3.1 Signed Measures

Def Let (X, M) be a measurable space. A _signed measure_ on (X, M) is a function $\nu: M \to [-\infty, \infty]$ such that

① $\nu(\phi) = 0$

② ν assumes at most one of the values $\pm\infty$

③ If $(E_n)_{n \in \mathbb{N}} \subset M$ is a disjoint collection, then
$$\nu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \nu(E_n)$$

where the series converges absolutely if the quantity on the left is finite. □

Every measure is a signed measure, and for emphasis we may call a measure a _positive measure_.

EX ① Let μ_1, μ_2 be measures on common measurable space, at least one which is finite. Then $\nu(E) := \mu_1(E) - \mu_2(E)$ is a signed measure. For $(E_n)_{n \in \mathbb{N}} \subset M$ disjoint, if $\nu(\bigcup E_n)$ is finite, then necessarily $\mu_j(\bigcup E_n) < \infty$ for $j = 1, 2$. Consequently
$$\sum_{n=1}^{\infty} |\nu(E_n)| \leq \sum_{n=1}^{\infty} \mu_1(E_n) + \mu_2(E_n) = \mu_1\left(\bigcup_{n=1}^{\infty} E_n\right) + \mu_2\left(\bigcup_{n=1}^{\infty} E_n\right) < \infty$$

so the series $\sum_{n=1}^{\infty} \nu(E_n)$ converges absolutely.

② Let (X, M, μ) be a measure space and let $f: X \to \bar{\mathbb{R}}$ be an M-measurable function such that at least one of $\int f^+ d\mu$ or $\int f^- d\mu < \infty$ is finite. Then $\nu(E) := \int_E f \, d\mu$ is a signed measure. In fact, $\nu = \mu_+ - \mu_-$ where $\mu_+(E) = \int_E f^+ d\mu$ and $\mu_-(E) = \int_E f^- d\mu$ are measures by Exercise 3 on Homework 5. □

We will see that these are really the _only_ examples of signed measures.

The exact same proofs that gave continuity from above and below for positive measures (Theorem 1.8) also give the following.

Proposition 3.1 Let ν be a signed measure on (X, M). If $E_1 \subseteq E_2 \subseteq \cdots$ with $E_n \in M$, $n \in \mathbb{N}$ then
$$\nu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n \to \infty} \nu(E_n)$$
If $E_1 \supseteq E_2 \supseteq \cdots$ with $E_n \in M$, $n \in \mathbb{N}$, and if $\nu(E_1)$ is finite then
$$\nu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n \to \infty} \nu(E_n)$$

Def Let ν be a signed measure on (X, M). We call $E \in M$ _positive_ (resp. _negative_) for ν if $\nu(F) \geq 0$ (resp. $\nu(F) \leq 0$) for all $F \subseteq E$ with $F \in M$. We call E a ν-_null_ set

if $\nu(F) = 0$ for all $F \subseteq E$ with $F \in \mathcal{M}$.

Remark If ν happens to be a positive measure, then a set $E \in \mathcal{M}$ is ν-null in the above sense iff $\nu(E) = 0$, since monotonicity holds for positive measures.

EX Let $\nu(E) = \int_E f \, d\mu$ be as in Example 2 above. Then $E \in \mathcal{M}$ is
- positive for ν iff $f \geq 0$ μ-a.e. on E
- negative for ν iff $f \leq 0$ μ-a.e. on E
- ν-null iff $f = 0$ μ-a.e. on E.

Lemma 3.2 Let ν be a signed measure on $(X, \mathcal{M})$.
① If $E \in \mathcal{M}$ is positive for ν, then every $F \subseteq E$ with $F \in \mathcal{M}$ is positive for ν.
② If $E_n \in \mathcal{M}$ is positive for ν for each $n \in \mathbb{N}$, then $\bigcup_{n=1}^{\infty} E_n$ is positive for ν.

Proof ①: This follows from the definition of being positive for ν.
②: For each $n \in \mathbb{N}$, let $F_n := E_n \setminus (E_1 \cup \cdots \cup E_n)$. Then $F_n \subseteq E_n$ is positive for ν by ①, and $(F_n)_{n \in \mathbb{N}}$ is a disjoint collection. Now if
$$F \subseteq \bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} F_n,$$
then
$$\nu(F) = \nu\left(\bigcup_{n=1}^{\infty} F \cap F_n\right) = \sum_{n=1}^{\infty} \nu(F \cap F_n) \geq 0$$
Hence $\bigcup_{n=1}^{\infty} E_n$ is positive for ν.

Theorem 3.3 (The Hahn Decomposition Theorem) If ν is a signed measure on $(X, \mathcal{M})$, then there exists a partition $X = P \cup N$ so that P is positive for ν and N is negative for ν. If $X = P' \cup N'$ is another such partition, then $P \triangle P' (= N \triangle N')$ is ν-null.

Proof Without loss of generality, we may assume ν does not take the value $+\infty$ (otherwise work with $-\nu$). Let
$$R := \sup \{\nu(E) : E \in \mathcal{M} \text{ is positive for } \nu\}.$$
Then we can find a sequence $(E_n)_{n \in \mathbb{N}} \subseteq \mathcal{M}$ of positive sets satisfying $\nu(E_n) \to R$. Set $P := \bigcup_{n=1}^{\infty} E_n$, which is positive for ν by Lemma 3.2.②, and $\nu(P) = R$ by Lemma 3.1. Thus $R < \infty$.
We claim $N := X \setminus P$ is negative for ν. First note N cannot contain any non-ν-null positive sets: if $E \subseteq N$ is positive then $E \cup P$ is positive with $\nu(E \cup P) = \nu(E) + \nu(P) > R$, a contradiction. Consequently, if $A \subseteq N$ satisfies $\nu(A) > 0$, then there must exist $B \subseteq A$ with $\nu(B) < 0$. Note that
$$\nu(A \setminus B) = \nu(A) - \nu(B) > \nu(A)$$
Hence any $A \subseteq N$ with $\nu(A) > 0$ has a subset with strictly larger measure. Now, suppose towards a contradiction that N is not negative for ν. Then N has subsets of strictly positive measure. Let $n_1 \in \mathbb{N}$ be the smallest integer such that $\exists B \subseteq N$ with $\nu(B) > \frac{1}{n_1}$, and let $A_1 := B$. We inductively define $(n_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$ and $(A_k)_{k \in \mathbb{N}} \subseteq N$ as follows. Let $n_{k+1} \in \mathbb{N}$ be the smallest integer such that $\exists B \subseteq A_k$ with $\nu(B) > \nu(A_k) + \frac{1}{n_{k+1}}$ (which

exists by our above observation) and let $A_{k+1} := B$. Note that $\nu(A_{k+1}) = \nu(A_k) + \frac{1}{n_{k+1}} > \cdots > \sum_{j=1}^{k+1} \frac{1}{n_k}$.
Denote $A := \bigcap_{k=1}^{\infty} A_k$, then by Proposition 3.1 we have

$$\infty > \nu(A) = \lim_{k \to \infty} \nu(A_k) > \sum_{k=1}^{\infty} \frac{1}{n_k}$$

Thus we must have $n_k \to \infty$. Since A has a subset with strictly larger measure, $\exists B \subset A$ and $n \in \mathbb{N}$ with $\nu(B) > \nu(A) + \frac{1}{n}$. Let $k \in \mathbb{N}$ be large enough so that $n < n_k$ and

$$\nu(B) > \nu(A_{k-1}) + \frac{1}{n}.$$

But then $B \subset A \subset A_{k-1}$ and $n < n_k$ contradict the definition of n_k. Thus N must be negative.
Finally, let $X = P' \cup N'$ be another partition such that P' is positive for ν and N' is negative for ν. Then $P \setminus P' \subset P$ and $P \setminus P' = P \cap (P')^c = P \cap N' \subset N'$. Thus $P \setminus P'$ is both positive and negative for ν, and is therefore ν-null. Similarly for $P' \setminus P$. Hence $P \triangle P' = P \setminus P' \cup P' \setminus P$ is ν-null. $\square$

Def Let ν be a signed measure on $(X, \mathcal{M})$. A partition $X = P \cup N$ such that P is positive for ν and N is negative for ν is called a <u>Hahn decomposition</u> for ν $\square$

While a Hahn decomposition is only unique up to ν-null sets by Theorem 3.3, it does lead to a unique representation of ν as a difference of two positive measures (see Theorem 3.4).

Def Given two signed measures μ and ν on $(X, \mathcal{M})$, we say μ and ν are <u>mutually singular</u> if there exists a partition $X = E \cup F$ such that E is μ-null and F is ν-null. In this case, we write: $\mu \perp \nu$. $\square$

Theorem 3.4 (The Jordan Decomposition Theorem) For a signed measure ν on $(X, \mathcal{M})$, there exist unique positive measures ν^+ and ν^- on $(X, \mathcal{M})$ such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.
Proof Let $X = P \cup N$ be a Hahn decomposition for ν and define

$$\nu^+(E) := \nu(E \cap P)$$
$$\nu^-(E) := -\nu(E \cap N).$$

Then $\nu(E) = \nu^+(E) - \nu^-(E)$ since $E = (E \cap P) \cup (E \cap N)$ and the union is disjoint. Also, $\nu^+(N) = 0$ so N is ν^+-null since ν^+ is a positive measure. Similarly P is ν^--null, and so $\nu^+ \perp \nu^-$.
Now suppose μ^+ and μ^- are another pair of positive measures satisfying $\nu = \mu^+ - \mu^-$ and $\mu^+ \perp \mu^-$. The latter condition implies there is a partition $X = E \cup F$ such that E is μ^--null and F is μ^+-null. But then for all $A \subset E$ we have

$$\nu(A) = \nu^+(A) - \nu^-(A) = \nu^+(A) \geq 0,$$

So E is positive for ν. Similarly, F is negative for ν. Hence $X = E \cup F$ is a Hahn decomposition for ν. So $P \triangle E = N \triangle F$ is ν-null by the Hahn-decomposition theorem (Theorem 3.3). Consequently for any $A \in \mathcal{M}$ we have

$$\nu^+(A) = \nu(A \cap P) = \nu(A \cap E) = \mu^+(A \cap E) - \mu^-(A \cap E) = \mu^+(A).$$

Similarly $\nu^- = \mu^-$. $\square$

Def Let ν be a signed measure on $(X, \mathcal{M})$. The unique positive measures ν^+ and ν^- on $(X, \mathcal{M})$ satisfying $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$ are called the <u>positive</u> and <u>negative</u> <u>variations</u> of ν, respectively, and $\nu = \nu^+ - \nu^-$ is called the <u>Jordan decomposition</u> of ν. The positive measure

$$|\nu| := \nu^+ + \nu^-$$

on $(X, \mathcal{M})$ is called the <u>total variation</u> of ν. □

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EX Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f: X \to \mathbb{R}$ be $\mathcal{M}$-measurable such that at least one of $\int f^+ d\mu$ or $\int f^- d\mu$ is finite. Then for the signed measure

$$\nu(E) = \int_E f \, d\mu,$$

$X = \{x \in X : f(x) \geq 0\} \cup \{x \in X : f(x) < 0\}$ is a Hahn decomposition for ν. One can also move any measurable subset of $f^{-1}(\{0\})$ over. The positive and negative variations of ν are

$$\nu^{\pm}(E) = \int_E f^{\pm}(E) d\mu,$$

and

$$|\nu|(E) = \int_E |f| \, d\mu$$

is the total variation of ν.

Conversely, if ν is any signed measure on $(X, \mathcal{M})$ with a Hahn decomposition $X = P \cup N$, then for $E \in \mathcal{M}$

$$\int_E \mathbb{1}_P - \mathbb{1}_N \, d|\nu| = |\nu|(E \cap P) - |\nu|(E \cap N) = \nu^+(E) - \nu^-(E) = \nu(E)$$

Thus $\nu = \int_E f \, d\mu$ where $f = \mathbb{1}_P - \mathbb{1}_N$ and $\mu = |\nu|$. □

Note that ν^+ is finite iff ν omits the value ∞, ν^- is finite iff ν omits the value $-\infty$, and $|\nu|$ is finite iff ν omits both $\pm\infty$.

Def Let ν be a signed measure on $(X, \mathcal{M})$. We say ν is finite (resp. σ-finite) if $|\nu|$ is finite (resp. σ-finite). We define

$$L^1(X, \nu) := L^1(X, \nu^+) \cap L^1(X, \nu^-)$$

and for $f \in L^1(X, \nu)$ we define

$$\int_X f \, d\nu := \int_X f \, d\nu^+ - \int_X f \, d\nu^-$$

□

EX Let $(X, \mathcal{M}, \mu)$ be a measure space. For an $\mathcal{M}$-measurable f, $\nu(E) = \int_E f \, d\mu$ is finite iff

$$|\nu|(E) = \int_E |f| \, d\mu$$

is finite iff $f \in L^1(X, \mu)$. For $g \in L^1(X, \nu)$, Exercise 3 on Homework 5 implies

$$\int_X g \, d\nu = \int_X g \, d\nu^+ - \int g \, d\nu^- = \int_X g f^+ d\mu - \int_X g f^- d\mu = \int_X g f \, d\mu.$$

□

3.2 The Lebesgue–Radon–Nikodym Theorem

Def Given two signed measures μ and ν on $(X, \mathcal{M})$, we say ν is <u>absolutely continuous</u> with respect to μ if whenever $E \in \mathcal{M}$ is μ-null it is also ν-null. In this case we write: $\nu \ll \mu$. □

Using the (positive, negative, total) varieties of ν and μ one can define absolute continuity in many equivalent ways:

$$\nu \ll \mu \quad \overset{②}{\Longleftrightarrow} \quad \nu \ll |\mu| \quad \overset{③}{\underset{\cancel{\Longrightarrow}}{\Longleftarrow}} \quad \nu \ll \mu^+ \text{ and } \nu \ll \mu^-$$

$$① \Updownarrow \qquad\qquad \Updownarrow$$

$$\nu^+ \ll \mu \text{ and } \nu^- \ll \mu \quad \Longleftrightarrow \quad |\nu| \ll \mu \quad \Longleftrightarrow \quad |\nu| \ll |\mu|$$

We leave most of these as exercises, but prove a few here:

①: Suppose $\nu \ll \mu$. Let $X = P \cup N$ be a Hahn decomposition for ν. If $E \in \mathcal{M}$ is μ-null, then so are $E \cap P$ and $E \cap N$. Hence
$$|\nu|(E) = \nu^+(E) + \nu^-(E) = \nu(E \cap P) - \nu(E \cap N) = 0 - 0 = 0$$
So E is $|\nu|$-null and $|\nu| \ll \mu$.
Conversely, if $|\nu| \ll \mu$ and $E \in \mathcal{M}$ is μ-null, then $|\nu|(F) = 0$ for any $F \subseteq E$ with $F \in \mathcal{M}$. Consequently, $\nu^\pm(F) = 0$ and so $\nu(F) = 0$. Thus E is ν-null and $\nu \ll \mu$. □

②: If $\nu \ll \mu$ and $E \in \mathcal{M}$ is $|\mu|$-null, then the above argument shows that E is then μ-null. Hence E is ν-null and $\nu \ll |\mu|$.
Conversely, suppose $\nu \ll |\mu|$. Let $X = P \cup N$ be a Hahn decomposition for μ. If E is μ-null, then so are $E \cap P$ and $E \cap N$. Consequently
$$|\mu|(E) = \mu^+(E) + \mu^-(E) = \mu(E \cap P) - \mu(E \cap N) = 0$$
So E is $|\mu|$-null and therefore ν-null. Thus $\nu \ll \mu$. □

③: One has $|\mu| \ll |\mu|$ by definition, but $|\mu| \ll \mu^+$ fails if μ^- is non-trivial non-trivial. Indeed, let $X = P \cup N$ be a Hahn decomposition for μ. Since μ^- is non-trivial, $\mu^-(N) > 0$ and hence $|\mu|(N) \geq \mu^-(N) > 0$. However, $\mu^+(N) = 0$. So $|\mu| \not\ll \mu^+$. Similarly, $|\mu| \not\ll \mu^-$ if μ^+ is non-trivial. □

In light of the above equivalences (namely $\nu \ll \mu \Leftrightarrow \nu \ll |\mu|$), we will usually assume μ is positive.

Another important observation is that $\nu \ll \mu$ and $\nu \perp \mu$ imply $\nu \equiv 0$. Indeed, let $X = E \cup F$ be a decomposition such that E is ν-null and F is μ-null. Then for any $A \in \mathcal{M}$,
$$\nu(A) = \nu(A \cap E) + \nu(A \cap F).$$

The former term is zero since E is ν-null, and the latter is zero since $\mu(A \cap F) = 0$.

Theorem 3.5 Let ν be a finite signed measure and μ a positive measure on $(X, \mathcal{M})$. Then $\nu \ll \mu$ if and only if $\forall \varepsilon > 0$ there exists $\delta > 0$ such that $|\nu(E)| < \varepsilon$ whenever $\mu(E) < \delta$.

Proof As observed above, $\nu \ll \mu$ iff $|\nu| \ll \mu$. Since $|\nu(E)| \leq |\nu|(E)$ for all $E \in \mathcal{M}$, it suffices to prove the theorem for $|\nu|$, and so we may just assume ν is a positive measure.

$\Longrightarrow$) We proceed by contrapositive and assume the $\varepsilon - \delta$ condition fails; that is, there exists $\varepsilon > 0$ such that for any $\delta > 0$ there exists $E \in \mathcal{M}$ satisfying $\mu(E) < \delta$ but $\nu(E) \geq \varepsilon$. In particular, for each $n \in \mathbb{N}$ we can find $E_n \in \mathcal{M}$ with $\mu(E_n) < 2^{-n}$ and $\nu(E_n) \geq \varepsilon$. Define

$$F_N := \bigcup_{n=N}^{\infty} E_n$$

so that $\mu(F_N) < 2^{-N+1}$ and $F_N \supset F_{N+1}$. Letting $F := \bigcap_{N=1}^{\infty} F_N$, we therefore have $\mu(F) = 0$ by continuity from above. On the other hand

$$\nu(F_N) \geq \nu(E_N) \geq \varepsilon$$

for each $N \in \mathbb{N}$. Since ν is finite, continuity from above implies $\nu(F) \geq \varepsilon$. Thus ν is not absolutely continuous with respect to μ.

$\Longleftarrow$): Assume the $\varepsilon - \delta$ condition holds. For each $\varepsilon = \frac{1}{n}$, let $\delta_n > 0$ be the corresponding quantity. If $\mu(E) = 0$, then $\mu(E) < \delta_n$ and so $\nu(E) < \frac{1}{n}$ for all $n \in \mathbb{N}$. Hence $\nu(E) = 0$. $\square$

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Def Let $(X, \mathcal{M}, \mu)$ be a measure space. We say an $\mathcal{M}$-measurable function $f : X \to \overline{\mathbb{R}}$ is an extended μ-integrable function if at least one of $\int f^+ d\mu$ or $\int f^- d\mu$ is finite. We write $d\nu = f d\mu$ to mean ν is the signed measure defined by $\nu(E) = \int_E f d\mu$. $\square$

If $d\nu = f d\mu$, then $\nu \ll \mu$ since if $\mu(E) = 0$ then $\mathbb{1}_E f = 0$ μ-a.e. and so

$$\nu(E) = \int_E f d\mu = \int_X \mathbb{1}_E f d\mu = 0$$

by Proposition 2.16. Also recall that ν is finite iff $f \in L^1(X, \mu)$. Thus the previous Theorem yields the following Corollary:

Corollary 3.6 Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f \in L^1(X, \mu)$. For every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left| \int_E f d\mu \right| < \varepsilon$$

whenever $\mu(E) < \delta$.

Remark Exercise 1 on Homework 6 asked you to show $t \mapsto \int_{-\infty}^{t} f d\mu$ was continuous for $f \in L^1(\mathbb{R}, m)$. Corollary 3.6 offers a very easy proof of this. $\square$

Lemma 3.7 Let ν and μ be finite measures on (X, M). Then either $\nu \perp \mu$ or there exists $\varepsilon > 0$ and $E \in M$ such that $\mu(E) > 0$ and E is a positive set for $\nu - \varepsilon \mu$.

Proof For each $n \in \mathbb{N}$, let $X = P_n \cup N_n$ be a Hahn decomposition for $\nu - \frac{1}{n} \mu$. Define

$$P := \bigcup_{n=1}^{\infty} P_n \qquad \text{and} \qquad N := \bigcap_{n=1}^{\infty} N_n = P^c$$

Then N is a negative set for each $\nu - \frac{1}{n} \mu$. In particular, $\nu(N) \leq \frac{1}{n} \mu(N)$ all $n \in \mathbb{N}$, and hence $\nu(N) = 0$. If $\mu(P) = 0$, then $\nu \perp \mu$. Otherwise $\mu(P) > 0$ and thus $\mu(P_n) > 0$ for some sufficiently large $n \in \mathbb{N}$. Take $E := P_n$. $\square$

Lemma 3.8 Let (X, M, μ) be a measure space, and let $(\nu_n)_{n \in \mathbb{N}}$ be signed measures on (X, M) such that $\nu := \sum_{n=1}^{\infty} \nu_n$ defines a signed measure.

① If $\nu_n \perp \mu$ for all $n \in \mathbb{N}$, then $\nu \perp \mu$

② If $\nu_n \ll \mu$ for all $n \in \mathbb{N}$, then $\nu \ll \mu$.

Proof ①: For each $n \in \mathbb{N}$, let $X = E_n \cup F_n$ be a partition such that E_n is ν_n-null and F_n is μ-null. Then $E := \bigcap_{n=1}^{\infty} E_n$ is ν_n-null for all $n \in \mathbb{N}$ and hence ν-null. Also $F := \bigcup_{n=1}^{\infty} F_n$ is μ-null by countable subadditivity. Thus $\nu \perp \mu$.

②: If $E \in M$ is μ-null, then for any measurable $F \subseteq E$, $\nu(F) = \sum_{n=1}^{\infty} \nu_n(F) = 0$. So $\nu \ll \mu$. $\square$

Theorem 3.9 (The Lebesgue-Radon-Nikodym Theorem) Let (X, M, μ) be a σ-finite measure space. For σ-finite signed measure ν on (X, M), there exists unique σ-finite signed measures λ, ρ on (X, M) satisfying

$$\nu = \lambda + \rho$$

where $\lambda \perp \mu$ and $\rho \ll \mu$. Moreover, there exists an extended μ-integrable function $f : X \to \mathbb{R}$ such that $d\rho = f \, d\mu$, and f is unique up to μ-almost everywhere equivalence.

Proof First suppose ν and μ are finite positive measures. Consider:

$$\mathcal{F} := \left\{ f : X \to [0, \infty] : \int_E f \, d\mu \leq \nu(E) \quad \forall E \in M \right\}$$

Then $\mathcal{F} \neq \phi$ since $0 \in \mathcal{F}$. Also if $f, g \in \mathcal{F}$, then $\max(f, g) \in \mathcal{F}$: if $A := \{x \in X : f(x) > g(x)\}$ then for $E \in M$

$$\int_E \max(f, g) \, d\mu = \int_{E \cap A} f \, d\mu + \int_{E \setminus A} g \, d\mu \leq \nu(E \cap A) + \nu(E \setminus A) = \nu(E).$$

Let $a := \sup \{ \int f \, d\mu : f \in \mathcal{F} \}$, and let $(f_n)_{n \in \mathbb{N}} \subseteq \mathcal{F}$ be such that $\int f_n \, d\mu \to a$. By replacing f_n with $\max(f_1, \ldots, f_n)$ we may — by our above observation — assume $f_n \leq f_{n+1}$. Then the monotone convergence theorem implies $f := \sup f_n \in \mathcal{F}$ with $\int f \, d\mu = a$. Since $a \leq \nu(X) < \infty$, we know f is finite μ-a.e. by Proposition 2.20, and so considering $f \cdot 1_{\{x : f(x) < \infty\}}$ if necessary, we may assume f is finitely valued.

Now, the measure $d\lambda := d\nu - f \, d\mu$ is positive since $f \in \mathcal{F}$. If $\lambda \not\perp \mu$, then using Lemma 3.7 we can find $\varepsilon > 0$ and $E \in M$ so that $\mu(E) > 0$ and E is positive for $\lambda - \varepsilon \mu$.

Consider $g := f_1 + \varepsilon \mathbb{1}_E$. We have for $F \in M$

$$\int_F g \, d\mu = \int_F f \, d\mu + \varepsilon \int_F \mathbb{1}_E \, d\mu$$
$$= \int_F f \, d\mu + \varepsilon \mu(E \cap F)$$
$$\leq \int_F f \, d\mu + \lambda(E \cap F)$$
$$= \int_F f \, d\mu + \lambda(F) = \nu(F)$$

Hence $g \in \mathcal{F}$, but

$$\int_X g \, d\mu = \int_X f \, d\mu + \varepsilon \int_X \mathbb{1}_E \, d\mu = a + \varepsilon \mu(E) > a,$$

contradicting the definition of a. Thus we must have $\lambda \perp \mu$. Letting $d\rho := f \, d\mu$, 11/6
we have shown existence of the claimed measures. Towards showing uniqueness, suppose
$\nu = \lambda' + \rho'$ for $\lambda' \perp \mu$ and $\rho' \ll \mu$ with $d\rho' = f' \, d\mu$. Then $\lambda + \rho = \nu = \lambda' + \rho'$ implies $\lambda - \lambda' = \rho' - \rho$,
and by Lemma 3.8 we have $(\lambda - \lambda') \perp \mu$ and $(\rho' - \rho) \ll \mu$. Hence $\lambda - \lambda' = \rho - \rho' = 0$ by our
observation preceding Theorem 3.5. In particular, $\int_E f \, d\mu = \int_E f' \, d\mu$ for all $E \in M$, and so
$f = f'$ μ a.e. by Proposition 2.23.

Next, assume ν and μ are σ-finite positive measures. By taking intersections of the respective finite-
measure decompositions of X for ν and μ, we have $X = \bigcup_{n=1}^{\infty} X_n$ where $\nu(X_n), \mu(X_n) < \infty$ for each $n \in \mathbb{N}$.
Define $\nu_n(E) := \nu(E \cap X_n)$ and $\mu_n(E) := \mu(E \cap X_n)$. The first part of the proof yields $\lambda_n \perp \mu_n$ and
$d\rho_n = f_n \, d\mu_n$ such that $\nu_n = \lambda_n + \rho_n$. Since $\nu_n(X_n^c) = \mu_n(X_n^c) = 0$, we have

$$\lambda_n(X_n^c) = \nu_n(X_n^c) - \rho_n(X_n^c) = 0,$$

and replacing f_n with $f_n \mathbb{1}_{X_n}$ if necessary, we may assume $f_n = 0$ on X_n^c. Define

$$\lambda := \sum_{n=1}^{\infty} \lambda_n, \quad \rho := \sum_{n=1}^{\infty} \rho_n, \quad \text{and} \quad f := \sum_{n=1}^{\infty} f_n.$$

Then $\lambda \perp \mu$ and $\rho \ll \mu$ by Lemma 3.8. These measures are also σ-finite since $\lambda(X_n) = \lambda_n(X_n)$ and
$\rho(X_n) = \rho_n(X_n)$ are finite. Also, $f \geq 0$ and so is automatically an extended μ-integrable function,
and the MCT (Theorem 2.14) implies $d\rho = f \, d\mu$. Uniqueness follows as in the first part of the proof.

Finally, for ν a σ-finite signed measure, apply the previous part to $\nu^{\pm}$ and take the differences
of the resulting measures. □

Remark Note that in the above proof, if ν was positive then $f \geq 0$. □

Def Let (X, M, μ) be a σ-finite measure space and let ν be a σ-finite signed
measure on (X, M). The **Lebesgue decomposition** of ν with respect to μ is
$$\nu = \lambda + \rho$$
where $\lambda \perp \mu$ and $\rho \ll \mu$. If $\nu \ll \mu$ so that $d\nu = f \, d\mu$ for some extended μ-integrable function, then
we call f the **Radon–Nikodym derivative** of ν with respect to μ and denote it $\frac{d\nu}{d\mu} := f$. □

① Let μ be a σ-finite signed measure on (X,M). Then $\mu \ll |\mu|$. If $X = P \cup N$ is a Hahn decomposition for μ, then $\frac{d\mu}{d|\mu|} = 1_P - 1_N$:

$$\int_E \frac{d\mu}{d|\mu|} d|\mu| = \int_E 1_P - 1_N \, d|\mu| = |\mu|(E \cap P) - |\mu|(E \cap N) = \mu^+(E) - \mu^-(E) = \mu(E)$$

Similarly $\frac{d\mu^+}{d|\mu|} = 1_P$ and $\frac{d\mu^-}{d|\mu|} = 1_N$. Note that

$$\frac{d(\mu^+ - \mu^-)}{d|\mu|} = \frac{d\mu}{d|\mu|} = 1_P - 1_N = \frac{d\mu^+}{d|\mu|} - \frac{d\mu^-}{d|\mu|}$$

If ν is a σ-finite signed measure on (X,M) with $\nu \ll \mu$, then $\nu \ll |\mu|$ and so we can consider $\frac{d\nu}{d|\mu|}$. If $\frac{d\nu}{d|\mu|} \in L^1(X,\mu)$ then $\frac{d\nu}{d|\mu|}(1_P - 1_N) \in L^1(X,\mu)$ with

$$\int_E \frac{d\nu}{d|\mu|}(1_P - 1_N) \, d\mu = \int_E \frac{d\nu}{d|\mu|} d|\mu| = \nu(E)$$

Thus we can define $\frac{d\nu}{d\mu} := \frac{d\nu}{d|\mu|}(1_P - 1_N)$.

② Let (X,M,μ) be a finite measure space, let $N \subset M$ be a σ-subalgebra, and define $\nu := \mu|_N$. For real-valued $f \in L^1(X,\mu)$, define a finite signed measure λ on (X,N) by:

$$\lambda(E) = \int_E f \, d\mu \qquad E \in N.$$

Then $\lambda \ll \nu$ since if $0 = \nu(E) = \mu(E)$, then for any $F \subset E$ with $F \in N$, $1_F f = 0$ μ-almost everywhere and so $\lambda(F) = 0$ by Proposition 2.16. Then $E(f|N) := \frac{d\lambda}{d\nu}$ is an N-measurable function called the _conditional expectation_ of f, and by Theorem 3.9 it is uniquely determined up to ν-almost everywhere equivalence by:

$$\int_E f \, d\mu = \int_E E(f|N) \, d\nu \qquad E \in N.$$

If $N = \{\phi, X\}$, then $E(f|N) = \int_X f \, d\mu$ ($= E(f)$ in probability notation).
If $N = M$, then $E(f|M) = f$.
Consider $(X \times Y, M \otimes N, \mu \times \nu)$ for finite measures μ, ν. Then $M \otimes \{\phi, Y\} \subset M \otimes N$ is a σ-subalgebra and for $f \in L^1(X \times Y, \mu \times \nu)$

$$E(f \mid M \otimes \{\phi, Y\}) = \int_Y f(\cdot, y) \, d\nu(y)$$

Indeed $E \in M \otimes \{\phi, Y\}$ is either empty or of the form $E = F \otimes Y$ for $F \in M$. Thus $1_E(x,y) = 1_F(x)$ and Fubini's theorem implies:

$$\int_E f \, d(\mu \times \nu) = \int_X \int_Y 1_E f \, d\nu \, d\mu = \int_X 1_F(x) \int_Y f(x,y) \, d\nu(y) \, d\mu(x) = \int_F \left[\int_Y f(x,y) d\nu(y) \right] d\mu(x).$$

Similarly $E(f \mid \{\phi, X\} \otimes N) = \int_X f(x, \cdot) \, d\mu(x)$.

③ On $(\mathbb{R}, B_{\mathbb{R}})$ we have $\delta_0 \perp m$. In particular we do _not_ have $\delta_0 \ll m$. However the Dirac delta function "$\frac{d\delta_0}{dm}$" is often used heuristically in physics as a function satisfying

$$\int_{\mathbb{R}} g(t) \frac{d\delta_0}{dm}(t) \, dt \text{ "}=\text{ "} g(0)$$

$\frac{d\delta_0}{dm}$

AS example ① (and the notation itself suggests) the Radon-Nikodym derivative shares many algebraic properties with the usual derivative:

$$\frac{d(\nu_1 + \nu_2)}{d\mu} = \frac{d\nu_1}{d\mu} + \frac{d\nu_2}{d\mu}.$$

We even have a chain-rule:

Theorem 3.10 On $(X, \mathcal{M})$, let ν be a σ-finite signed measure and let μ, λ be σ-finite measures. Suppose $\nu \ll \mu$ and $\mu \ll \lambda$.

① $\nu \ll \lambda$ with
$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \frac{d\mu}{d\lambda} \qquad \lambda\text{-a.e.}$$

② For $g \in L^1(X, \nu)$, $g \frac{d\nu}{d\mu} \in L^1(X, \mu)$ with
$$\int_X g \frac{d\nu}{d\mu} d\mu = \int_\lambda g \, d\nu.$$

Proof ①: If $E \in \mathcal{M}$ is λ-null, then it is μ-null by $\mu \ll \lambda$, and hence it is ν-null by $\nu \ll \mu$. Thus $\nu \ll \lambda$. Now, by Exercise 3 on Homework 5 we have

$$\int_E \frac{d\nu^\pm}{d\mu} \frac{d\mu}{d\lambda} d\lambda = \int_E \frac{d\nu^\pm}{d\mu} d\mu = \nu^\pm(E) = \int_E \frac{d\nu^\pm}{d\lambda} d\lambda.$$

Thus $\frac{d\nu^\pm}{d\mu} \cdot \frac{d\mu}{d\lambda} = \frac{d\nu^\pm}{d\lambda}$ λ-a.e. by Proposition 2.23. Using $\nu = \nu^+ - \nu^-$ and our observation preceding the statement of the theorem gives

$$\frac{d\nu}{d\lambda} = \frac{d\nu^+}{d\lambda} - \frac{d\nu^-}{d\lambda} = \frac{d\nu^+}{d\mu}\frac{d\mu}{d\lambda} - \frac{d\nu^-}{d\mu}\frac{d\mu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}$$

②: For $g \in L^1(X, \nu^\pm)$ we have
$$\int_X g \frac{d\nu^\pm}{d\mu} d\mu = \int_X g \, d\nu^\pm$$

by Exercise 3 on Homework 5. This then extends to $g \in L^1(X, \nu) = L^1(X, \nu^+) \cap L^1(X, \nu^-)$ by taking the appropriate complex combination of integrals of $\text{Re}(g)^\pm$, $\text{Im}(g)^\pm$. Finally, using $d\nu/d\mu = d\nu^+/d\mu - d\nu^-/d\mu$ we have

$$\int_X g \frac{d\nu}{d\mu} d\mu = \int_X g \frac{d\nu^+}{d\mu} d\mu - \int_X g \frac{d\nu^-}{d\mu} d\mu = \int_X g \, d\nu^+ - \int_X g \, d\nu^- = \int_X g \, d\nu$$

for $g \in L^1(X, \nu)$. □

Def For signed measures μ and ν on $(X, \mathcal{M})$, we say μ and ν are <u>equivalent</u> if $\mu \ll \nu$ and $\nu \ll \mu$. In this case we write $\mu \sim \nu$. □

For $\mu \sim \nu$, $E \in \mathcal{M}$ is μ-null iff it is ν-null. Thus μ-almost everywhere has the same meaning as ν-almost everywhere. We have already seen $\nu \sim |\nu|$.

Corollary 3.11 For σ-finite measures μ and ν on $(X, \mathcal{M})$, if $\mu \sim \nu$ then $\frac{d\mu}{d\nu} \frac{d\nu}{d\mu} = 1$ μ-a.e. (equivalently ν-a.e.)

Proof Since $\frac{d\mu}{d\mu} = 1$ μ-almost everywhere, this is immediate from Theorem 3.10. ① $\square$

Proposition 3.12 For measures $\mu_1, \dots, \mu_n$ on $(X, \mathcal{M})$, $\mu := \sum_{j=1}^{n} \mu_j$ satisfies $\mu_j \ll \mu$ for $j = 1, \dots, n$.

Proof This follows from $0 \le \mu_j(E) \le \mu(E)$ for all $E \in \mathcal{M}$. $\square$

3.3 Complex Measures

Def A _complex measure_ on a measurable space (X, M) is a map $\nu : M \to \mathbb{C}$ such that

① $\nu(\phi) = 0$

② If $(E_n)_{n \in \mathbb{N}} \subset M$ are disjoint then
$$\nu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \nu(E_n)$$

where the series converges absolutely

The signed measures defined by $\nu_r(E) := \text{Re}(\nu(E))$ and $\nu_i(E) := \text{Im}(\nu(E))$ are called the _real_ and _imaginary parts_ of ν, respectively. We define
$$L^1(X, \nu) := L^1(X, \nu_r) \cap L^1(X, \nu_i)$$
and for $f \in L^1(X, \nu)$ we define its integral with respect to ν as
$$\int_X f \, d\nu := \int_X f \, d\nu_r + i \int_X f \, d\nu_i$$

Note that a complex measure does _not_ take any infinite values. Hence ν_r and ν_i are also finite and $\nu(M)$ is a _bounded_ subset of $\mathbb{C}$ (Exercise check this).

Ex Let (X, M, μ) be a measure space. For any $f \in L^1(X, \mu)$,
$$\nu(E) := \int_E f \, d\mu$$

defines a complex measure since for $(E_n)_{n \in \mathbb{N}} \subset M$ the monotone convergence theorem implies
$$\sum_{n=1}^{\infty} |\nu(E_n)| \leq \sum_{n=1}^{\infty} \int_{E_n} |f| \, d\mu = \int_{\cup E_n} |f| \, d\mu < \infty$$

and so the dominated convergence implies
$$\sum_{n=1}^{\infty} \nu(E_n) = \sum_{n=1}^{\infty} \int_{E_n} f \, d\mu = \int_{\cup E_n} f \, d\mu = \nu\left(\bigcup_{n=1}^{\infty} E_n\right).$$

The real and imaginary parts are:
$$\nu_r(E) = \int_E \text{Re}(f) \, d\mu \qquad \text{and} \qquad \nu_i(E) = \int_E \text{Im}(f) \, d\mu.$$

Thus $\nu_r, \nu_i \ll \mu$ with $\frac{d\nu_r}{d\mu} = \text{Re}(f)$ and $\frac{d\nu_i}{d\mu} = \text{Im}(f)$, so by Theorem 3.10.② we have for $g \in L^1(X, \nu)$
$$\int_X g \, d\nu = \int_X g \, d\nu_r + i \int_X g \, d\nu_i = \int_X g \, \text{Re}(f) \, d\mu + i \int_X g \, \text{Im}(f) \, d\mu = \int_X g f \, d\mu.$$

We write $d\nu = f \, d\mu$ for the measure defined in the previous example. Note that $\mu(E) = 0$ implies $\nu(E) = 0$.

Def For a complex measure ν on (X, M), we say $E \in M$ is a ν-_null_ set if $\nu(F) = 0$ for all $F \subset E$ with $F \in M$. If μ is a complex (or signed) measure on (X, M), we say ν and

μ are mutually singular and write $\nu \perp \mu$ if there exists a partition $X = E \cup F$ so that E is ν-null and F is μ-null. We say ν is absolutely continuous with respect to μ and write $\nu \ll \mu$ if every μ-null set is also a ν-null set. We say ν and μ are equivalent and write $\nu \sim \mu$ if $\nu \ll \mu$ and $\mu \ll \nu$.

For a complex measure ν on $(X, \mathcal{M})$, $E \in \mathcal{M}$ is ν-null iff E is ν_r-null and ν_i-null. Consequently

$$\nu \perp \mu \iff \nu_r \perp \mu \text{ and } \nu_i \perp \mu$$
$$\nu \ll \mu \iff \nu_r \ll \mu \text{ and } \nu_i \ll \mu$$

Theorem 3.13 (The (complex) Lebesgue-Radon-Nikodym Theorem) Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. For a complex measure ν on $(X, \mathcal{M})$, there exists unique complex measures λ, ρ on $(X, \mathcal{M})$ satisfying

$$\nu = \lambda + \rho$$

where $\lambda \perp \mu$ and $\rho \ll \mu$. Moreover, there exists $f \in L^1(X, \mu)$ such that $d\rho = f d\mu$, and f is unique up to μ-almost everywhere equivalence.

Proof This follows from applying Theorem 3.9 to ν_r and ν_i separately. That f is integrable follows from ρ (and hence ρ_r, ρ_i) being finite. □

Def Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. For ν a complex measure on $(X, \mathcal{M})$ satisfying $\nu \ll \mu$, the function f such that $d\nu = f d\mu$ is called the Radon-Nikodym derivative of ν with respect to μ, and is denoted $\frac{d\nu}{d\mu} := f$.

Our computation in the above example therefore gives:

Corollary 3.14 Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. If ν is a complex measure on $(X, \mathcal{M})$ satisfying $\nu \ll \mu$, then for $g \in L^1(X, \nu)$ one has $g \frac{d\nu}{d\mu} \in L^1(X, \mu)$ with

$$\int_X g \, d\nu = \int_X g \, \frac{d\nu}{d\mu} \, d\mu.$$

Given a complex measure ν on $(X, \mathcal{M})$, we wish to define its total variation $|\nu|$. In the signed measure case, this was easy to do because the positive and negative variations were mutually singular. But ν_r and ν_i are not mutually singular, and so it is not clear that the naive formula

$$|\nu|(E) \overset{?}{=} \left(|\nu_r|(E)^2 + |\nu_i|(E)^2 \right)^{1/2}$$

even defines a measure. However, the Radon-Nikodym allows us to play with formulas like this for functions. In particular, recall from Exercise 5.(a) on Homework 8 that if ν is signed measure and $\nu \ll \mu$ for a σ-finite measure, then $\frac{d|\nu|}{d\mu} = |\frac{d\nu}{d\mu}|$. In the complex measure case, we always have $\nu \ll |\nu_r| + |\nu_i|$ (check this) and $|\nu_r| + |\nu_i|$ is finite.

Def For a complex measure ν on (X,M), the ~~total variation~~ of ν is the measure

$$d|\nu| := \left|\frac{d\nu}{d(|\nu_r| + |\nu_i|)}\right| d(|\nu_r| + |\nu_i|)$$

We denote by $M(X,M)$ the $\mathbb{C}$-vector space of all complex measures on (X,M), and for $\nu \in M(X,M)$ the ~~total variation norm~~ of ν is $\|\nu\| := |\nu|(X)$.

By Exercise 5 on Homework 9, the total variation norm is indeed a norm, and moreover $M(X,M)$ is a Banach space with this norm: it is complete with respect to the metric $\|\nu - \mu\|$.

Proposition 3.15 Let ν be a complex measure on (X,M). For any σ-finite measure μ on (X,M) satisfying $\nu \ll \mu$, one has

$$d|\nu| = \left|\frac{d\nu}{d\mu}\right| d\mu.$$

Proof Let $\lambda := |\nu_r| + |\nu_i|$ (so that $d|\nu| = |d\nu/d\lambda| d\lambda$ by definition) and let $\rho := \lambda + \mu$. Then we have $\nu \ll \mu, \lambda \ll \rho$ and so by Theorem 3.10.① we have

$$\frac{d\nu}{d\mu}\frac{d\mu}{d\rho} = \frac{d\nu}{d\rho} = \frac{d\nu}{d\lambda}\frac{d\lambda}{d\rho}$$

ρ-almost everywhere. Therefore

$$\left|\frac{d\nu}{d\mu}\right|\frac{d\mu}{d\rho} = \left|\frac{d\nu}{d\mu}\frac{d\mu}{d\rho}\right| = \left|\frac{d\nu}{d\lambda}\frac{d\lambda}{d\rho}\right| = \left|\frac{d\nu}{d\lambda}\right|\frac{d\lambda}{d\rho}$$

ρ-almost everywhere and so

$$d|\nu| = \left|\frac{d\nu}{d\lambda}\right| d\lambda = \left|\frac{d\nu}{d\lambda}\right|\frac{d\lambda}{d\rho} d\rho = \left|\frac{d\nu}{d\mu}\right|\frac{d\mu}{d\rho} d\rho = \left|\frac{d\nu}{d\mu}\right| d\mu.$$

Proposition 3.16 Let ν be a complex measure on (X,M).
① $|\nu(E)| \le |\nu|(E)$ for all $E \in M$
② $\nu \sim |\nu|$ and $\left|\frac{d\nu}{d|\nu|}\right| = 1$ $|\nu|$-a.e.
③ $L'(X,\nu) = L'(X,|\nu|)$, and for $f \in L'(X,\nu)$

$$\left|\int_X f \, d\nu\right| \le \int_X |f| \, d|\nu|$$

Proof ① : Let $\mu := |\nu_r| + |\nu_i|$. Then by Proposition 2.22 we have

$$|\nu(E)| = \left|\int_E \frac{d\nu}{d\mu} d\mu\right| \le \int_E \left|\frac{d\nu}{d\mu}\right| d\mu = |\nu|(E)$$

② : The previous part shows $\nu \ll |\nu|$, and $|\nu| \ll \mu \ll \nu$ implies $\nu \sim |\nu|$. Observe that for $E \in M$

$$\int_E \frac{d\nu}{d\mu} d\mu = \nu(E) = \int_E \frac{d\nu}{d|\nu|} d|\nu| = \int_E \frac{d\nu}{d|\nu|}\left|\frac{d\nu}{d\mu}\right| d\mu,$$

and so $\frac{d\nu}{d\mu} = \frac{d\nu}{d|\nu|}\left|\frac{d\nu}{d\mu}\right|$ μ-almost everywhere (and hence $|\nu|$-a.e. since $\mu \sim \nu \sim |\nu|$). Observe that if $E := \left\{ x \in X : \left|\frac{d\nu}{d\mu}(x)\right| = 0 \right\}$, then

$$|\nu|(E) = \int_E \left|\frac{d\nu}{d\mu}\right| d\mu = \int_E 0 \, d\mu = 0$$

Thus $\left|\frac{d\nu}{d\mu}\right| > 0$ $|\nu|$-a.e., and so $\left|\frac{d\nu}{d\mu}\right| = \left|\frac{d\nu}{d|\nu|}\right| \left|\frac{d|\nu|}{d\mu}\right|$ $|\nu|$-a.e. implies $\left|\frac{d\nu}{d|\nu|}\right| = 1$ $|\nu|$-a.e.

③: Part ① implies $|\nu_r| \leq |\nu|$ and $|\nu_i| \leq |\nu|$. Hence by Exercise 1.(a) on Homework 8 11/13

$$L^1(X, |\nu|) \subset L^1(X, |\nu_r|) \cap L^1(X, |\nu_i|) = L^1(X, \nu_r) \cap L^1(X, \nu_i) = L^1(X, \nu)$$

Conversely, Exercise 5.(a) on Homework 8 gives:

$$|\nu|(E) = \int_E \left|\frac{d\nu}{d|\nu|}\right| d|\nu| \leq \int_E \left|\frac{d\nu_r}{d|\nu|}\right| + \left|\frac{d\nu_i}{d|\nu|}\right| d|\nu| = \int_E \frac{d|\nu_r|}{d|\nu|} + \frac{d|\nu_i|}{d|\nu|} d|\nu| = |\nu_r|(E) + |\nu_i|(E)$$

So $|\nu| \leq |\nu_r| + |\nu_i|$, which implies

$$L^1(X, \nu) = L^1(X, |\nu_r|) \cap L^1(X, |\nu_i|) \subset L^1(X, |\nu|).$$

Hence $L^1(X, \nu) = L^1(X, |\nu|)$.

Now, for a simple function with standard representation $\phi = \sum_{j=1}^{n} \alpha_j \mathbb{1}_{E_j}$, we have by ① that:

$$\left|\int_X \phi \, d\nu\right| = \left|\sum_{j=1}^{n} \alpha_j \nu(E_j)\right| \leq \sum_{j=1}^{n} |\alpha_j| |\nu|(E_j) = \int_X |\phi| \, d|\nu|.$$

The general inequality then follows by approximating $f \in L^1(X, \nu)$ by simple functions and using the dominated convergence theorem (Theorem 2.24). $\qquad\square$

Since $\nu \sim |\nu|$ by Part 2 of the above theorem, we have the following characterizations for mutual singularity, absolute continuity and equivalence of complex measures ν, μ:

$$\nu \perp \mu \iff |\nu| \perp |\mu|$$
$$\nu \ll \mu \iff |\nu| \ll |\mu|$$
$$\nu \sim \mu \iff |\nu| \sim |\mu|.$$

Proposition 3.17 If ν, μ are complex measures on (X, M), then $|\nu + \mu| \leq |\nu| + |\mu|$.

Proof Since $\nu \sim |\nu| \leq |\nu| + |\mu|$ we have $\nu \ll |\nu| + |\mu|$ and so by Proposition 3.15

$$d|\nu| = \left|\frac{d\nu}{d(|\nu|+|\mu|)}\right| d(|\nu| + |\mu|).$$

Similarly for $|\mu|$. Thus for $E \in M$

$$|\nu + \mu|(E) = \int_E \left|\frac{d(\nu+\mu)}{d(|\nu|+|\mu|)}\right| d(|\nu| + |\mu|)$$

$$\leq \int_E \left|\frac{d\nu}{d(|\nu|+|\mu|)}\right| + \left|\frac{d\mu}{d(|\nu|+|\mu|)}\right| d(|\nu|+|\mu|) = |\nu|(E) + |\mu|(E) \qquad\square$$

3.4 Differentiation on Euclidean Space

Recall that the fundamental theorem of calculus (or at least one of its parts) says that if $f: [a,b] \to \mathbb{R}$ is continuous, then $F: [a,b] \to \mathbb{R}$ defined by

$$F(x) := \int_a^x f(t)\, dt$$

is differentiable on (a,b) with $F' = f$. Equivalently, for each $x \in (a,b)$

$$\frac{1}{m(B(x,\varepsilon))} \int_{B(x,\varepsilon)} f(t)\, dt = \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} f(t)\, dt = \frac{F(x+\varepsilon) - F(x-\varepsilon)}{2\varepsilon} \xrightarrow{\varepsilon \to 0} F'(x) = f(x).$$

Our main goal of this section is to use the Lebesgue measure on $\mathbb{R}^n$ to show this limit holds for a much broader class of functions.

Throughout, we fix $n \in \mathbb{N}$ and consider the measure space $(\mathbb{R}^n, L^n, m^n)$. We will typically write m for m^n, and "measurable", "integrable", and "almost everywhere" will mean with respect to this measure space. We first require a technical geometric lemma:

Lemma 3.18 Let $\mathcal{C}$ be a collection of open balls in $\mathbb{R}^n$, and denote

$$\mathcal{U} := \bigcup_{B \in \mathcal{C}} B$$

If $m(\mathcal{U}) > c > 0$, then there exists disjoint $B_1, \ldots, B_k \in \mathcal{C}$ such that

$$m(B_1) + \cdots + m(B_k) > 3^{-n} c.$$

Proof Using Theorem 2.38 we can find $K \subset \mathcal{U}$ compact with $m(K) > c$. Since $\mathcal{C}$ covers K, the compactness implies we can find a finite subcover, say by $A_1, \ldots, A_m \in \mathcal{C}$. Let B_1 be the A_i with largest radius, let B_2 be the A_i with the largest radius that is disjoint from B_1, let B_3 be the A_i with the largest radius that is disjoint from B_1 and B_2, and so on until there are no A_i's disjoint from $B_1, B_2, \ldots, B_d$. Now if A_i does not appear in the list $B_1, \ldots, B_d$ then, $A_i \cap B_j \neq \emptyset$ for some j. Moreover, if j is the smallest index with this property (so that $A_i \cap B_{j-1} = \emptyset$) then the radius of A_i must be less than or equal to the radius of B_j (by definition of B_j). Consequently, if B_j^* is the ball with radius three times that of B_j but with the same center, then $A_i \subset B_j^*$. So

$$K \subset A_1 \cup \cdots \cup A_m \subset B_1^* \cup \cdots \cup B_d^*$$

and therefore by Theorem 2.40

$$c < m(K) \le \sum_{j=1}^{d} m(B_j^*) = \sum_{j=1}^{d} 3^n\, m(B_j).$$

B_j A_i

$d < \text{rad}(B_j) + 2\,\text{rad}(A_i) \le 3\,\text{rad}(B_j)$
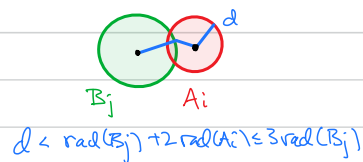
Def We call an L^n-measurable function $f: \mathbb{R}^n \to \mathbb{C}$ **locally integrable** with respect to m if

$$\int_K |f| \, dm < \infty$$

for all bounded measurable sets $K \subset \mathbb{R}^n$. We denote the set of such functions by $L^1_{loc}(\mathbb{R}^n, m)$. Given $f \in L^1_{loc}(\mathbb{R}^n, m)$, $x \in \mathbb{R}^n$, and $r > 0$ we denote

$$A_r f(x) := \frac{1}{m(B(x,r))} \int_{B(x,r)} f \, dm$$

and call this quantity the **average** of f over $B(x,r)$. □

Lemma 3.19 For $f \in L^1_{loc}(\mathbb{R}^n, m)$, the map

$$\mathbb{R}^n \times (0, \infty) \ni (x, r) \longmapsto A_r f(x)$$

is continuous.

Proof If $(x_n, r_n) \to (x_0, r_0)$ then $\mathbb{1}_{B(x_n, r_n)} \to \mathbb{1}_{B(x_0, r_0)}$ pointwise on $\mathbb{R}^n \setminus S(x_0, r_0)$ where $S(x_0, r_0) = \{y \in \mathbb{R}^n : |x-y| = r\}$. Since we showed $m(S(x_0, r_0)) = 0$ at the end of Chapter 2, we see that $\mathbb{1}_{B(x_n, r_n)} f \to \mathbb{1}_{B(x_0, r_0)} f$ m-almost everywhere. By dropping sufficiently many terms at the beginning of our sequence, we may assume

$$|x_n - x_0| < \tfrac{1}{2} \quad \text{and} \quad |r_n - r_0| < \tfrac{1}{2} \qquad \forall n \in \mathbb{N}.$$

Hence $B(x_n, r_n) \subset B(x_0, r_0 + 1)$ for all $n \in \mathbb{N}$, and so $|f \mathbb{1}_{B(x_n, r_n)}| \leq |f| \mathbb{1}_{B(x_0, r_0 + 1)}$. The latter function is integrable since f is locally integrable, and so the dominated convergence theorem (Theorem 2.24) implies

$$\lim_{n \to \infty} \int_{B(x_n, r_n)} f \, dm = \int_{B(x_0, r_0)} f \, dm.$$

Thus $(x, r) \mapsto \int_{B(x,r)} f \, dm$ is continuous. Also $\frac{1}{m(B(x,r))} = \frac{1}{c} r^{-n}$ for some $c > 0$ by the example at the end of Chapter 2. Thus

$$A_r f(x) = \frac{1}{c} r^{-n} \int_{B(x,r)} f \, dm$$

is continuous as the product of continuous functions. □

Def For $f \in L^1_{loc}(\mathbb{R}^n, m)$, its **Hardy–Littlewood maximal function** $Hf: \mathbb{R}^n \to [0, \infty]$ is defined by

$$Hf(x) := \sup_{r > 0} A_r |f|(x) = \sup_{r > 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f| \, dm$$

□

Observe that Hf is always L^n-measurable since

$$(Hf)^{-1}((a, \infty)) = \bigcup_{r > 0} (A_r |f|)^{-1}((a, \infty))$$

(if $Hf(x) > a$, then $A_r |f|(x) > a$ for some $r > 0$), and $(A_r |f|)^{-1}((a, \infty))$ are open by Lemma 3.19.

Theorem 3.20 (The Hardy-Littlewood Maximal Inequality) There exists $C > 0$ such that for all $f \in L^1(\mathbb{R}^n, m)$ and all $\varepsilon > 0$

$$m(\{x \in X : Hf(x) > \varepsilon\}) \leq \frac{C}{\varepsilon} \int_{\mathbb{R}^n} |f| \, dm$$

Proof Let $E := \{x \in X : Hf(x) > \varepsilon\}$. Then for each $x \in E$ there exists $r(x) > 0$ so that $A_{r(x)}|f| > \varepsilon$, or equivalently

$$m(B(x, r(x))) \leq \frac{1}{\varepsilon} \int_{B(x, r(x))} |f| \, dm. \qquad \textcolor{red}{*}$$

Now, since

$$E \subset \bigcup_{x \in E} B(x, r(x)),$$

if $m(E) > c$ for some $c > 0$, then by Lemma 3.18 there exists $x_1, \ldots, x_d \in E$ so that the $B(x_j, r(x_j))$ are disjoint and

$$\sum_{j=1}^{d} m(B(x_j, r(x_j))) > 3^{-n} c.$$

Combining this with $(\textcolor{red}{*})$ gives:

$$c < 3^n \sum_{j=1}^{d} m(B(x_j, r(x_j))) \leq \frac{3^n}{\varepsilon} \sum_{j=1}^{d} \int_{B(x_j, r(x_j))} |f| \, dm \leq \frac{3^n}{\varepsilon} \int_{\mathbb{R}^n} |f| \, dm.$$

Letting $c \uparrow m(E)$ yields the claimed inequality. $\qquad \square$

Recall that for a metric space (X, d) and a function $f : X \to \mathbb{R}$, one defines the limit inferior/superior for f at $x_0 \in X$ by:

$$\liminf_{x \to x_0} f(x) = \lim_{\varepsilon \to 0} \inf_{d(x, x_0) < \varepsilon} f(x) \qquad \text{and} \qquad \limsup_{x \to x_0} f(x) = \lim_{\varepsilon \to 0} \sup_{d(x, x_0) < \varepsilon} f(x)$$

Exercise Show $\lim_{x \to x_0} f(x)$ exists iff $\limsup_{x \to x_0} f(x) = \liminf_{x \to x_0} f(x)$ iff there exists $c \in \mathbb{R}$ so that $\limsup_{x \to x_0} |f(x) - c| = 0$, in which case $\lim_{x \to x_0} f(x) = c$.

Note in particular that for $f \in L^1_{loc}(\mathbb{R}^n, m)$ and $x \in \mathbb{R}^n$

$$\limsup_{r \to 0} |A_r f(x)| \leq \lim_{\varepsilon \to 0} \sup_{0 < r < \varepsilon} A_r |f|(x) \leq Hf(x)$$

Theorem 3.21 For $f \in L^1_{loc}(\mathbb{R}^n, m)$,

$$f(x) = \lim_{r \to 0} A_r f(x) = \lim_{r \to 0} \frac{1}{m(B(x, r))} \int_{B(x, r)} f \, dm$$

for m-almost every $x \in \mathbb{R}^n$.

Proof We first note that it suffices to prove this for functions in $L^1(\mathbb{R}^n, m)$. Indeed, for $f \in L^1_{loc}(\mathbb{R}^n, m)$, if $N \in \mathbb{N}$, then for $r \leq 1$ and m-almost every $x \in B(0, N)$

$$A_r f(x) = A_r(f \, \mathbb{1}_{B(0, N+1)})(x) \xrightarrow{r \to 0} (f \, \mathbb{1}_{B(0, N+1)})(x) = f(x).$$

since $f \, \mathbb{1}_{B(0, N+1)} \in L^1(\mathbb{R}^n, m)$. Thus

$$\{x \in \mathbb{R}^n : A_r f(x) \not\to f(x)\} = \overset{\circ}{\underset{N=1}{\bigcup}} \{x \in B(0,N) : A_r f(x) \not\to f(x)\}$$

is an m-null set.

So let $f \in L^1(\mathbb{R}^n, m)$ and let $\varepsilon > 0$. Using Theorem 2.39 we can find a continuous function $g : \mathbb{R}^n \to \mathbb{C}$ satisfying

$$\int_{\mathbb{R}^n} |f - g| \, dm < \varepsilon.$$

Given $x \in \mathbb{R}^n$, the continuity of g implies there exists $\delta > 0$ so that $|x - y| < \delta$ implies $|g(x) - g(y)| < \varepsilon$. Hence for $r \leq \delta$

$$|A_r g(x) - g(x)| \leq \frac{1}{m(B(x,r))} \int_{B(x,r)} |g(y) - g(x)| \, dm(y) \leq \varepsilon.$$

Thus $A_r g(x) \to g(x)$ as $r \to 0$ for every $x \in \mathbb{R}^n$. So we have

$$\limsup_{r \to 0} |A_r f(x) - f(x)| = \limsup_{r \to 0} |A_r(f-g)(x) + (A_r g - g)(x) + (g - f)(x)|$$

$$\leq H(f-g)(x) + 0 + |f(x) - g(x)|.$$

For $\alpha > 0$, denote

$$E_\alpha := \{x \in \mathbb{R}^n : \limsup_{r \to 0} |A_r f(x) - f(x)| \geq \alpha\} \quad \text{and} \quad F_\alpha := \{x \in \mathbb{R}^n : |f(x) - g(x)| \geq \alpha\}.$$

Then the above estimate implies

$$E_\alpha \subset F_{\alpha/2} \cup \{x \in \mathbb{R}^n : H(f-g)(x) \geq \tfrac{\alpha}{2}\}.$$

Using

$$m(F_{\alpha/2}) \leq \frac{2}{\alpha} \int_{F_{\alpha/2}} |f - g| \, dm \leq \frac{2\varepsilon}{\alpha}$$

and the Hardy–Littlewood maximal inequality (Theorem 3.20), we have

$$m(E_\alpha) \leq \frac{2\varepsilon}{\alpha} + \frac{2C\varepsilon}{\alpha}$$

for some $C > 0$. Since $\varepsilon > 0$ was arbitrary, we have $m(E_\alpha) = 0$ for all $\alpha > 0$. Hence

$$\{x \in \mathbb{R}^n : \limsup_{r \to 0} |A_r f(x) - f(x)| > 0\} = \overset{\circ}{\underset{n=1}{\bigcup}} E_{1/n}$$

is m-null, and so $A_r f(x) \to f(x)$ m-almost everywhere. $\square$

Since $f(x) = \frac{1}{m(B(x,r))} \int_{B(x,r)} f(x) \, dm(y)$, the previous theorem is equivalent to

$$\lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(x) - f(y) \, dm(y) = 0$$

for m-almost every $x \in \mathbb{R}^n$. We will strengthen this in Theorem 3.22 below by replacing the integrand with its absolute value. We first introduce some terminology.

Def   The Lebesgue set of $f \in L^1_{loc}(\mathbb{R}^n, m)$ is the set

$$L_f := \left\{ x \in \mathbb{R}^n : \lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(x) - f(y)| \, dm(y) = 0 \right\}$$

Theorem 3.22   For $f \in L^1_{loc}(\mathbb{R}^n, m)$, $m(L_f^c) = 0$.

Proof For each $z \in \mathbb{C}$, Theorem 3.21 applied to $g_z(x) := |f(x) - z| \in L^1_{loc}(\mathbb{R}^n, m)$ $\leftarrow \notin L^1(\mathbb{R}^n, m)$ shows that

$$\frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - z| \, dm(y) \xrightarrow{r \to 0} |f(x) - z|$$

except for x in an m-null set $E_z \in \mathbb{R}^n$. Then

$$E := \bigcup_{z \in \mathbb{Q} + i\mathbb{Q}} E_z$$

is m-null since $\mathbb{Q} + i\mathbb{Q}$ is countable. For $x \notin E$ and $\varepsilon > 0$, we can find $z \in \mathbb{Q} + i\mathbb{Q}$ so that $|f(x) - z| \leq \varepsilon$ and hence

$$\frac{1}{m(B(x,r))} \int_{B(x,r)} |f(x) - f(y)| \, dm(y) \leq |f(x) - z| + \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - z| \, dm(y) \longrightarrow 2|f(x) - z| < 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that $x \in L_f$. Thus $E^c \subset L_f$ and so $m(L_f^c) \leq m(E) = 0$. $\square$

We continue to strengthen this result by replacing $B(x,r)$ with a more generic class of sets:

Def We say a family of Borel subsets $\{E_r \subset \mathbb{R}^n : r > 0\}$ **shrinks nicely** to $x \in \mathbb{R}^n$ if $E_r \subset B(x,r)$ for all $r > 0$ and

$$\inf_{r > 0} \frac{m(E_r)}{m(B(x,r))} > 0 .$$

$\square$

Note that if $\alpha > 0$ is the infimum in the above definition, then $\alpha \cdot m(B(x,r)) \leq m(E_r) \leq m(B(x,r))$ for all $r > 0$.

EX Let $U \subset B(0,1)$ be any Borel subset with $m(U) > 0$. For $x \in \mathbb{R}^n$ and $r > 0$, if
$$E_r := \{x + ry : y \in U\} = U \cdot r + x$$
then $\{E_r : r > 0\}$ shrinks nicely to x with

$$\inf_{r > 0} \frac{m(E_r)}{B(x,r)} = \frac{m(U)}{m(B(0,1))} > 0.$$

Note that if $0 \notin U$ then $x \notin E_r$ for any $r > 0$. $\square$

Theorem 3.23 (The Lebesgue Differentiation Theorem)
Let $f \in L^1_{loc}(\mathbb{R}^n, m)$. For every $x \in L_f$ the Lebesgue set of f — in particular, for almost every $x \in \mathbb{R}^n$ — we have

$$\lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r} |f(x) - f(y)| \, dm(y) = 0$$

and

$$\lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r} f \, dm = f(x)$$

for every family $\{E_r : r > 0\}$ that shrinks nicely to x.

Proof Denote

$$\alpha := \inf_{r > 0} \frac{m(E_r)}{m(B(x,r))}$$

So that $m(E_r) \geq \alpha \, m(B(x,r))$ for all $r > 0$. We have for $x \in L_f$ that

$$\frac{1}{m(E_r)} \int_{E_r} |f(x) - f(y)| \, dm(y) \leq \frac{1}{m(E_r)} \int_{B(x,r)} |f(x) - f(y)| \, dm(y) \leq \frac{1}{\alpha \, m(B(x,r))} \int_{B(x,r)} |f(x) - f(y)| \, dm(y) \longrightarrow 0 .$$

This proves the first limit. The second limit follows from

$$\left| f(x) - \frac{1}{m(E_r)} \int_{E_r} f(y) \, dm(y) \right| = \left| \frac{1}{m(E_r)} \int_{E_r} f(x) - f(y) \, dm(y) \right| \leq \frac{1}{m(E_r)} \int_{E_r} |f(x) - f(y)| \, dm(y),$$

and the first limit. □

We can interpret the above results in terms of measures: Suppose $\nu \in M(\mathbb{R}^i, \mathcal{B}_{\mathbb{R}^i})$ with $\nu \ll m$ and $\frac{d\nu}{dm} \in L^1_{loc}(\mathbb{R}^n, m)$. Then Theorem 3.23 implies

$$\lim_{r \to 0} \frac{\nu(E_r)}{m(E_r)} = \frac{d\nu}{dm}(x)$$

for any family $\{E_r : r > 0\}$ that shrinks nicely to a point $x \in L_{\frac{d\nu}{dm}}$. Let us generalize this a bit more.

Def We call a Borel measure ν on $\mathbb{R}^n$ **regular** if
① $\nu(K) < \infty$ for every compact $K \subset \mathbb{R}^n$;
② for every $E \in \mathcal{B}_{\mathbb{R}^n}$
$$\nu(E) = \inf \{ \nu(U) : U \supset E \text{ open} \}$$

A signed or complex Borel measure ν is **regular** if $|\nu|$ is regular in the above sense. □

Note that ① implies any regular signed or complex Borel measure is σ-finite. Consequently one can consider the Lebesgue decomposition

$$\nu = \lambda + \rho$$

of ν with respect to m: $\lambda \perp m$ and $\rho \ll m$.

Theorem 3.24 Let ν be a regular signed or complex Borel measure on $\mathbb{R}^n$ with Lebesgue decomposition $\nu = \lambda + \rho$ with respect to m, where $\lambda \perp m$ and $\rho \ll m$. Then for m-almost every $x \in \mathbb{R}^n$,

$$\lim_{r \to 0} \frac{\nu(E_r)}{m(E_r)} = \frac{d\rho}{dm}(x)$$

for every family $\{E_r : r > 0\}$ that shrinks nicely to x.

Before proving this theorem, we first require a lemma:

Lemma 3.25 Let ρ be a signed or complex Borel measure on $\mathbb{R}^n$ satisfying $\rho \ll m$. Then ρ is regular iff $\frac{d\rho}{dm} \in L^1_{loc}(\mathbb{R}^n, m)$.

Proof Recall that in both the signed measure and complex measure cases $\left| \frac{d\rho}{dm} \right| = \frac{d|\rho|}{dm}$ (by Exercise 3.6(a) on Homework 8 and by Proposition 3.15, respectively). Thus $\frac{d\rho}{dm} \in L^1_{loc}(\mathbb{R}^n, m)$

iff
$$|\rho|(K) = \int_K \frac{d|\rho|}{dm} \, dm = \int_K \left|\frac{d\rho}{dm}\right| dm < \infty.$$

for all bounded $K \in \mathcal{B}_{\mathbb{R}^n}$. Since every bounded $K \subset \mathbb{R}^n$ is contained in a compact set (namely its closure) this is further equivalent to $|\rho|(K) < \infty$ for all compact $K \subset \mathbb{R}^n$. So it remains to show that $\frac{d\rho}{dm} \in L^1_{loc}(\mathbb{R}^n, m)$ implies

$$|\rho|(E) = \inf \{|\rho|(U) : U \supset E \text{ open}\} \qquad \textcolor{red}{*}$$

for all $E \in \mathcal{B}_{\mathbb{R}^n}$. First suppose E is bounded, say $E \subset B(0, R)$ for $R > 0$. Note that $f := \left|\frac{d\rho}{dm}\right| 1_{B(0,R)} \in L^1(\mathbb{R}^n, m)$, and so by Corollary 3.6 given $\varepsilon > 0$ there exists $\delta > 0$ so that $m(A) < \delta$ implies

$$\int_A f \, dm < \varepsilon.$$

By Theorem 2.38, $\exists \ U \supset E$ open so that $m(U) < m(E) + \delta$, and hence $m(U \setminus E) < \delta$ since $m(E) < \infty$ by virtue of E being bounded. By considering $U \cap B(0, R)$, we can assume $U \setminus E \subset B(0, R)$. Thus

$$|\rho|(U) = |\rho|(E) + \int_{U \setminus E} \left|\frac{d\rho}{dm}\right| dm = |\rho|(E) + \int_{U \setminus E} f \, dm < |\rho|(E) + \varepsilon,$$

and $(\textcolor{red}{*})$ follows.

For potentially unbounded $E \in \mathcal{B}_{\mathbb{R}^n}$, let $E_n := E \cap B(0, n)$. Using the above we can find $U_n \supset E_n$ open such that $|\rho|(U_n \setminus E_n) < \frac{1}{2^n}\varepsilon$ for some $\varepsilon > 0$. Then

$$U := \bigcup_{n=1}^\infty U_n$$

is open, contains E, and satisfies $|\rho|(U \setminus E) < \varepsilon$. Thus $|\rho|(U) < |\rho|(E) + \varepsilon.$ $\qquad \square$

Remark If $\rho \in M(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$ satisfies $\rho \ll m$, then $\frac{d\rho}{dm} \in L^1(\mathbb{R}^n, m)$ since ρ is finite. Thus the previous lemma implies _every_ complex Borel measure on $\mathbb{R}^n$ that is absolutely continuous with respect to the Lebesgue measure m is regular. $\square$

Proof (Theorem 3.24) By Homework 9, λ and ρ are both regular. Lemma 3.25 and the Lebesgue differentiation theorem (Theorem 3.23) imply for m-almost every $x \in \mathbb{R}^n$

$$\lim_{r \to 0} \frac{\rho(E_r)}{m(E_r)} = \lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r} \frac{d\rho}{dm} \, dm = \frac{d\rho}{dm}(x)$$

for every family $\{E_r : r > 0\}$ that shrinks nicely to x. Thus it suffices to show $\frac{\lambda(E_r)}{m(E_r)} \to 0$ as $r \to 0$. Let

$$\alpha := \inf_{r > 0} \frac{m(E_r)}{m(B(x, r))}.$$

Then

$$\left|\frac{\lambda(E_r)}{m(E_r)}\right| \leq \frac{|\lambda|(E_r)}{m(E_r)} \leq \frac{|\lambda|(B(x, r))}{m(E_r)} \leq \frac{|\lambda|(B(x, r))}{\alpha \, m(B(x, r))}.$$

So it suffices to show $|\lambda|(B(x, r)) / m(B(x, r)) \to 0$ as $r \to 0$. Recall $\lambda \perp m$ implies $|\lambda| \perp m$, and so there exists a partition $\mathbb{R}^n = A \cup B$ so that A is $|\lambda|$-null and B is m-null. Define

$$F_k := \left\{x \in A : \limsup_{r \to 0} \frac{|\lambda|(B(x, r))}{m(B(x, r))} > \frac{1}{k}\right\},$$

so that showing $m(F_k)=0$ for all $k \in \mathbb{N}$ will complete the proof. Given $\varepsilon > 0$, the regularity of $|\lambda|$ implies there exists $U_\varepsilon \supset A$ open such that $|\lambda|(U_\varepsilon) < \varepsilon$. For each $x \in F_k$, there exists $r(x)$ so that $B(x, r(x)) \subset U_\varepsilon$ and $|\lambda|(B(x, r(x))) > \frac{1}{k} m(B(x, r(x)))$. Let

$$V_\varepsilon := \bigcup_{x \in F_k} B(x, r(x)),$$

which is contained in U_ε. If $m(V_\varepsilon) > c > 0$, then lemma 3.18 implies there exists $x_1, \ldots, x_d \in F_k$ so that $B(x_1, r(x_1)), \ldots, B(x_d, r(x_d))$ are disjoint and

$$3^{-n} c < \sum_{j=1}^{d} m(B(x_j, r(x_j))) < \sum_{j=1}^{d} k\,|\lambda|(B(x_j, r(x_j))) \leq k\,|\lambda|(V_\varepsilon) \leq k\,|\lambda|(U_\varepsilon) < k\varepsilon.$$

Hence $c < 3^n k\varepsilon$. Since this held for arbitrary $c < m(V_\varepsilon)$, we see that $m(F_k) \leq m(V_\varepsilon) < 3^n k\varepsilon$. Letting $\varepsilon \to 0$ shows $m(F_k) = 0$. $\qquad \square$

3.5 Functions of Bounded Variation

In this section, we will analyze regular signed or complex Borel measures on $\mathbb{R}$ ($n=1$). We have actually already conducted this analysis for regular positive Borel measures on $\mathbb{R}$ back in Section 1.5. Indeed, Theorem 1.16 implies any such measure μ is of the form μ_F for $F:\mathbb{R}\to\mathbb{R}$ increasing and right-continuous. Conversely, Theorems 1.16 and 1.18 imply any increasing right-continuous function $F:\mathbb{R}\to\mathbb{R}$ yields a regular Borel measure μ_F. One of our objectives is to extend this correspondence to regular __complex__ Borel measures and a new class of $\mathbb{C}$-valued functions. This correspondence will yield decomposition theorems for such functions paralleling those we obtained for measures. Another objective is to understand which $\mathbb{C}$-valued functions F yield complex measures ν_F satisfying $\nu_F \perp m$ and $\nu_F \ll m$. This will provide new insights into Theorem 3.24 in the $n=1$ case, where genuine derivatives now appear:

EX Given a function $F:\mathbb{R}\to\mathbb{C}$ suppose there exists a regular complex Borel measure ν_F satisfying
$$\nu_F((a,b]) = F(b) - F(a)$$
for $a, b \in \mathbb{R}$ with $a < b$. Let $\nu_F = \lambda + \rho$ be the Lebesgue decomposition with respect to m. Observe that
$$F(x+h) - F(x) = \begin{cases} \nu_F((x, x+h]) & \text{if } h > 0 \\ -\nu_F((x+h, x]) & \text{if } h < 0 \end{cases}$$
and the families $\{(x, x+h]: h>0\}$ and $\{(x+h, x]: h<0\}$ shrink nicely to x as $|h| \to 0$. Thus Theorem 3.24 says for m-almost every $x \in \mathbb{R}$ that
$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{\nu_F((x, x+h])}{m((x, x+h])} = \frac{d\rho}{dm}(x)$$
and
$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{\nu_F((x+h, x])}{m((x+h, x])} = \frac{d\rho}{dm}(x)$$
Thus F is differentiable m-almost everywhere with $F' = \frac{d\rho}{dm}$. If $\nu_F \ll m$ so that $\rho = \nu_F$, then
$$\int_{(a,b]} F'\, dm = \int_{(a,b]} \frac{d\nu_F}{dm}\, dm = \nu_F((a,b]) = F(b) - F(a)$$
is a generalization of the fundamental theorem of calculus (or one of its other parts). □

Theorem 3.26 Let $F:\mathbb{R}\to\mathbb{R}$ be increasing, and define $G:\mathbb{R}\to\mathbb{R}$ by
$$G(x) := \lim_{a \to x} F(a) = \inf_{a > x} F(a)$$

① G is increasing and right-continuous.
② F has a countable discontinuity set.
③ F and G are differentiable m-almost everywhere with $F' = G'$ m-a.e.

Proof ①: For $x < y$, the containment $(x, \infty) \supset (y, \infty)$ implies $G(x) \le G(y)$. Also
$$\inf_{a > x} G(a) = \inf_{a > x} \inf_{b > a} F(b) = \inf_{b > x} F(b) = G(x)$$
implies G is right-continuous.
②: Since F is increasing, for $x \in \mathbb{R}$

$$F(x^-) := \sup_{a<x} F(a) \le F(x) \le \inf_{a>x} F(x) =: F(x^+)$$

and F is discontinuous iff $F(x^-) < F(x^+)$. For $N \in \mathbb{N}$, let $D_N := \{x \in (-N, N) : F \text{ is discontinuous at } x\}$. Then each $x \in D_N$ satisfies $F(-N) \le F(x^-) \le F(x^+) \le F(N)$. Moreover if $x, x' \in D_N$ are distinct, say with $x < x'$, then $F(x^+) \le F(x'^-)$. Thus

$$\bigcup_{x \in D_N} (F(x^-), F(x^+)) \subseteq [F(-N), F(N)]$$

and the union is disjoint. Hence

$$\sum_{x \in D_N} F(x^+) - F(x^-) \le F(N) - F(-N)$$

Since each term in the sum is positive, D_N is necessarily countable. Therefore $\bigcup_{N \in \mathbb{N}} D_N$ is countable.
③: Let μ_G be the unique Borel measure on $\mathbb{R}$ satisfying $\mu_G((a,b]) = G(b) - G(a)$. Then μ_G is regular $^{1/27}$ by Theorem 1.18 and the example preceding the theorem implies G is differentiable m-almost everywhere. Consider

$$H := G - F$$

Note that $H(x) = 0$ whenever F is continuous at x, and by ② this fails to hold for countably many points. Let $\{x_n : n \in \mathbb{N}\}$ be an enumeration of $\{x \in \mathbb{R} : H(x) \ne 0\}$. Note that

$$F(x^-) \le F(x) \le F(x^+) = G(x)$$

implies $H(x_n) > 0$. Also, arguing as in ② we have for each $N \in \mathbb{N}$

$$\sum_{|x_n| < N} H(x_n) = \sum_{|x_n| < N} F(x_n^+) - F(x_n) \le \sum_{|x_n| < N} F(x_n^+) - F(x_n^-) \le F(N) - F(-N) < \infty.$$

For each $n \in \mathbb{N}$, let δ_n be the point mass at x_n and define a Borel measure by

$$\mu := \sum_{n \in \mathbb{N}} H(x_n) \delta_n.$$

The above estimate shows μ is finite on bounded sets, and hence regular by Theorems 1.16 and 1.18. Moreover, the partition $\mathbb{R} = \{x_n : n \in \mathbb{N}\} \cup \{x_n : n \in \mathbb{N}\}^c$ shows $\mu \perp m$. So by Theorem 3.24 we have for m-almost every $x \in \mathbb{R}$ that

$$\left| \frac{H(x+h) - H(x)}{h} \right| \le \frac{H(x+h) + H(x)}{|h|} \le \mu\left(\frac{(x - 2|h|, x + 2|h|)}{|h|} \right) = 4 \frac{\mu((x-2|h|, x+2|h|))}{m((x-2|h|, x+2|h|))} \xrightarrow{h \to 0} 0 .$$

Thus H is differentiable m-almost everywhere with $H'(x) = 0$. Therefore $F = G - H$ is differentiable m-almost everywhere with $F'(x) = G'(x) - H'(x) = G'(x)$. $\qquad \square$

We will now define a class of functions which will correspond to regular complex Borel measures. Recall from Exercise 2 on Homework 9 that for a complex measure ν

$$|\nu|(E) = \sup\left\{ \sum_{j=1}^n |\nu(E_j)| : n \in \mathbb{N}, E = E_1 \cup \cdots \cup E_n \text{ is a partition}\right\}$$

Def For $F: \mathbb{R} \to \mathbb{C}$, the _total variation function_ of F is the function $T_F : \mathbb{R} \to [0, \infty]$ defined by

$$T_F(x) := \sup\left\{ \sum_{j=1}^n |F(x_j) - F(x_{j-1})| : n \in \mathbb{N}, -\infty < x_0 < x_1 < \cdots x_n = x\right\},$$

and the __total variation__ of F on $\mathbb{R}$ is the quantity $\sup_{x \in \mathbb{R}} T_F(x)$. We say F is of __bounded variation__ on $\mathbb{R}$ its total variation is finite (equivalently, T_F is bounded), and we denote by BV the set of all such functions. □

The relation between F and T_F is like the relation between v and $|v|$ (which we will soon see even more clearly). Adding more points to the partition

$$(-\infty, x] = (-\infty, x_0] \cup (x_0, x_1] \cup \cdots \cup (x_{n-1}, x_n]$$

can only increase the sum

$$\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})|.$$

So if $x < y$, then we can always assume the partitions of $(-\infty, y]$ include x, and consequently

$$T_F(y) - T_F(x) = \sup\left\{ \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| : n \in \mathbb{N}, \; x = x_0 < x_1 < \cdots < x_n = y \right\} \geq 0. \quad \text{✱}$$

Hence T_F is an increasing function, and the total variation of F equals $\lim_{x \to \infty} T_F(x)$.

Remark For an increasing function $G : \mathbb{R} \to \mathbb{R}$, we will denote

$$G(-\infty) := \lim_{x \to -\infty} G(x) = \inf_{x \in \mathbb{R}} G(x)$$

$$G(\infty) := \lim_{x \to \infty} G(x) = \sup_{x \in \mathbb{R}} G(x)$$

Thus $F \in BV$ iff $T_F(\infty) < \infty$. □

EX ① Let $F : \mathbb{R} \to \mathbb{R}$ be increasing. Then for $-\infty < x_0 < x_1 < \cdots < x_n = x$ we have

$$\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| = \sum_{j=1}^{n} F(x_j) - F(x_{j-1}) = F(x) - F(x_0).$$

Thus if $F(-\infty)$ is finite then $T_F(x) = F(x) - F(-\infty)$, and otherwise $T_F \equiv \infty$. In the former case, $T_F(\infty) = F(\infty) - F(-\infty)$ which is finite iff F is bounded. Thus $F \in BV$ iff F is bounded.

② For $F, G \in BV$ and $a, b \in \mathbb{C}$

$$T_{aF + bG} \leq |a| T_F + |b| T_G$$

Hence $aF + bG \in BV$.

③ Let $a, b \in \mathbb{R}$ with $a \leq b$. Suppose $F : [a, b] \to \mathbb{C}$ satisfies

$$\sup\left\{ \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| : n \in \mathbb{N}, \; a = x_0 < x_1 < \cdots < x_n = b \right\} < \infty.$$

Then $\tilde{F} \in BV$ where

$$\tilde{F}(x) := \begin{cases} F(a) & \text{if } x < a \\ F(x) & \text{if } a \leq x \leq b \\ F(b) & \text{if } x > b, \end{cases}$$

since $T_{\tilde{F}}(x)$ equals the above supremum for all $x \geq b$, and so $T_{\tilde{F}}(\infty)$ also equals the above supremum. (Also $T_{\tilde{F}}(x) = 0$ for $x \leq a$). □

Example ③ motivates the following definition:

Def For $a, b \in \mathbb{R}$ with $a < b$ and $F: [a,b] \longrightarrow \mathbb{C}$, the __total variation__ of F on $[a,b]$ is the quantity

$$\sup\left\{ \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| : n \in \mathbb{N}, \ a = x_0 < x_1 < \cdots < x_n = b \right\}.$$

We denote by $BV([a,b])$ the set of functions with finite total variation on $[a,b]$. ☐

Heuristically, the total variation of F on $[a,b]$ measures the total distance traveled by a particle whose position at time t is $F(t)$.

EX ④ Suppose $F: [a,b] \longrightarrow \mathbb{C}$ is differentiable with F' bounded by M, then for any $a = x_0 < x_1 < \cdots < x_n = b$, the mean value theorem implies

$$\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| = \sum_{j=1}^{n} |F'(y_j)| (x_j - x_{j-1}) \leq M \sum_{j=1}^{n} (x_j - x_{j-1}) = M(b-a).$$
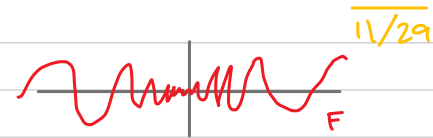
So $F \in BV([a,b])$. In fact, if F' is Riemann integrable then the above is a Riemann sum and the total variation equals:

$$\int_a^b |F'(t)| \, dt$$

Recall from calculus that this agrees with the heuristic picture discussed above. In particular, $\sin(x) \in BV([a,b])$ for any finite interval $[a,b]$. However, $\sin(x) \notin BV$ since

$$\sum_{k=1}^{n} \left| \sin\left(\frac{(2k+1)\pi}{2}\right) - \sin\left(\frac{(2k-1)}{2}\pi\right) \right| = 2n$$

⑤ Define
$$F(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$
Then $F \notin BV([a,b])$ for any interval containing zero. __Exercise__ check this

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F

⑥ For $F \in BV$, $F \in BV([a,b])$ for any interval $[a,b] \subset \mathbb{R}$ since (✗) implies the total variation of F on $[a,b]$ is:

$$T_F(b) - T_F(a) \leq T_F(b) \leq T_F(\infty) < \infty.$$ ☐

From Examples ③ and ⑥, we see that everything we will show for functions in BV also applies to functions in $BV([a,b])$ for any interval $[a,b] \subset \mathbb{R}$.

Lemma 3.27 If $F \in BV$ is $\mathbb{R}$-valued, then $T_F + F$ and $T_F - F$ are both increasing.

Proof Let $x < y$ and $\varepsilon > 0$. Choose $x_0 < x_1 < \cdots < x_n = x$ so that

$$\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| \geq T_F(x) - \varepsilon.$$

Then

$$T_F(y) \pm F(y) \geq \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| + |F(y) - F(x)| \pm F(y)$$

$$= \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| + |F(y) - F(x)| \pm [F(y) - F(x)] \pm F(x)$$

$$\geq T_F(x) - \varepsilon + 0 \pm F(x)$$

Since $\varepsilon > 0$ was arbitrary, we see that $T_F(x) \pm F(x) \leq T_F(y) \pm F(y)$. $\square$

Theorem 3.28 Let $F: \mathbb{R} \to \mathbb{C}$.

① $F \in BV$ iff $\operatorname{Re} F \in BV$ and $\operatorname{Im} F \in BV$

② If F is $\mathbb{R}$-valued, then $F \in BV$ iff F is the difference of two bounded increasing functions. In this case, $F = \frac{1}{2}(T_F + F) - \frac{1}{2}(T_F - F)$ and $\frac{1}{2}(T_F \pm F)$ are bounded increasing functions.

③ If $F \in BV$, then $F(x^+) := \lim_{a \to x} F(a)$ and $F(x^-) := \lim_{a \to x} F(x)$ exist for all $x \in \mathbb{R}$, as do $F(\pm \infty) := \lim_{a \to \pm \infty} F(a)$.

④ If $F \in BV$, then F has a countable discontinuity set.

⑤ If $F \in BV$ and $G(x) := F(x^+)$, then $G \in BV$ and F', G' exist and are equal m-almost everywhere.

Proof ①: This follows from
$$T_{\operatorname{Re} F}, T_{\operatorname{Im} F} \leq T_F \leq T_{\operatorname{Re} F} + T_{\operatorname{Im} F}.$$

②: ($\Longrightarrow$) Suppose $F \in BV$, then
$$F = \frac{1}{2}(T_F + F) - \frac{1}{2}(T_F - F)$$
and by Lemma 3.27 these functions are increasing. We also know T_F is bounded by definition of BV, and so it remains to show F is bounded. Observe that for $x < y$
$$T_F(x) \pm F(x) \leq T_F(y) \pm F(y) \implies \pm (F(x) - F(y)) \leq T_F(y) - T_F(x)$$
$$\implies |F(x) - F(y)| \leq T_F(y) - T_F(x) \leq T_F(\infty) < \infty.$$

Hence F is bounded.

($\Longleftarrow$) Suppose $F = G - H$ for $G, H : \mathbb{R} \to \mathbb{R}$ bounded and increasing. Then $G, H \in BV$ by Example ①, and hence $F \in BV$ by Example ②.

③, ④, ⑤: Using ① and ② we can find bounded increasing functions F_1, F_2, F_3, F_4 so that
$$F = F_1 - F_2 + i F_3 - F_4.$$
Then ③ is immediate and ④ follows from Theorem 3.26. Also
$$G(x) = F_1(x^+) - F_2(x^+) + i F_3(x^+) - F_4(x^+).$$
So $G \in BV$ by ① and ②, and $F' = G'$ m-almost everywhere by Theorem 3.26. $\square$

Def For an $\mathbb{R}$-valued $F \in BV$, $\frac{1}{2}(T_F + F)$ and $\frac{1}{2}(T_F - F)$ are called the positive and negative variations of F, respectively, and
$$F = \frac{1}{2}(T_F + F) - \frac{1}{2}(T_F - F)$$
is called the Jordan decomposition of F. $\square$

Observe that $\frac{1}{2}(T_F + F) + \frac{1}{2}(T_F - F) = T_F$, just like for measures.

Remark For $x \in \mathbb{R}$, denote $x^+ := \max\{x, 0\}$ and $x^- := \max\{-x, 0\}$. Then
$$x^+ = \frac{1}{2}(|x| + x)$$
$$x^- = \frac{1}{2}(|x| - x)$$
and thus for $x_0 < x_1 < \cdots < x_n = x$ and $F: \mathbb{R} \to \mathbb{R}$
$$\sum_{j=1}^{n} [F(x_j) - F(x_{j-1})]^{\pm} = \frac{1}{2}\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| \pm (F(x_j) - F(x_{j-1}))$$
$$= \frac{1}{2}\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| \pm \frac{1}{2}(F(x) - F(x_0))$$

It follows that
$$\frac{1}{2}(T_F \pm F)(x) = \sup\left\{\sum_{j=1}^{n}[F(x_j) - F(x_{j-1})]^{\pm} \pm \frac{1}{2}F(x_0) : -\infty < x_0 < x_1 < \cdots < x_n = x\right\}$$
$$= \sup\left\{\sum_{j=1}^{n}[F(x_j) - F(x_{j-1})]^{\pm} : -\infty < x_0 < x_1 < \cdots < x_n = x\right\} \pm \frac{1}{2}F(-\infty).$$

Theorem 3.28 tells us we can guarantee right-continuity for $F \in BV$ by replacing $F(x)$ with $F(x^+)$ if necessary, which only modifies F at countably many points. Moreover, the theorem also tells us
$$F = F_1 - F_2 + iF_3 - iF_4$$
for $F_1, \dots, F_4$ bounded increasing functions. We will see below that the F_j can be chosen to be right-continuous (see Lemma 3.29), and hence there exist unique Borel measures μ_j satisfying $\mu_j((a, b]) = F_j(b) - F_j(a)$ by Theorem 1.16. As F_j is bounded, μ_j is finite and therefore
$$\nu := \mu_1 - \mu_2 + i\mu_3 - i\mu_4$$
is a complex Borel measure satisfying $\nu((a, b]) = F(b) - F(a)$. Recall that up to adding a constant, F_j corresponded uniquely to μ_j. So by choosing an appropriate normalization, the correspondence $F \leftrightarrow \nu$ will be one-to-one. We therefore make the following definition:

Def We say $F \in BV$ is _normalized_ if it is right-continuous and $F(-\infty) = 0$, and we denote the set of all such functions by NBV.

Observe that for $F \in BV$, $G(x) := F(x^+) - F(-\infty) \in NBV$ by Theorem 3.28 (and Example ②).

Lemma 3.29 For $F \in BV$, $T_F(-\infty) = 0$. If F is right-continuous then so is T_F. In particular, for $F \in NBV$, $\frac{1}{2}(T_{\mathrm{Re}F} \pm \mathrm{Re}F)$, $\frac{1}{2}(T_{\mathrm{Im}F} \pm \mathrm{Im}F) \in NBV$.

Proof For $x \in \mathbb{R}$ and $\varepsilon > 0$, let $x_0 < x_1 < \cdots < x_n = x$ be such that
$$T_F(x) \le \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| + \varepsilon$$

Since $T_F(x) - T_F(x_0)$ gives the total variation of F on $[x_0, x]$ (see Example ⑥), it follows that

$$T_F(x) = T_F(x) - T_F(x_0) + \varepsilon,$$

or $T_F(x_0) \leq \varepsilon$. Since T_F is increasing, we have $T_F(y) \leq \varepsilon$ for all $y \leq x_0$. Thus $T_F(-\infty) = 0$.

Now, suppose F is right-continuous. For $x \in \mathbb{R}$, let $\alpha := T_F(x^+) - T_F(x)$ (which we must show is zero). For $\varepsilon > 0$, let $\delta > 0$ be such that $0 < h < \delta$ implies

$$|F(x+h) - F(x)| < \varepsilon$$

and

$$T_F(x+h) - T_F(x^+) < \varepsilon.$$

Fix $0 < h < \delta$ and let $x = x_0 < x_1 < \cdots < x_n = x + \varepsilon$ be such that

$$\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| \geq \frac{3}{4}\left(T_F(x+h) - T_F(x)\right) \geq \frac{3}{4}\alpha$$

Note that $|F(x_1) - F(x)| < \varepsilon$ since $x < x_1 < x + h$, and so we have

$$\frac{3}{4}\alpha \leq \varepsilon + \sum_{j=2}^{n} |F(x_j) - F(x_{j-1})|$$

$$\leq \varepsilon + T_F(x+h) - T_F(x_1)$$

$$\leq \varepsilon + T_F(x+h) - T_F(x^+) < 2\varepsilon.$$

Thus $\alpha < \frac{8}{3}\varepsilon$, and consequently $\alpha = 0$.

Finally, $F \in NBV$ implies $\operatorname{Re} F, \operatorname{Im} F \in NBV$ and the above implies $T_F \in NBV$. Hence the positive and negative variations of $\operatorname{Re} F$ and $\operatorname{Im} F$ belong to NBV. $\square$

Theorem 3.30 For a regular complex Borel measure μ on $\mathbb{R}$, $F_\mu(x) := \mu((-\infty, x]) \in NBV$. Conversely, for any $F \in NBV$ there is a unique regular complex Borel measure μ_F on $\mathbb{R}$ such that $F(x) = \mu((-\infty, x])$. Moreover, one has

$$|\mu_F| = \mu_{T_F}$$

$$(\mu_F)_r = \mu_{\operatorname{Re} F} \qquad\qquad (\mu_F)_i = \mu_{\operatorname{Im} F}$$

$$(\mu_F)_r^{\pm} = \mu_{\frac{1}{2}(T_{\operatorname{Re}F} \pm \operatorname{Re}F)} \qquad\qquad (\mu_F)_i^{\pm} = \mu_{\frac{1}{2}(T_{\operatorname{Im}F} \pm \operatorname{Im}F)}$$

Proof Let $F_r^{\pm}(x) := \mu_r^{\pm}((-\infty, x])$ and $F_i^{\pm}(x) := \mu_i^{\pm}((-\infty, x])$. Then $F_r^{\pm}, F_i^{\pm}$ are increasing and bounded (since $\mu_r^{\pm}, \mu_i^{\pm}$ are finite), and hence belong to BV by Example ①. Thus

$$F_\mu(x) := \mu((-\infty, x]) = (\mu_r^+ - \mu_r^- + i\mu_i^+ - i\mu_i^-)((-\infty, x]) = F_r^+(x) - F_r^-(x) + iF_i^+(x) - iF_i^-(x) \in BV$$

by Theorem 3.28. Moreover, $F_r^{\pm}$ and $F_i^{\pm}$ are right-continuous and zero at $-\infty$ by continuity from above. Hence $F_\mu \in NBV$.

Conversely, for $F \in NBV$ we have

$$F = \underbrace{\frac{1}{2}(T_{\operatorname{Re}F} + \operatorname{Re}F)}_{=: F_1} - \underbrace{\frac{1}{2}(T_{\operatorname{Re}F} - \operatorname{Re}F)}_{=: F_2} + i\underbrace{\frac{1}{2}(T_{\operatorname{Im}F} + \operatorname{Im}F)}_{=: F_3} - i\underbrace{\frac{1}{2}(T_{\operatorname{Im}F} - \operatorname{Im}F)}_{=: F_4}$$

and $F_1, F_2, F_3, F_4 \in NBV$ by Lemma 3.29. For each $j = 1, 2, 3, 4$, let μ_j be the unique Borel measure from Theorem 1.16 satisfying $\mu_j((a, b]) = F_j(b) - F_j(a)$. Since $F_j(-\infty) = 0$, it follows that $\mu_j((-\infty, x]) = F_j(x)$. Also note that each μ_j is regular by Theorem 1.18, and finite since F_j is bounded. Hence

$$\mu_F := \mu_1 - \mu_2 + i\mu_3 - i\mu_4$$

defines a regular complex Borel measure satisfying $\mu_F((-\infty, x]) = F(x)$. The uniqueness follows from the

uniqueness of $\mu_1, \mu_2, \mu_3, \mu_4$.

Finally, we must show show $|\mu_F| = \mu_{T_F}$. (The remaining formulas then follow directly from this.) First observe that by Exercise 2 on Homework 9

$$\mu_{T_F}((-\infty, x]) = T_F(x) = \sup\left\{\underbrace{\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})|}_{\mu_F((x_{j-1}, x_j])} : n \in \mathbb{N}, x_0 < x_1 \cdots < x_n = x\right\}$$

$$\leq \sup\left\{\sum_{j=1}^{n} |\mu_F(E_j)| : n \in \mathbb{N}, (-\infty, x] = E_1 \cup \cdots \cup E_n \text{ is a partition}\right\} = |\mu_F|((-\infty, x]).$$

On the other hand, for disjoint intervals $(a_n, b_n) \subseteq \mathbb{R}$, $n \in \mathbb{N}$, we have by Exercise 5 on Homework 3

$$\left|\mu_F\left(\bigcup_{n=1}^{\infty} (a_n, b_n)\right)\right| \leq \sum_{n=1}^{\infty} |F(b_n) - F(a_n)| \leq \sum_{n=1}^{\infty} T_F(b_n) - T_F(a_n) = \mu_{T_F}\left(\bigcup_{n=1}^{\infty} (a_n, b_n)\right)$$

The regularity of μ_F and μ_{T_F} therefore implies $|\mu_F|(E)| \leq \mu_{T_F}(E)$ for all Borel sets $E \subseteq \mathbb{R}$. Thus if $(-\infty, x] = E_1 \cup \cdots \cup E_n$ is a partition of by Borel sets, then

$$\sum_{j=1}^{n} |\mu_F(E_j)| \leq \sum_{j=1}^{n} \mu_{T_F}(E_j) = \mu_{T_F}((-\infty, x]).$$

So $|\mu_F|((-\infty, x]) \leq \mu_{T_F}((-\infty, x])$ by Exercise 2 on Homework 9 again. Hence $|\mu_F|$ and μ_{T_F} agree on $(-\infty, x]$, and so are equal by the uniqueness of μ_{T_F}. □

This completes the first objective of this section. We now move on to understanding which functions $F \in NBV$ satisfy $\mu_F \perp m$ or $\mu_F \ll m$.

Proposition 3.31 If $F \in NBV$ then $F' \in L^1(\mathbb{R}, m)$. Moreover, $\mu_F \perp m$ if and only if $F' = 0$ m-almost everywhere, and $\mu_F \ll m$ if and only if $F(x) = \int_{(-\infty, x]} F' \, dm$ for all $x \in \mathbb{R}$.

Proof Let $\mu_F = \lambda + \rho$ be the Lebesgue decomposition with respect to m, where $\lambda \perp m$ and $\rho \ll m$. By Example ①, $F' = \frac{d\rho}{dm} \in L^1(\mathbb{R}, m)$ since ρ is finite. We have $\mu_F \perp m$ iff $\mu_F = \lambda$ iff $\rho \equiv 0$ it $F' = \frac{d\rho}{dm} = 0$ m-a.e. Also, $\mu_F \ll m$ iff $\mu_F = \rho$ iff $d\mu_F = F' dm$. The last statement is equivalent to

$$F(x) = \mu_F((-\infty, x]) = \int_{(-\infty, x]} F' \, dm$$

by the uniqueness of μ_F. □

There is another way to characterize $\mu_F \ll m$ that does not rely on computing F' or its integrals. To motivate it, recall that $\mu_F \ll m$ iff $|\mu_F| \ll m$, and by Theorem 3.5 this is further equivalent to: $\forall \varepsilon > 0$ $\exists \delta > 0$ so that $m(E) < \delta$ implies $|\mu_F|(E) < \varepsilon$. In particular, if $E = (a_1, b_1) \cup \cdots \cup (a_n, b_n)$ is a disjoint union, then

$$\sum_{j=1}^{n} (b_j - a_j) = m(E) < \delta$$

implies by Exercise 2 on Homework 9 that

$$\sum_{j=1}^{n} |F(b_j) - F(a_j)| = \sum_{j=1}^{n} |\mu_F((a_j, b_j])| \leq |\mu_F|(E) < \varepsilon.$$

Note that we have used $\mu_F \ll m$ to deduce $\mu_F((a, b)) = \mu_F((a, b]) = F(b) - F(a)$.

Def We say $F: \mathbb{R} \to \mathbb{C}$ is <u>absolutely continuous</u> if for every $\varepsilon > 0$ there exists $\delta > 0$ so that for any disjoint intervals $(a_1, b_1), \ldots, (a_n, b_n) \subset \mathbb{R}$

$$\sum_{j=1}^{n} (b_j - a_j) < \delta \implies \sum_{j=1}^{n} |F(b_j) - F(a_j)| < \varepsilon.$$

For $F: [a, b] \to \mathbb{C}$, we say F is <u>absolutely continuous</u> on $[a, b]$ if the above holds for $(a_j, b_j) \subset [a, b]$. □

Note that absolute continuity implies uniform continuity by considering $n=1$ in the above definition. We have shown above that any $F \in NBV$ with $\mu_F \ll m$ is absolutely continuous, and the converse will be shown in Proposition 3.32 below.

EX Suppose $F: \mathbb{R} \to \mathbb{C}$ is differentiable with bounded derivative. Then the mean value theorem implies

$$\sum_{j=1}^{n} |F(b_j) - F(a_j)| \le \sum_{j=1}^{n} \sup_{x \in \mathbb{R}} |F'(x)| \, |b_j - a_j|.$$

So given $\varepsilon > 0$, choosing $\delta := \frac{\varepsilon}{\sup |F'|}$ shows F is absolutely continuous. □

Proposition 3.32 For $F \in NBV$, $\mu_F \ll m$ if and only if F is absolutely continuous.

Proof We have already shown the "only if" direction, so suppose F is absolutely continuous. Let $E \subset \mathbb{R}$ be a Borel set with $m(E) = 0$, let $\varepsilon > 0$, and let $\delta > 0$ be as in the definition of absolute continuity. Using the regularity of m and μ_F, we can find $U \supseteq E$ open such that $m(U) < \delta$ and $|\mu_F(U) - \mu_F(E)| \le \varepsilon$. Since it is open, U is a countable union of disjoint open intervals:

$$U = \bigcup_{j=1}^{\infty} (a_j, b_j).$$

Then for $n \in \mathbb{N}$, $\sum_{j=1}^{n} (b_j - a_j) \le m(U) < \delta$ implies $\sum_{j=1}^{n} |F(b_j) - F(a_j)| < \varepsilon$. Letting $n \to \infty$ yields:

$$|\mu_F(U)| \le \sum_{j=1}^{\infty} |F(b_j) - F(a_j)| \le \varepsilon.$$

Hence $|\mu_F(E)| \le |\mu_F(E) - \mu_F(U)| + |\mu_F(U)| \le 2\varepsilon$. Letting $\varepsilon \to 0$ shows $\mu_F(E) = 0$ and so $\mu_F \ll m$. □

Combining the last part of Proposition 3.31 with Proposition 3.32 gives

Corollary 3.33 For $f \in L^1(\mathbb{R}, m)$, $F(x) := \int_{(-\infty, x]} f \, dm \in NBV$ and is absolutely continuous. Conversely, if $F \in NBV$ is absolutely continuous, then $F' \in L^1(\mathbb{R}, m)$ with $F(x) = \int_{(-\infty, x]} F' \, dm$.

If we restrict ourselves to an interval $[a, b] \subset \mathbb{R}$, then the hypothesis that $F \in NBV$ can be dropped in the above corollary. We first need a lemma:

Lemma 3.34 If F is absolutely continuous on $[a, b]$, then $F \in BV([a, b])$.

Proof For $\varepsilon = 1$, let $\delta > 0$ be as in the definition of absolute continuity. Fix $N \in \mathbb{N}$ with $N \ge \frac{b-a}{\delta}$, and let $t_k := a + \frac{k}{N}(b-a)$ for $k = 0, 1, \ldots, N$. Given $a = x_0 < x_1 < \cdots < x_N = b$, the sum $\sum |F(x_j) - F(x_{j+1})|$ is only made larger by adding $t_0, t_1, \ldots, t_N$, and in this case we have

$$\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| = \sum_{k=1}^{N} \sum_{t_{k-1} \leq x_j < t_k} |F(x_j) - F(x_{j-1})| \leq \sum_{k=1}^{N} 1 = N.$$

Thus the total variation of F on $[a,b]$ is bounded by N. $\qquad \square$

Theorem 3.35 (The Fundamental Theorem of Calculus for Lebesgue Integrals)

For $[a,b] \subset \mathbb{R}$ and $F : [a,b] \to \mathbb{C}$, the following are equivalent:

① F is absolutely continuous on $[a,b]$

② There exists $f \in L^1([a,b],m)$ so that
$$F(x) = F(a) + \int_{[a,x]} f \, dm$$

③ F is differentiable m-almost everywhere on $[a,b]$ with $F' \in L^1([a,b],m)$, and $F(x) = F(a) + \int_{[a,x]} F' \, dm$.

Proof ③ $\Rightarrow$ ② is immediate, and ② $\Rightarrow$ ① follows from Corollary 3.33 after extending f to $\mathbb{R}$ by letting $f(t) = 0$ for all $t \notin [a,b]$.

① $\Rightarrow$ ③ : By Lemma 3.34, $F \in BV([a,b])$. And so extending F to $\mathbb{R}$ as in Example ③, we have $F \in BV$. Since F is continuous by virtue of being absolutely continuous, it follows that $F - F(a) \in NBV$. Hence Corollary 3.33 implies $F' = (F - F(a))' \in L^1(\mathbb{R}, m)$ with
$$F(x) - F(a) = \int_{(-\infty, x]} (F - F(a))' \, dm = \int_{(a,x]} F' \, dm.$$
$\qquad \square$

Def Let $F \in NBV$, and let μ_F be the unique regular complex Borel measure satisfying $F(x) = \mu_F((-\infty, x])$. For $g \in L^1(\mathbb{R}, \mu_F)$, we denote
$$\int_{\mathbb{R}} g \, dF := \int_{\mathbb{R}} g \, d\mu_F,$$

and call this a <u>Lebesgue-Stieltjes integral</u> of g with respect to F. $\qquad \square$

The final results of this section show that the Lebesgue Stieltjes integrals satisfy an integration-by-parts formula:

Theorem 3.36 Let $F, G \in NBV$ with at least one of them continuous. For $a, b \in \mathbb{R}$ with $a < b$,
$$\int_{(a,b]} F \, dG + \int_{(a,b]} G \, dF = F(b)G(b) - F(a)G(a)$$

Proof By decomposing F and G into linear combinations of bounded increasing functions (and μ_F and μ_G into positive measures), we may assume F and G are $\mathbb{R}$-valued, bounded, and increasing. Now, without loss of generality, suppose G is continuous. Consider the set
$$\Omega := \{(x,y) \in \mathbb{R}^2 : a < x \leq y \leq b\}.$$
Using Tonelli's theorem (Theorem 2.36), we have
$$\mu_F \times \mu_G(\Omega) = \int_{(a,b]} \int_{(a,b]} 1_\Omega(x,y) \, dF(x) \, dG(y) = \int_{(a,b]} \int_{(a,y]} dF \, dG(y)$$

$$= \int_{(a,b]} F(y) - F(a) \, dG(y) = \int_{(a,b]} F \, dG - F(a)[G(b) - G(a)],$$

and computing the iterated integral in the other order gives:

$$\mu_F \times \mu_G(\Omega) = \int_{(a,b]} \int_{(a,b]} \mathbb{1}_\Omega(x,y) \, dG(y) \, dF(x) = \int_{(a,b]} \int_{(x,b]} dG \, dF(x)$$

$$= \int_{(a,b]} G(b) - G(x) \, dF(x) = G(b)[F(b) - F(a)] - \int_{(a,b]} G \, dF,$$

where we have used the continuity of G to assert $G(x^-) = G(x)$. Subtracting these two computations and rearranging terms yields the claimed equality. $\square$

Cor 3.37 If F and G are absolutely continuous on $[a,b]$, then

$$\int_{(a,b]} F G' + F' G \, dm = F(b) G(b) - F(a) G(a)$$

Proof First observe that

$$\int_{(a,b]} (F - F(a)) (G - G(a))' + (F - F(a))' (G - G(a)) \, dm = \int_{(a,b]} F G' - F(a) G' + F' G - F' G(a) \, dm$$

$$= \int_{(a,b]} F G' + F' G \, dm - F(a)(G(b) - G(a)) - (F(b) - F(a)) G(a)$$

and

$$(F(b) - F(a)) (G(b) - G(a)) = F(b) G(b) - F(b) G(a) - F(a)(G(b) - G(a))$$

$$= F(b) G(b) - F(a) G(a) - (F(b) - F(a)) G(a) - F(a)(G(b) - G(a))$$

So it suffices to assume $F(a) = G(a) = 0$. Lemma 3.34 implies $F, G \in BV([a,b])$ and extending them to $\mathbb{R}$ as in Example ③ we have $F, G \in NBV$. Since both are continuous, Theorem 3.36 implies

$$F(b) G(b) = \int_{(a,b]} F \, dG + \int_{(a,b]} G \, dF = \int_{(a,b]} F G' + G F' \, dm,$$

where we have used $F' = \frac{d\mu_F}{dm}$ and $G' = \frac{d\mu_G}{dm}$. $\square$