Exercises: (Chapter 1.1-2)

1. Consider $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} \frac{1}{n} & \text{if } t = \frac{m}{n} \text{ with } m \in \mathbb{Z},\ n \in \mathbb{N} \text{ sharing no common factors} \\ 0 & \text{if } t \in \mathbb{R} \setminus \mathbb{Q} \end{cases}.$$

   (a) Show that f is discontinuous at every $t \in \mathbb{Q}$.
   (b) Show that f is continuous at every $t \in \mathbb{R} \setminus \mathbb{Q}$.
   (c) Show that $1_{\mathbb{Q}}$ is discontinuous at every $t \in \mathbb{R}$.

2. Show that if $E \subset \mathbb{R}$ is countable then E is a null set.

3. Let X be a set and let $(E_n)_{n\in\mathbb{N}}$ be a sequence of subsets. Recall that the **limit inferior** and **limit superior** of this sequence of sets are defined as

$$\liminf E_n := \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k \qquad \text{and} \qquad \limsup E_n := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k,$$

respectively. Show that for all $x \in X$,

$$1_{\liminf E_n}(x) = \liminf_{n\to\infty} 1_{E_n}(x) \qquad \text{and} \qquad 1_{\limsup E_n}(x) = \limsup_{n\to\infty} 1_{E_n}(x).$$

4. Let X be an uncountable set. Show that

$$\mathcal{C} := \{E \subset X \colon E \text{ or } E^c \text{ is countable}\}$$

is a σ-algebra on X.

5. Let $\mathcal{B}_{\mathbb{R}}$ be the Borel σ-algebra on $\mathbb{R}$, and consider the following collections of subsets of $\mathbb{R}$:

$$\mathcal{E}_1 := \{(a,b) \colon a,b \in \mathbb{R},\ a < b\}$$
$$\mathcal{E}_2 := \{[a,\infty) \colon a \in \mathbb{R}\}.$$

Show that $\mathcal{M}(\mathcal{E}_1) = \mathcal{M}(\mathcal{E}_2) = \mathcal{B}_{\mathbb{R}}$.

Solutions:

1. (a) Fix $t = \frac{m}{n} \in \mathbb{Q}$ and let $\epsilon := \frac{1}{n}$. For any $\delta > 0$, the density of the irrationals implies there exists $s \in (t-\delta, t+\delta) \setminus \mathbb{Q}$. Hence $|t-s| < \delta$, but

$$|f(t) - f(s)| = \frac{1}{n} = \epsilon.$$

Hence f is discontinuous at t. □

   (b) Fix $t \in \mathbb{R} \setminus \mathbb{Q}$ and let $\epsilon > 0$. Let $M \in \mathbb{N}$ satisfy $M \geq \frac{1}{\epsilon}$. Note that for each $m \in \{1, \ldots, M\}$, there are only finitely many $n \in \mathbb{Z}$ satisfying $\frac{n}{m} \in [t-1, t+1]$ (since this requires $m(t-1) \leq n \leq m(t+1)$). Thus the following is a *finite* set:

$$F := \left\{ \frac{n}{m} \in [t-1, t+1] \mid 1 \leq m \leq M,\ n \in \mathbb{Z} \right\}.$$

Since $F \subset \mathbb{Q}$, we know $t \notin F$ and therefore

$$\delta := \min_{s\in F} |t - s| > 0.$$

Note that $\delta < 1$ since there exists $n \in \mathbb{Z} \cap F$ with $n \leq t < n+1$.

Now, we claim that if $s \in \mathbb{R}$ satisfies $|t-s| < \delta$, then $|f(t) - f(s)| < \epsilon$. Since $f(t) = 0$, if $s \in \mathbb{R} \setminus \mathbb{Q}$ then $|f(t) - f(s)| = |0 - 0| = 0 < \epsilon$. So now assume $s \in \mathbb{Q}$, and say $s = \frac{n}{m}$ for $n \in \mathbb{Z}$ and $m \in \mathbb{N}$ with no common factors. If $m > M$, then we have

$$|f(t) - f(s)| = |0 - \frac{1}{m}| = \frac{1}{m} < \frac{1}{M} \leq \epsilon,$$

as needed. If $m \leq M$, then since $\delta < 1$ we have $s \in [t-1, t+1]$ and hence $s \in F$. But then $|t-s| < \delta \leq |t-s|$ is a contradiction. Thus we cannot have $m \leq M$ and so in all cases we have have shown $|f(t) - f(s)| < \epsilon$. □

(c) Fix $t \in \mathbb{R}$ and let $\epsilon = 1$. Since both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$, for any $\delta > 0$ we can find $s \in (t-\delta, t+\delta)$ that is either rational if t is irrational or irrational if t is rational. In either case we have $|t-s| < \delta$ while

$$|f(t) - f(s)| = 1 = \epsilon.$$

Hence f is discontinuous at t. □

2. Let $\epsilon > 0$ and let $E = \{t_n : n \in \mathbb{N}\}$ be an enumeration of E. For each $n \in \mathbb{N}$, define

$$a_n := t_n - \frac{\epsilon}{2^{n+1}} \qquad b_n := t_n + \frac{\epsilon}{2^{n+1}}$$

so that $t_n \in (a_n, b_n)$ and $b_n - a_n = \frac{2\epsilon}{2^{n+1}} = \frac{\epsilon}{2^n}$. Then

$$E \subset \bigcup_{n \in \mathbb{N}} (a_n, b_n)$$

and

$$\sum_{n=1}^{\infty} b_n - a_n = \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon \sum_{n=1}^{\infty} 2^{-n} = \epsilon.$$

Thus E is a null set. □

3. First suppose $x \in \liminf E_n$. By definition, this means there exists $n_0 \geq 1$ so that $x \in \bigcap_{k=n_0}^{\infty} E_k$. Consequently, $1_{E_k}(x) = 1$ for all $k \geq n_0$. Hence

$$\liminf_{n \to \infty} 1_{E_n}(x) = \sup_{n \in \mathbb{N}} \inf_{k \geq n} 1_{E_k}(x) \geq \inf_{k \geq n_0} 1_{E_k}(x) = 1.$$

Since $1_{E_n}(x) \leq 1$ for all n, it follows that $\liminf 1_{E_n}(x) = 1 = 1_{\liminf E_n}(x)$. Next, suppose $x \notin \liminf E_n$. Again by definition we have for all $n \geq 1$ that $x \notin \bigcap_{k=n}^{\infty} E_k$; that is, for all $n \geq 1$ there exists $k(n) \geq n$ so that $x \notin E_{k(n)}$. Thus

$$\liminf_{n \to \infty} 1_{E_n}(x) = \sup_{n \in \mathbb{N}} \inf_{k \geq n} 1_{E_k}(x) \leq \sup_{n \in \mathbb{N}} 1_{E_{k(n)}}(x) = 0.$$

Since $1_{E_n}(x) \geq 0$ for all n, it follows that $\liminf 1_{E_n}(x) = 0 = 1_{\liminf E_n}(x)$. This establishes the first equality.

To prove the second equality, one can proceed among more or less the same lines as above. Alternatively, observe that

$$(\limsup E_n)^c = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k^c = \liminf E_n^c,$$

and so

$$1 - 1_{\limsup E_n}(x) = 1_{(\limsup E_n)^c}(x) = 1_{\liminf E_n^c}(x) = \liminf_{n \to \infty} 1_{E_n^c}(x)$$

by the first part of the proof. Hence

$$1_{\limsup E_n}(x) = 1 - \liminf_{n \to \infty} 1_{E_n^c}(x) = \limsup_{n \to \infty} (1 - 1_{E_n^c}(x) = \limsup_{n \to \infty} 1_{E_n}(x).$$

□

4. First observe that $\mathcal{C}$ is nonempty since $X^c = \emptyset$ implies $X \in \mathcal{C}$. Since the definition of $\mathcal{C}$ is symmetric with respect to E and E^c, it follows that $\mathcal{C}$ is closed under taking complements. Finally, suppose $E_1, E_2, \ldots \in \mathcal{C}$. If each E_n is countable, then so is their union and hence

$$\bigcup_{n=1}^\infty E_n \in \mathcal{C}.$$

Otherwise, at least one E_n is not countable and therefore E_n^c must be countable. But then

$$\left(\bigcup_{n=1}^\infty E_n\right)^c = \bigcap_{n=1}^\infty E_n^c \subset E_n^c,$$

and so the complement of the union is countable. Hence the union belongs to $\mathcal{C}$, which is therefore a σ-algebra. □

5. We will use Lemma 1.1 to show the inclusions

$$\mathcal{M}(\mathcal{E}_1) \subset \mathcal{M}(\mathcal{E}_2) \subset \mathcal{B}_\mathbb{R} \subset \mathcal{M}(\mathcal{E}_1).$$

Since σ-algebras are closed under complements, we have $(-\infty, a) \subset \mathcal{M}(\mathcal{E}_2)$ for all $a \in \mathbb{R}$. Since σ-algebras are also closed under countable intersections, we have

$$[a, b) = [a, \infty) \cap (-\infty, b) \in \mathcal{M}(\mathcal{E}_2)$$

for all $a < b$. Observe that

$$(a, b) = \bigcup_{n=1}^\infty [a + \frac{1}{n}, b).$$

Indeed, the union is clearly contained in the interval, and if $a < x < b$ then there exists a sufficiently large $n \in \mathbb{N}$ so that $a + \frac{1}{n} \leq x$. Thus we have $\mathcal{E}_1 \subset \mathcal{M}(\mathcal{E}_2)$ since σ-algebras are closed under countable unions. Lemma 1.1 then gives the first of our claimed inclusions.

Next, we note that $\mathcal{B}_\mathbb{R}$ contains all closed subsets of $\mathbb{R}$ since it contains all open subsets and is closed under taking complements. Consequently, $\mathcal{E}_2 \subset \mathcal{B}_\mathbb{R}$ and Lemma 1.1 yields the second of our claimed inclusions.

Finally, we claim every open subset $U \subset \mathbb{R}$ is a *countable* union of open intervals, in which case $U \in \mathcal{M}(\mathcal{E}_1)$. Since $\mathcal{B}_\mathbb{R}$ is the σ-algebra generated by the open subsets of $\mathbb{R}$, Lemma 1.1 will give us the last of our claimed inclusions. Indeed, for every $x \in U$ there exists an open interval satisfying $x \in (a_x, b_x) \subset U$. Enumerate $\mathbb{Q} \cap U = \{q_n : n \in \mathbb{N}\}$. The density of the rationals implies each (a_x, b_x) contains at least one rational number (in fact they will all contain an infinite number of them), and so

$$U = \bigcup_{x \in U} (a_x, b_x) = \bigcup_{n=1}^\infty \bigcup_{x\colon (a_x, b_x) \ni q_n} (a_x, b_x)$$

For each $n \in \mathbb{N}$, denote

$$U_n := \bigcup_{x\colon (a_x, b_x) \ni q_n} (a_x, b_x).$$

Then U_n is open as a union of open sets, and is connected as the union of connected sets with a common point (namely q_n). Therefore $U_n = (a_n, b_n)$ is an interval, and U is a countable union of intervals. □