Exercises: (Section 3.5)

1. Define
$$F(x) := \begin{cases} x^2\sin(\frac{1}{x}) & \text{if } x\neq 0 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad G(x) := \begin{cases} x^2\sin(\frac{1}{x^2}) & \text{if } x\neq 0 \\ 0 & \text{otherwise} \end{cases}.$$
   (a) Compute F' and G'.
   (b) Show that $F \in BV([-1,1])$ but $G \notin BV([-1,1])$.

2. Suppose $(F_n)_{n\in\mathbb{N}} \subset BV$ converges pointwise to a function $F\in BV$. Show that $T_F \leq \liminf_{n\to\infty} T_{F_n}$.

3. For $F\in BV$ define
$$\|F\|_{BV} := |F(0)| + T_F(\infty).$$
   (a) Show that $\|\cdot\|_{BV}$ defines a norm on BV.
   (b) Show that BV is complete with respect to the metric $\|F-G\|_{BV}$.
   (c) Show that for any $x_0\in\mathbb{R}$ one has
$$\frac{1}{2}\|F\|_{BV} \leq |F(x_0)| + T_F(\infty) \leq 2\|F\|_{BV}$$
   for all $F\in BV$.

4. Suppose $F,G\colon [a,b]\to\mathbb{C}$ are absolutely continuous functions and that $G(x)\neq 0$ for all $x\in[a,b]$. Show that the quotient $\frac{F}{G}$ is absolutely continuous.

5. Let $G\colon [a,b]\to[c,d]$ be a continuous increasing surjection.
   (a) For a Borel set $E\subset[c,d]$, show that $m(E) = \mu_G(G^{-1}(E))$.
   [**Hint:** first consider when E is an open then closed.]
   (b) For $f\in L^1([c,d],\mathcal{B}_{[c,d]},m)$, show that
$$\int_{[c,d]} f\ dm = \int_{[a,b]} f\circ G\ d\mu_G.$$
   (c) Suppose G is absolutely continuous. Show that the above integrals also equal $\int_{[a,b]}(f\circ G)G'\ dm$.

6. [1] For $(a,b)\subset\mathbb{R}$ (possibly equal), we say a function $F\colon (a,b)\to\mathbb{R}$ is **convex** if
$$F(\lambda s + (1-\lambda)t) \leq \lambda F(s) + (1-\lambda)F(t)$$
   for all $s,t\in(a,b)$ and $\lambda\in(0,1)$.
   (a) Show that F is convex if and only if for all $s,s',t,t'\in(a,b)$ satisfying $s\leq s'<t'$ and $s<t\leq t'$ one has
$$\frac{F(t)-F(s)}{t-s} \leq \frac{F(t')-F(s')}{t'-s'}.$$
   (b) Show that F is convex if and only if F is absolutely continuous on every compact subinterval of (a,b) and F' is increasing on the set where it is defined.
   (c) For convex F and $t_0\in(a,b)$, show that there exists $\beta\in\mathbb{R}$ satisfying $F(t)-F(t_0)\geq\beta(t-t_0)$ for all $t\in(a,b)$.

[1] Not collected

(d) (**Jensen's Inequality**) Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) = 1$. Suppose $g \in L^1(X, \mu)$ is valued in (a, b) and F is convex on this interval. Show that

$$F\left(\int_X g\ d\mu\right) \leq \int_X F \circ g\ d\mu.$$

[**Hint:** use part (c) with $t_0 = \int g\ d\mu$ and $t = g(x)$.]

Solutions:

1. (a) For $x \neq 0$, we have the following from calculus:

$$F'(x) = 2x\sin(\frac{1}{x}) - \cos(\frac{1}{x})$$
$$G'(x) = 2x\sin(\frac{1}{x^2}) - \frac{1}{x}\cos(\frac{1}{x^2}).$$

For $x = 0$, the we have

$$\lim_{h\to 0}\left|\frac{F(h) - F(0)}{h}\right| = \lim_{h\to 0}\left|\frac{h^2\sin(\frac{1}{h}) - 0}{h}\right| \leq \lim_{h\to 0}|h| = 0$$
$$\lim_{h\to 0}\left|\frac{G(h) - G(0)}{h}\right| = \lim_{h\to 0}\left|\frac{h^2\sin(\frac{1}{h^2}) - 0}{h}\right| \leq \lim_{h\to 0}|h| = 0$$

Hence $F'(0) = G'(0) = 0$. □

(b) Since F' is bounded on $[-1, 1]$, $F \in BV([-1, 1])$ by an example from lecture. To see that $G \notin BV([-1, 1])$, for each integer $j \geq 0$ let $x_j \in [0, 1]$ be such that

$$\frac{1}{x_j^2} = \frac{(2j+1)\pi}{2}.$$

Then $1 \geq x_0 > x_1 > x_2 > \cdots$, and $\sin(\frac{1}{x_j^2}) = \pm 1$ if j is odd or even, respectively. Consequently, the total variation of G on $[-1, 1]$ is bounded below by

$$\sum_{j=0}^{N}|G(x_j) - G(x_{j-1})| = \sum_{j=0}^{N}\frac{1}{x_j^2} - \frac{1}{x_{j-1}^2} = \frac{1}{x_N^2} - \frac{1}{x_0^2} = \frac{(2N+1)\pi}{2} - \frac{\pi}{2},$$

for each $N \in \mathbb{N}$. □

2. For $x \in \mathbb{R}$ and $\epsilon > 0$, let $-\infty < x_0 < x_1 < \cdots < x_m = x$ such that

$$T_F(x) \leq \sum_{j=1}^{m}|F(x_j) - F(x_{j-1})| + \epsilon.$$

Using the pointwise convergence, we can find $N \in \mathbb{N}$ so that

$$|[F(x_j) - F(x_{j-1})] - [F_n(x_j) - F_n(x_{j-1})]| \leq |F(x_j) - F_n(x_j)| + |F(x_{j-1}) - F_n(x_{j-1})| < \frac{\epsilon}{n}$$

for each $j = 1, \ldots, m$ and all $n \geq N$. We therefore have

$$T_F(x) \leq \sum_{j=1}^{m}|F_n(x_j) - F_n(x_{j-1})| + 2\epsilon \leq T_{F_n}(x) + 2\epsilon$$

for all $n \geq N$. Hence $T_F(x) \leq \liminf_{n\to\infty} T_{F_n}(x) + 2\epsilon$. Since $x \in \mathbb{R}$ and $\epsilon > 0$ were arbitrary, the claimed inequality holds. □

3. (a) For $F, G \in BV$, recall from lecture that $T_{F+G} \leq T_F + T_G$. Using this and the triangle inequality for $\mathbb{C}$ one has

$$\|F+G\|_{BV} = |F(0)+G(0)| + T_{F+G}(\infty) \leq |F(0)| + |G(0)| + T_F(\infty) + T_G(\infty) \leq \|F\|_{BV} + \|G\|_{BV}.$$

Next for $\alpha \in \mathbb{C}$ we observe that for any $-\infty < x_0 < x_1 < \cdots < x_n$ one has

$$\sum_{j=1}^{n} |\alpha F(x_j) - \alpha F(x_{j-1})| = |\alpha| \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})|.$$

It follows that $T_{\alpha F} = |\alpha| T_F$ and hence

$$\|\alpha F\|_{BV} = |\alpha F(0)| + T_{\alpha F}(\infty) = |\alpha|(|F(0)| + T_F(\infty)) = |\alpha| \|F\|_{BV}.$$

Finally, if $F = 0$ then $\|F\|_{BV} = 0$ is clear, and on the other hand $\|F\|_{BV} = 0$ implies $F(0) = 0$ and $T_F(\infty) = 0$. Since T_F is increasing it must be that $T_F \equiv 0$ and therefore F is constant. But then $F(0) = 0$ yields $F \equiv 0$. Thus $\| \cdot \|_{BV}$ is norm on BV. □

(b) Suppose $(F_n)_{n \in \mathbb{N}} \subset BV$ is Cauchy with respect to this norm. For any $x \in \mathbb{R}$, one has

$$\begin{aligned} |(F_n - F_m)(x)| &\leq |(F_n - F_m)(0)| + |(F_n - F_m)(x) - (F_n - F_m)(0)| \\ &\leq |(F_n - F_m)(0)| + T_{F_n - F_m}(\infty) = \|F_n - F_m\|_{BV}. \end{aligned}$$

Consequently, $(F_n(x))_{n \in \mathbb{N}} \subset \mathbb{C}$ is a Cauchy sequence so that we may define

$$F(x) := \lim_{n \to \infty} F_n(x)$$

for each $x \in \mathbb{R}$. Let $\epsilon > 0$ and let $N \in \mathbb{N}$ be large enough so that $\|F_n - F_m\|_{BV} < \epsilon$ for all $n, m \geq N$. We claim that $\|F - F_n\|_{BV} \leq 2\epsilon$ for all $n \in \mathbb{N}$, which we also note implies $F \in BV$ since $T_F(\infty) \leq \|F\|_{BV} \leq \|F - F_n\|_{BV} + \|F_n\|_{BV} < \infty$. For $-\infty < x_0 < x_1 < \cdots < x_d < \infty$, the definition of F allows us to find $m \in \mathbb{N}$ large enough so that

$$|(F - F_m)(x_j) - (F - F_m)(x_{j-1})| \leq |F(x_j) - F_m(x_j)| + |F(x_{j-1}) - F_m(x_{j-1})| < \frac{\epsilon}{d+1}.$$

Increasing m if necessary, we can also ensure $|(F - F_m)(0)| < \frac{\epsilon}{d+1}$ and $m \geq N$. For $n \geq N$, one then has

$$\begin{aligned} |(F - F_n)(0)| &+ \sum_{j=1}^{d} |(F - F_n)(x_j) - (F - F_n)(x_{j-1})| \\ &\leq |(F - F_m)(0)| + \sum_{j=1}^{d} |(F - F_m)(x_j) - (F - F_m)(x_{j-1})| \\ &\quad + |(F_m - F_n)(0)| + \sum_{j=1}^{d} |(F_m - F_n)(x_j) - (F_m - F_n)(x_{j-1})| \\ &\leq \frac{\epsilon}{d+1} + \sum_{j=1}^{d} \frac{\epsilon}{d+1} + \|F_m - F_n\|_{BV} < 2\epsilon. \end{aligned}$$

Taking the supremum over all $-\infty < x_0 < x_1 < \cdots < x_d < \infty$ then yields $\|F - F_n\|_{BV} \leq 2\epsilon$. □

(c) This follows from $|F(0) - F(x_0)| \leq T_F(\max\{0, x_0\}) \leq T_F(\infty)$. Indeed, one then has

$$|F(x_0)| + T_F(\infty) \leq |F(0)| + |F(x_0) - F(0)| + T_F(\infty) \leq |F(0)| + 2T_F(\infty) \leq 2\|F\|_{BV}.$$

The other inequality is similar. □

4. Since F and G are in particular continuous on the compact set $[a,b]$, we have

$$R := \sup_{a\leq t\leq b} |F(t)| < \infty \qquad \text{and} \qquad r := \inf_{a\leq t\leq b} |G(t)| = \min_{a\leq t\leq b} |G(t)| > 0.$$

Given $\epsilon > 0$ let $\delta_F > 0$ be as in the definition of absolute continuity for F corresponding to $\frac{r\epsilon}{2}$, and let $\delta_G > 0$ be as in the definition of absolute continuity for G corresponding to $\frac{r^2\epsilon}{2R}$. Set $\delta := \min\{\delta_F, \delta_G\}$. Then if $(a_1,b_1),\ldots,(a_n,b_n) \subset \mathbb{R}$ are disjoint intervals satisfying

$$\sum_{j=1}^n (b_j - a_j) < \delta,$$

then one has

$$\begin{aligned}\sum_{j=1}^n \left|\frac{F(b_j)}{G(b_j)} - \frac{F(a_j)}{G(a_j)}\right| &\leq \sum_{j=1}^n \left|\frac{F(b_j)-F(a_j)}{G(b_j)}\right| + \left|F(a_j)\frac{G(a_j)-G(b_j)}{G(b_j)G(a_j)}\right| \\ &\leq \frac{1}{r}\sum_{j=1}^n |F(b_j)-F(a_j)| + \frac{R}{r^2}\sum_{j=1}^n |G(b_j)-G(a_j)| < \frac{1}{r}\frac{r\epsilon}{2} + \frac{R}{r^2}\frac{r^2\epsilon}{2R} = \epsilon.\end{aligned}$$

Hence $\frac{F}{G}$ is absolutely continuous. □

5. (a) First suppose $E = (x,y)$ is an interval. Then $G^{-1}(x,y)$ is open by the continuity of G and connected since G is increasing. Hence $G^{-1}(x,y) = (s,t)$ for some $s,t \in [a,b]$. These properties of G also imply

$$G(s) = \inf_{r>s} G(r) \geq x$$
$$G(t) = \sup_{r<t} G(t) \leq y.$$

If these inequalities were strict then $G((s,t))$ would be a strict subinterval of (x,y), contradicting the surjectivity of G. Hence $G(s) = x$ and $G(t) = y$, and therefore

$$\mu_G(G^{-1}(x,y)) = \mu_G((s,t)) = G(t) - G(s) = y - x = m((x,y)).$$

It follows that $m(U) = \mu_G(G^{-1}(U))$ for any open $U \subset [c,d]$, since we can write U as a disjoint union of countably many open intervals. Next, if $V \subset [c,d]$ is closed, then $U := [c,d] \setminus V$ is open and

$$[a,b] \setminus G^{-1}(U) = G^{-1}(V).$$

Thus

$$m(V) = d - c - m(U) = G(b) - G(a) - \mu_G(G^{-1}(U)) = \mu_G([a,b]\setminus G^{-1}(U)) = \mu_G(G^{-1}(V)).$$

Now, for a Borel set $E \subset [c,d]$ and $\epsilon > 0$ the regularity of m allows us to find $K \subset E$ compact and $U \supset E$ open so that $mu(U) - \epsilon \leq m(E) \leq m(K) + \epsilon$. Since $G^{-1}(K) \subset G^{-1}(E) \subset G^{-1}(U)$ and K is in particular closed, we have

$$\mu_G(G^{-1}(E)) \leq \mu_G(G^{-1}(U)) = m(U) \leq m(E) + \epsilon$$

and

$$\mu_G(G^{-1}(E)) \geq \mu_G(G^{-1}(K)) = m(K) \geq m(E) - \epsilon.$$

Letting $\epsilon \to 0$ yields $\mu_G(G^{-1}(E)) = m(E)$. □

(b) First suppose f is a simple function with standard representation $\sum_{j=1}^n \alpha_n 1_{E_j}$. Then by part (a) we have

$$\int_{[c,d]} f\ dm = \sum_{j=1}^n \alpha_j m(E_j) = \sum_{j=1}^n \alpha_j \mu_G(G^{-1}(E_j)) = \int_{[c,d]} \sum_{j=1}^n \alpha_j 1_{G^{-1}(E_j)}\ d\mu_G.$$

Note that $1_{E_j} \circ G(x) = 1$ iff $G(x) \in E_j$ iff $x \in G^{-1}(E_j)$ iff $1_{G^{-1}(E_j)}(x) = 1$. Hence the integrand in the above integral is actually $f \circ G$. For general f, we can approximate it pointwise by simple functions dominated by $|f|$ and use the dominated convergence theorem. □

(c) Absolute continuity of G implies $\mu_G \ll m$, and we have seen in lecture that in this case $\frac{d\mu_G}{dm} = G'$ m-almost everywhere. Hence we have

$$\int_{[a,b]} (f \circ G)G' \; dm = \int_{[a,b]} (f \circ G) \frac{d\mu_G}{dm} \; dm = \int_{[a,b]} f \circ G \; d\mu_G.$$

□

6. (a) ($\Longrightarrow$) : Suppose F is convex. We first consider the case when $t' = t$ and $s \leq s' < t$. Then $\lambda := \frac{t-s'}{t-s} \in (0,1)$ and

$$\lambda s + (1-\lambda)t = \lambda(s-t) + t = s' - t + t = s'.$$

Thus by convexity we have

$$\frac{F(t) - F(s')}{t - s'} = \frac{F(t) - F(\lambda s + (1-\lambda)t)}{\lambda(t-s)} \geq \frac{F(t) - \lambda F(s) - (1-\lambda)F(t)}{\lambda(t-s)} = \frac{F(t) - F(s)}{t-s}.$$

Next we consider the case when $s = s'$ and $s' < t \leq t'$. Then $\lambda := \frac{t-s'}{t'-s'} \in (0,1)$ and

$$\lambda t' + (1-\lambda)s' = \lambda(t' - s') + s' = t - s' + s' = t.$$

Thus by convexity we have

$$\frac{F(t) - F(s')}{t - s'} = \frac{F(\lambda t' + (1-\lambda)s') - F(s')}{\lambda(t'-s')} \leq \frac{\lambda F(t') + (1-\lambda)F(s') - F(s')}{\lambda(t'-s')} = \frac{F(t') - F(s')}{t'-s'}.$$

For the general case we combine these two special cases to get:

$$\frac{F(t) - F(s)}{t-s} \leq \frac{F(t') - F(s)}{t'-s} \leq \frac{F(t') - F(s')}{t'-s'}.$$

($\Longleftarrow$) : Given $\lambda \in (0,1)$ let $s' := \lambda s + (1-\lambda)t$. Then $s < s' < t$ and

$$\frac{t - s'}{t-s} = \frac{t - \lambda s - (1-\lambda)t}{t-s} = \lambda.$$

The assumed property therefore implies

$$\begin{aligned} F(\lambda s + (1-\lambda)t) &= -[F(t) - F(s')] + F(t) = -(t-s')\frac{F(t) - F(s')}{t-s'} + F(t) \\ &\leq -(t-s')\frac{F(t) - F(s)}{t-s} + F(t) = -\lambda(F(t) - F(s)) + F(t) = \lambda F(s) + (1-\lambda)F(t). \end{aligned}$$

Hence F is convex. □

(b) ($\Longrightarrow$) : Suppose F is convex. Fix a compact subinterval $[c,d] \subset (a,b)$ (i.e. just a bounded closed interval), and let $\rho > 0$ be such that $[c-\rho, d+\rho] \subset (a,b)$. For $c \leq s < t \leq d$, part (a) implies

$$\frac{F(c) - F(c-\rho)}{\rho} \leq \frac{F(t) - F(s)}{t-s} \leq \frac{F(d+\rho) - F(d)}{\rho}.$$

Thus for

$$M := \rho^{-1} \max\{|F(c) - F(c-\rho)|, |F(d+\rho) - F(d)\},$$

we have $|F(t) - F(s)| \leq M|t-s|$ and so F is absolutely continuous by letting $\delta := \frac{\epsilon}{M}$ for any $\epsilon > 0$. Moreover, if $F'(s)$ and $F'(t)$ exist then for $\epsilon > 0$ let $s' < t$ and $t' > s$ be such that

$$\left| \frac{F(s) - F(s')}{s - s'} - F'(s) \right| < \epsilon$$
$$\left| \frac{F(t) - F(t')}{t - t'} - F'(t) \right| < \epsilon.$$

Then using part (a) again we have

$$F'(s) < \frac{F(s) - F(s')}{s - s'} + \epsilon \leq \frac{F(t) - F(t')}{t - t'} + \epsilon < F'(t) + 2\epsilon.$$

Hence $F'(s) \leq F'(t)$ and F' is increasing.

($\Longleftarrow$) : Let $s, s', t, t' \in (a, b)$ satisfy $s \leq s' < t$ and $s' < t \leq t'$. By assumption F is absolutely continuous on the compact subinterval $[s, t'] \subset (a, b)$, and hence $F' \in L^1([s, t'], dm)$ by the fundamental theorem of calculus for Lebesgue integrals (Theorem 3.35 from lecture). Consider $G\colon [s, t] \to [s', t']$ defined by

$$G(x) = \frac{t' - s'}{t - s}(x - s) + s',$$

which is continuous, increasing (since $\frac{t'-s'}{t-s} > 0$), and a surjection. In fact, G is absolutely continuous on $[s, t]$ (since $|G'(x)| = \frac{t'-s'}{t-s}$ is uniformly bounded) and so by Exercise 5.(c) we have

$$\int_{[s',t']} F' \; dm = \int_{[s,t]} (F' \circ G) G' \; dm = \int_{[s,t]} F' \left(\frac{t' - s'}{t - s}(x - s) + s' \right) \frac{t' - s'}{t - s} \; dm(x).$$

Using this and the formula from Theorem 3.35 we have

$$\frac{F(t') - F(s')}{t' - s'} = \frac{1}{t' - s'} \int_{[s',t']} F' \; dm = \frac{1}{t - s} \int_{[s,t]} F' \left(\frac{t' - s'}{t - s}(x - s) + s' \right) \; dm(x).$$

Now, we claim that $\frac{t'-s'}{t-s}(x - s) + s' \geq x$ holds on $[s, t]$. Indeed, at $x = s$ it reduces to $s' \geq s$ and at $x = t$ it reduces to $t' \geq t$. Thus the inequality holds on $[s, t]$ since both sides are linear. Therefore we can continue the above computation using the fact that F' is increasing:

$$\frac{F(t') - F(s')}{t' - s'} \geq \frac{1}{t - s} \int_{[s,t]} F'(x) \; dm(x) = \frac{F(t) - F(s)}{t - s},$$

where the last equality follows from Theorem 3.35 again. So by part (a), we have that F is convex. □

(c) By part (b), F' exists almost everywhere on (a, b) and is increasing where it is defined, so we can choose $\beta \in \mathbb{R}$ satisfying

$$\sup\{F'(s) \colon s \leq t_0\} \leq \beta \leq \inf\{F'(t) \colon t \geq t_0\}.$$

Now, for $t = t_0$ the inequality is immediate. For $t > t_0$, using Theorem 3.35 we have

$$F(t) - F(t_0) = \int_{[t_0,t]} F' dm \geq \int_{[t_0,t]} \beta \; dm = \beta(t - t_0).$$

For $t < t_0$ we have

$$F(t_0) - F(t) = \int_{[t,t_0]} F' \; dm \leq \int_{[t,t_0]} \beta \; dm = \beta(t_0 - t_0),$$

and multiplying by negative one yields $F(t) - F(t_0) \geq \beta(t - t_0)$. □

(d) Following the hint we set $t_0 := \int g\, d\mu$ and let β be as in part (c). Then for $t = g(x)$ we have

$$F \circ g(x) - F\left(\int_X g\ d\mu\right) \geq \beta\left(g(x) - \int_X g d\mu\right).$$

Integrating with respect to x (and using $\mu(X) = 1$ so that $\int_X c\ d\mu = c$ for a constant $c \in \mathbb{C}$) yields

$$\int_X F \circ g\ d\mu - F\left(\int_X g\ d\mu\right) \geq \beta\left(\int_X g\ d\mu - \int_X g\ d\mu\right) = 0.$$

□