Show that you must have $m = N$ if $\mathrm{conv}(T)$ has interior (which is "commonly the case" as soon as $|T| \geq N+1$). However, prove that there exist $A \in \mathbb{R}^{m\times N}$ with $m \leq C' \log(|T|)/\epsilon^2$ for some constant $C' > 0$ such that $\mathbf{y} := \arg\min_{\mathbf{x}\in\mathrm{conv}(T)} \|A\mathbf{x}-\mathbf{b}\|_2$ always satisfies $\|\mathbf{y}-\mathbf{x}'\|_2 \leq \epsilon$ whenever $\mathbf{b} = A\mathbf{x}'$ for some $\mathbf{x}' \in \mathrm{conv}(T)$.

40. Let $A \in \mathbb{R}^{m\times N}$, and $T \subset \mathbb{R}^N$ be the union of M d-dimensional affine subspaces. Suppose you want to stably solve $A\mathbf{x} = \mathbf{b}$ subject to the constraint $\mathbf{b} \in A(T)$. Show that you must have $m \geq C(d + \log M)$ for some constant $C > 0$. Furthermore, prove that there exist $A \in \mathbb{R}^{m\times N}$ with $m \leq C'(d + \log M)$ for some constant $C' > 0$ such that $A\mathbf{x} = \mathbf{b}$ subject to the constraint $\mathbf{b} \in A(T)$ is always solvable.

41. Denote the set of all s-sparse vectors in $\mathbb{R}^N$ by $\Sigma_s := \left\{\mathbf{x} \in \mathbb{R}^N \mid \mathbf{x} \text{ only has } s \text{ nonzero entries}\right\}$. Suppose that $A \in \mathbb{R}^{m\times N}$ is an ϵ-JL map of Σ_s into $\mathbb{R}^m$ (recall Definition 3.1.1 from the notes `https://math.msu.edu/~iwenmark/Notes_Fall2020_Iwen_Classes.pdf`). Prove that every $m \times s$ submatrix of A consisting of just s-columns of A is full rank, and that all its top s singular values always belong to the interval $[1-\epsilon, 1+\epsilon]$.