Exercises: (Chapter 1.2-3)

1. Fix $n \in \mathbb{N}$ and denote the Borel σ-algebra on $\mathbb{R}^n$ by $\mathcal{B}$.

   (a) Show that $\mathcal{B}$ is generated by the collection of *open boxes*

   $$(a_1, b_1) \times \cdots \times (a_n, b_n)$$

   for $a_1, b_1, \ldots, a_n, b_n \in \mathbb{R}$ with $a_1 < b_1, \ldots a_n < b_n$.

   (b) Fix $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n$. For a Borel set $E \subset \mathbb{R}^n$, show that

   $$E + \mathbf{t} := \{(x_1 + t_1, \ldots, x_n + t_n) \colon (x_1, \ldots, x_n) \in E\}$$

   is also a Borel set. We say $\mathcal{B}$ is **translation invariant**.
   [**Hint:** Consider the collection $\{E \in \mathcal{B} \colon E + \mathbf{t} \in \mathcal{B}\}$.]

   (c) Fix $\mathbf{t} = (t_1, \ldots, t_n) \in (\mathbb{R} \setminus \{0\})^n$. For a Borel set $E \subset \mathbb{R}^n$, show that

   $$E \cdot \mathbf{t} := \{(x_1 t_1, \ldots, x_n t_n) \colon (x_1, \ldots, x_n) \in E\}$$

   is also a Borel set. We say $\mathcal{B}$ is **dilation invariant**.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space with $E_n \in \mathcal{M}$ for each $n \in \mathbb{N}$.

   (a) Show that
   $$\mu(\liminf E_n) \leq \liminf_{n \to \infty} \mu(E_n).$$

   (b) Suppose $\mu(\bigcup_n E_n) < \infty$. Show that

   $$\mu(\limsup E_n) \geq \limsup_{n \to \infty} \mu(E_n).$$

3. Let μ be a finitely additive measure on a measurable space $(X, \mathcal{M})$.

   (a) Show that μ is a measure if and only if it satisfies continuity from below.

   (b) If $\mu(X) < \infty$, show that μ is a measure if and only if it satisfies continuity from above.

4. Let $(X, \mathcal{M}, \mu)$ be a measure space.

   (a) Suppose μ is σ-finite. Show that μ is semifinite.

   (b) Suppose μ is semifinite. Show that for $E \in \mathcal{M}$ with $\mu(E) = \infty$ and any $C > 0$, there exists $F \subset E$ with $C < \mu(F) < \infty$.

5. Let $(X, \mathcal{M}, \mu)$ be a measure space and for $E \in \mathcal{M}$ define

   $$\mu_0(E) := \sup\{\mu(F) \colon F \subset E \text{ with } \mu(F) < \infty\}.$$

   We call μ_0 the **seminfinite part** of μ.

   (a) Show that μ_0 is a semifinite measure.

   (b) Show that if μ is itself semifinite, then $\mu = \mu_0$.

   (c) [1] We say $E \in \mathcal{M}$ is **μ-semifinite** if for any $F \subset E$ with $\mu(F) = \infty$ there exists $G \subset F$ with $0 < \mu(G) < \infty$. Show that

   $$\nu(E) := \begin{cases} 0 & \text{if } E \text{ is } \mu\text{-semifinite} \\ \infty & \text{otherwise} \end{cases}$$

   defines a measure on $(X, \mathcal{M})$ satisfying $\mu = \mu_0 + \nu$.

[1] not collected

Solutions:

1. (a) Let $\mathcal{E}$ denote the collection of open intervals in $\mathbb{R}$, then the collection of open boxes in $\mathbb{R}^n$ is precisely

$$\{\prod_{j=1}^{n} E_j \colon E_j \in \mathcal{E}\}.$$

Also denote $\tilde{\mathcal{E}} := \mathcal{E} \cup \{\mathbb{R}\}$. Then $\mathcal{E} \subset \tilde{\mathcal{E}}$ and

$$\mathbb{R} = \bigcup_{k=1}^{\infty} (-k, k)$$

imply $\mathcal{M}(\mathcal{E}) = \mathcal{M}(\tilde{\mathcal{E}})$ and

$$\mathcal{M}(\{\prod_{j=1}^{n} E_j \colon E_j \in \mathcal{E}\}) = \mathcal{M}(\{\prod_{j=1}^{n} E_j \colon E_j \in \tilde{\mathcal{E}}\})$$

by Lemma 1.1 from lecture. Since $\mathbb{R} \in \tilde{\mathcal{E}}$, we can apply the second part of Proposition 1.4 to get that the above σ-algebra equals $\bigotimes_{j=1}^{n} \mathcal{M}(\tilde{\mathcal{E}}) = \bigotimes_{j=1}^{n} \mathcal{M}(\mathcal{E})$. Since $\mathcal{M}(\mathcal{E}) = \mathcal{B}_{\mathbb{R}}$ by Exercise 5 from Homework 1, we thus have

$$\mathcal{M}(\{\prod_{j=1}^{n} E_j \colon E_j \in \mathcal{E}\}) = \bigotimes_{j=1}^{n} \mathcal{M}(\mathcal{E}) = \bigotimes_{j=1}^{n} \mathcal{B}_{\mathbb{R}}.$$

Finally, by Corollary 1.6 from lecture the above σ-algebra is exactly $\mathcal{B}$. □

(b) Denote $\mathcal{B}' := \{E \in \mathcal{B} \colon E + \mathbf{t} \in \mathcal{B}\}$ as in the hint. Then clearly $\mathcal{B}'$ contains all the open boxes, and it is a σ-algebra since the map $E \mapsto E + \mathbf{t}$ being invertible implies it commutes with taking complements and unions. Lemma 1.1 therefore tells us that $\mathcal{B}'$ contains the σ-algebra generated by open boxes, which is $\mathcal{B}$ by part (a). On the other hand, $\mathcal{B}' \subset \mathcal{B}$ by definition. Hence $\mathcal{B}' = \mathcal{B}$, and so the translation of any Borel set is a Borel set. □

(c) Let $\mathcal{B}'' := \{E \in \mathcal{B} \colon E \cdot \mathbf{t} \in \mathcal{B}\}$. Once again $\mathcal{B}''$ is a σ-algebra containing the open boxes, and so arguing exactly as in the previous part we see that $\mathcal{B}'' = \mathcal{B}$. Hence the dilation of any Borel set is a Borel set. □

2. (a) Denote

$$F_n := \bigcap_{n=k}^{\infty} E_k.$$

Then $\liminf E_n = \bigcup_n F_n$, and $F_n \subset F_{n+1}$. Thus continuity from below implies

$$\mu(\liminf E_n) = \lim_{n\to\infty} \mu(F_n).$$

Also, since $F_n \subset E_k$ for all $k \geq n$, monotonicity gives us $\mu(F_n) \leq \mu(E_k)$. Thus

$$\mu(\liminf E_n) \leq \lim_{n\to\infty} \inf_{k\geq n} \mu(E_k) = \liminf_{n\to\infty} \mu(E_n).$$

□

(b) Denote

$$G_n := \bigcup_{k=n}^{\infty} E_k.$$

Then $\limsup E_n = \bigcap_n G_n$, and $G_n \supset G_{n+1}$. Since

$$\mu(G_1) = \mu(\bigcup_{n=1}^{\infty} E_n) < \infty$$

by assumption, continuity from above implies

$$\mu(\limsup E_n) = \lim_{n\to\infty} \mu(G_n).$$

Now, $\mu(G_n) \geq \mu(E_k)$ for all $k \geq n$ by monotonicity, and thus

$$\mu(\limsup E_N) \geq \lim_{n\to\infty} \sup_{k\geq n} \mu(E_k) = \limsup_{n\to\infty} \mu(E_n).$$

□

3. Note that the forward directions of both parts follow from Theorem 1.8 in lecture, so it suffices to prove only the backward directions. Additionally, by definition of a finitely additive measure, it suffices in each part to show that μ is countably additive.

   (a) Suppose μ satisfies continuity from below and let $\{E_n : n \in \mathbb{N}\} \subset \mathcal{M}$ be a disjoint collection. Letting $F_n := E_1 \cup \cdots \cup E_n$ for each $n \in \mathbb{N}$, we see that

   $$\mu(F_n) = \mu(E_1) + \cdots + \mu(E_n),$$

   and $F_1 \subset F_2 \subset \cdots$. Thus by continuity from below we have

   $$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \lim_{n\to\infty} \mu(F_n) = \lim_{n\to\infty} (\mu(E_1) + \cdots + \mu(E_n)) = \sum_{n=1}^{\infty} \mu(E_n).$$

   Hence μ is a measure. □

   (b) Suppose μ satisfies continuity from above, $\mu(X) < \infty$, and let $\{E_n : n \in \mathbb{N}\} \subset \mathcal{M}$ be a disjoint collection. Note that by finite additivity

   $$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu(E_1) + \cdots + \mu(E_{k-1}) + \mu\left(\bigcup_{n=k}^{\infty} E_n\right) \tag{1}$$

   for each $k \in \mathbb{N}$. Define $G_k := \bigcup_{n=k}^{\infty} E_n$ for each $k \in \mathbb{N}$ so that $G_1 \supset G_2 \supset \cdots$. Also note that

   $$\bigcap_{k=1}^{\infty} G_k = \limsup E_n = \emptyset,$$

   since the E_n are disjoint and therefore no x belongs to infinitely many E_n. We also have $\mu(G_1) \leq \mu(G_1) + \mu(G_1^c) = \mu(X) < \infty$, so by continuity from above:

   $$\lim_{k\to\infty} \mu(G_k) = \mu(\bigcap_{k=1}^{\infty} G_k) = \mu(\emptyset) = 0.$$

   Letting $k \to \infty$ in (1) then gives

   $$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{k\to\infty} \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{k\to\infty} \mu(E_1) + \cdots + \mu(E_{k-1}) + \mu(G_k) = \sum_{n=1}^{\infty} \mu(E_n).$$

   Hence μ is a measure. □

4. (a) Let $(X,\mathcal{M},\mu)$ be a σ-finite measure space, and let $E\in\mathcal{M}$ with $\mu(E)=\infty$. By σ-finiteness, there exists $\{G_n\}_{n\in\mathbb{N}}\subset\mathcal{M}$ satisfying $\bigcup_n G_n = X$ and $\mu(G_n)<\infty$ for all $n\in\mathbb{N}$. We claim $0<\mu(E\cap G_n)$ for some $n\in\mathbb{N}$. Indeed, if not then by subadditivity we have

$$\mu(E)=\mu(\bigcup_{n=1}^\infty E\cap G_n)\leq\sum_{n=1}^\infty \mu(E\cap G_n)=0,$$

contradicting $\mu(E)=\infty$. Thus $0<\mu(E\cap G_n)$ for some $n\in\mathbb{N}$, and by monotonicity we also have $\mu(E\cap G_n)<\infty$. Thus μ is semifinite. $\square$

(b) Note that by semifiniteness,

$$\mathcal{F}:=\{F\in\mathcal{M}\colon F\subset E,\ 0<\mu(F)<\infty\}$$

is non-empty, and therefore $\alpha:=\sup_{F\in\mathcal{F}}\mu(F)>0$. Suppose, towards a contradiction, that $\alpha<\infty$. Let $(F_n)_{n\in\mathbb{N}}\subset\mathcal{F}$ be a sequence satisfying $\mu(F_n)\to\alpha$. Note that $F_1\cup\cdots\cup F_n\in\mathcal{F}$ for each $n\in\mathbb{N}$ by subadditivity, and hence $\mu(F_1\cup\cdots\cup F_n)\leq\alpha$. By continuity from below, we therefore have

$$\mu(\bigcup_{n=1}^\infty F_n)=\lim_{n\to\infty}\mu(F_1\cup\cdots\cup F_n)\leq\alpha.$$

On the other hand, $\mu(F_1\cup\cdot\cup F_n)\geq\mu(F_n)$ by monotonicity. It follows that $G:=\bigcup_n F_n\in\mathcal{F}$ with $\mu(G)=\alpha$. Now,

$$\mu(E)=\mu(E\setminus G)+\mu(G)$$

implies $\mu(E\setminus G)=\infty$. As μ is semifinite, there exists $F\subset E\setminus G$ with $0<\mu(F)<\infty$. But then $\mu(F\cup G)=\mu(F)+\mu(G)\in(\alpha,\infty)$ so that $F\cup G\in\mathcal{F}$ with $\mu(F\cup G)>\alpha$, contradicting the definition of α.

Thus $\alpha=\infty$ and hence for any $C>0$ one can find $F\in\mathcal{F}$ with $\mu(F)>C$. $\square$

5. (a) We first show that μ_0 is a measure. For $E=\emptyset$, any subset is necessarily the empty set and hence

$$\mu_0(\emptyset)=\sup\{\mu(\emptyset)\}=0.$$

If $\{E_n\colon n\in\mathbb{N}\}\subset\mathcal{M}$ is a disjoint collection, then for any $F\subset\bigcup_{n=1}^\infty E_n$ with $\mu(F)<\infty$ one has that $\mu(F\cap E_n)<\infty$ for all $n\in\mathbb{N}$ by monotonicity. Thus

$$\mu(F)=\sum_{n=1}^\infty\mu(F\cap E_n)\leq\sum_{n=1}^\infty\mu_0(E_n),$$

and taking the supremum over all finite measure subsets $F\subset\bigcup E_n$ one obtains

$$\mu_0\left(\bigcup_{n=1}^\infty E_n\right)\leq\sum_{n=1}^\infty\mu_0(E_n).$$

Conversely, if $\mu_0(E_n)=\infty$ for any $n\in\mathbb{N}$, then there exists $F_R\subset E_n$ with $\mu(F_R)\geq R$ for all $R>0$. Since $F_R\subset\bigcup E_n$, this implies $\mu_0(\bigcup E_n)=\infty$. Now suppose $\mu(E_n)<\infty$ for all $n\in\mathbb{N}$. Let $\epsilon>0$ and for each $n\in\mathbb{N}$ let $F_n\subset E_n$ be such that $\mu(F_n)<\infty$ and $\mu_0(E_n)-\frac{\epsilon}{2^n}<\mu(F_n)$. Then for each $N\in\mathbb{N}$ we have

$$\bigcup_{n=1}^N F_n\subset\bigcup_{n=1}^\infty E_n$$

and is of finite measure. Thus

$$\sum_{n=1}^N\mu_0(E_n)\leq\sum_{n=1}^N\mu(F_n)+\frac{\epsilon}{2^n}=\mu\left(\bigcup_{n=1}^N F_n\right)+\epsilon(1-(1/2)^N)<\mu_0\left(\bigcup_{n=1}^\infty E_n\right)+\epsilon.$$

Letting $N\to\infty$ and $\epsilon\to0$ yields the inequality needed to show μ_0 is countably additive.

Next, we must show μ_0 is semifinite. Suppose $E\in\mathcal{M}$ satisfies $\mu_0(E)=\infty$. The definition of μ_0 implies there exists $F\subset E$ with $0<\mu(F)<\infty$. Then for any $G\subset F$ one has $\mu(G)\leq\mu(F)$ and hence $\mu_0(F)=\mu(F)\in(0,\infty)$. Thus μ_0 is semifinite. $\square$

(b) Suppose μ is semifinite and let $E \in \mathcal{M}$. If $\mu(E) < \infty$, then the argument at the end of part (a) shows $\mu_0(E) = \mu(E)$. If $\mu(E) = \infty$, then by Exercise 4.(a) for any $C > 0$ there exists $F \subset E$ with $C < \mu(F) < \mu(E)$. Hence $\mu_0(E) > C$ for all $C > 0$, and therefore $\mu_0(E) = \infty = \mu(E)$. $\square$

(c) We will first show $\mu = \mu_0 + \nu$. First claim that if E is μ-semifinite, then $\mu(E) = \mu_0(E)$. Indeed, if $\mu(E) < \infty$ then this follows from the argument at the end of part (a). Otherwise, arguing as in Exercise 4.(b) we see that

$$\{\mu(F) \colon F \subset E,\ 0 < \mu(F) < \infty\}$$

is unbounded, and hence $\mu_0(E) = \infty$ as the supremum of the above set. This proves the claim and consequently, $\mu(E) = \mu_0(E) = \mu_0(E) + \nu(E)$ for any μ-semifinite E. If E is not μ-semifinite, then necessarily $\mu(E) = \infty = \nu(E) = \mu_0(E) + \nu(E)$.

Now we show ν is a measure. Since $\mu(\emptyset) = 0 < \infty$, we see that $\emptyset$ is μ-semifinite and therefore $\nu(\emptyset) = 0$. Next, let $\{E_n \colon n \in \mathbb{N}\} \subset \mathcal{M}$ be a disjoint collection. If E_k is not μ-semifinite for some $k \in \mathbb{N}$, then there exists $F \subset E_k$ with $\mu(F) = \infty$ but $\mu(G) \in \{0, \infty\}$ for all $G \subset F$. This same F is a subset for $\bigcup E_n$, and so we see that $\bigcup E_n$ is also not μ-semifinite in this case and hence

$$\sum_{n=1}^{\infty} \nu(E_n) = \infty = \nu\left(\bigcup_{n=1}^{\infty} E_n\right).$$

Otherwise, E_n is μ-semifinite for all $n \in \mathbb{N}$ and therefore

$$\sum_{n=1}^{\infty} \nu(E_n) = \sum_{n=1}^{\infty} 0 = 0.$$

So we must show that in this case, $\bigcup E_n$ is also μ-semifinite. Let $F \subset \bigcup E_n$ with $\mu(F) = \infty$. Note that

$$\infty = \mu(F) = \mu\left(\bigcup_{n=1}^{\infty} E_n \cap F\right) = \sum_{n=1}^{\infty} \mu(E_n \cap F),$$

and so $\mu(E_k \cap F) > 0$ for at least one $k \in \mathbb{N}$. If $\mu(E_k \cap F) < \infty$, we take $G := E_k \cap F$. Otherwise, $\mu(E_k \cap F) = \infty$, and then invoking the μ-semifiniteness of E_k we can find $G \subset E_k \cap F$ with $0 < \mu(G) < \infty$. This G is of course also a subset of F, and so $\bigcup E_n$ is μ-semifinite. $\square$