## Exercises: (Sections 3.1, 3.2)

1. Let $\nu$ be a signed measure on a measurable space $(X, \mathcal{M})$.
   (a) Show $L^1(X, \nu) = L^1(X, |\nu|)$.
   (b) For $f \in L^1(X, \nu)$, show $|\int_X f \, d\nu| \leq \int_X |f| \, d|\nu|$.
   (c) For $E \in \mathcal{M}$, prove the following formulas:
      (i) $\nu^+(E) = \sup\{\nu(F) \colon F \subset E, F \in \mathcal{M}\}$
      (ii) $\nu^-(E) = -\inf\{\nu(F) \colon F \subset E, F \in \mathcal{M}\}$
      (iii) assuming $\nu$ is $\sigma$-finite, $|\nu|(E) = \sup\{|\int_E f \, d\nu| \colon f \in L^1(X, \nu) \text{ and } |f| \leq 1\}$
      (iv) $|\nu|(E) = \sup\{|\nu(E_1)| + \cdots + |\nu(E_n)| \colon n \in \mathbb{N},\ E = E_1 \cup \cdots \cup E_n \text{ is a partition}\}$

2. Let $(\nu_n)_{n\in\mathbb{N}}$ be a sequence of positive measures and let $\mu$ be a positive measure, all defined on the same measurable space $(X, \mathcal{M})$. Denote $\nu = \sum_{n=1}^{\infty} \nu_n$.
   (a) Show that if $\nu_n \perp \mu$ for all $n \in \mathbb{N}$, then $\nu \perp \mu$.
   (b) Show that if $\nu_n \ll \mu$ for all $n \in \mathbb{N}$, then $\nu \ll \mu$.

3. For $j = 1, 2$, let $\mu_j$, $\nu_j$ be $\sigma$-finite measures on $(X_j, \mathcal{M}_j)$ with $\nu_j \ll \mu_j$. Show that $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$ with
$$\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x_1, x_2) = \frac{d\nu_1}{d\mu_1}(x_1)\frac{d\nu_2}{d\mu_2}(x_2)$$
for $(\mu_1 \times \mu_2)$-almost every $(x_1, x_2) \in X_1 \times X_2$.

4. On $([0, 1], \mathcal{B}_{[0,1]})$, let $m$ be the Lebesgue measure and let $\nu$ be the counting measure.
   (a) Show that $m \ll \nu$, but $dm \neq f \, d\nu$ for any function $f$.
   (b) Show that there does **not** exist $\lambda \perp m$ and $\rho \ll m$ so that $\nu = \lambda + \rho$.

   [**Note:** this shows the $\sigma$-finiteness assumption in the Lebesgue–Radon–Nikodym theorem is necessary.]

5. Let $(X, \mathcal{M}, \mu)$ be a $\sigma$-finite measure space, and let $\nu$ be a $\sigma$-finite signed measure on $(X, \mathcal{M})$ with $\nu \ll \mu$.
   (a) Show that $\left|\frac{d\nu}{d\mu}\right| = \frac{d|\nu|}{d\mu}$.
   (b) Show that $\frac{d\nu}{d\mu} \in L^1(X, \mu)$ if and only if $\nu$ is finite.
   (c) Suppose $\nu$ is positive and let $\lambda := \nu + \mu$. Show that $0 \leq \frac{d\nu}{d\lambda} < 1$ $\mu$-almost everywhere and that
$$\frac{d\nu}{d\mu} = \frac{\frac{d\nu}{d\lambda}}{1 - \frac{d\nu}{d\lambda}}.$$

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## Solutions:

1. (a) By definition $L^1(X, \nu) = L^1(X, \nu^+) \cap L^1(X, \nu^-)$. So for $f \in L^1(X, \nu)$ we have
$$\int_X |f| \, d|\nu| = \int_X |f| \, d\nu^+ + \int_X |f| \, d\nu^- < \infty,$$
and so $f \in L^1(X, |\nu|)$. Conversely, if $f \in L^1(X, |\nu|)$, then
$$\int_X |f| \, d\nu^+ + \int_X |f| \, d\nu^- = \int_X |f| \, d|\nu| < \infty$$
implies $f \in L^1(X, \nu^+) \cap L^1(X, \nu^-) = L^1(X, \nu)$. □

(b) Using Proposition 2.22 we have

$$\begin{aligned}
\left|\int_X f\ d\nu\right| &= \left|\int_X f\ d\nu^+ - \int_X f\ d\nu^-\right| \\
&\leq \left|\int_X f\ d\nu^+\right| + \left|\int_X f\ d\nu^-\right| \\
&\leq \int_X |f|\ d\nu^+ + \int_X |f|\ d\nu^- = \int_X |f|\ d|\nu|
\end{aligned}$$

(c) Let $X = P \cup N$ be a partition so that $P$ and $N$ are positive and negative for $\nu$, respectively.

(i) We have $\nu^+(E) = \nu(E \cap P)$ and so $\nu^+(E)$ is bounded above by the supremum. Conversely, for measurable $F \subset E$, we have $\nu(F) = \nu^+(F) - \nu^-(F) \leq \nu^+(F) \leq \nu^+(E)$.

(ii) We have $\nu^-(E) = -\nu(E \cap N)$ and so $\nu^-(E)$ is bounded above by the negative infimum. Conversely, for measurable $F \subset E$, we have $-\nu(F) = -\nu^+(F) + \nu^-(F) \leq \nu^-(F) \leq \nu^-(E)$.

(iii) Using part (b), if $|f| \leq 1$ then $|\int_E f\ d\nu| \leq \int_E |f|\ d|\nu| \leq |\nu|(E)$. So $|\nu|(E)$ is bounds the supremum above. On the other hand, recall $\nu$ being $\sigma$-finite means $|\nu|$ is $\sigma$-finite and so we have $X = \bigcup_{n=1}^\infty F_n$ with $|\nu|(F_n) < \infty$ and $F_n \subset F_{n+1}$ for each $n \in \mathbb{N}$. Consider $f_n := 1_{P\cap F_n} - 1_{N\cap F_n}$, which satisfies

$$\int_X |f_n|\ d|\nu| = \int_X 1_{F_n}\ d|\nu| = |\nu|(F_n) < \infty.$$

Thus $f_n \in L^1(X, |\nu|)$, and hence $f_n \in L^1(X, \nu)$ by part (a). Now, $f_n = 1_{P\cap F_n}$ $\nu^+$-a.e. and $f = -1_{N\cap F_n}$ $\nu^-$-a.e. and therefore

$$\begin{aligned}
\int_E f_n\ d\nu = \int_E f_n\ d\nu^+ - \int_E f_n\ d\nu^- = \int_E 1_{P\cap F_n}\ d\nu^+ + \int_E 1_{N\cap F_n}\ d\nu^- \\
= \nu^+(E\cap P\cap F_n) + \nu^-(E\cap N\cap F_n) = \nu^+(E\cap F_n) + \nu^-(E\cap F_n) = |\nu|(E\cap F_n).
\end{aligned}$$

Since the $F_n$'s increase to $X$, taking supremum of the above quantity over $n \in \mathbb{N}$ yields $|\nu|(E)$ by continuity from below.

(iv) For any partition $E = E_1 \cup \cdots \cup E_n$, we have

$$|\nu(E_1)| + \cdots + |\nu(E_n)| \leq |\nu|(E_1) + \cdots + |\nu|(E_n) = |\nu|(E_1 \cup \cdots \cup E_n) = |\nu|(E).$$

Hence $|\nu|(E)$ bounds the supremum above. Conversely, let $X = P \cup N$ be a Hahn decomposition for $\nu$. Then

$$|\nu|(E) = \nu^+(E) + \nu^-(E) = |\nu(E\cap P)| + |\nu(E\cap N)|,$$

and so $|\nu|(E)$ is bounded by the supremum. □

2. (a) For each $n \in \mathbb{N}$, let $X = E_n \sqcup F_n$ be a partition such that $E_n$ is $\nu_n$-null and $F_n$ is $\mu$-null. Define

$$E := \bigcap_{n=1}^\infty E_n \qquad \text{and} \qquad F := \bigcup_{n=1}^\infty F_n.$$

Then

$$E^c = \bigcup_{n=1}^\infty E_n^c = \bigcup_{n=1}^\infty F_n = F,$$

so that $X = E \sqcup F$ is a partition. Additionally, $F$ is $\mu$-null as the countable union of $\mu$-null sets. Finallly, $E \subset E_n$ so that $E$ is $\nu_n$-null for all $n \in \mathbb{N}$. Consequently,

$$\nu(E) = \sum_{n=1}^\infty \nu_n(E) = \sum_{n=1}^\infty 0 = 0.$$

That is, $E$ is $\nu$-null and therefore $\nu \perp \mu$. □

(b) Let $E \in \mathcal{M}$ be $\mu$-null. By assumption it is $\nu_n$-null for all $n \in \mathbb{N}$, and hence

$$\nu(E) = \sum_{n=1}^{\infty} \nu_n(E) = \sum_{n=1}^{\infty} 0 = 0.$$

Thus $\nu \ll \mu$. $\square$

3. Let $E \in \mathcal{M}_1 \otimes \mathcal{M}_2$. Then Tonelli's theorem (applied twice) and Exercise 3 on Homework 5 imply

$$(\nu_1 \times \nu_2)(E) = \int 1_E \; d(\nu_1 \otimes \nu_2) = \iint 1_E \; d\nu_1 d\nu_2 = \iint 1_E \frac{d\nu_1}{d\mu_1} d\mu_1 \frac{d\nu_2}{d\mu_2} d\mu_2 = \int_E \frac{d\nu_1}{d\mu_1}\frac{d\nu_2}{d\mu_2} \; d(\mu_1 \times \mu_2).$$

Thus if $(\mu_1 \times \mu_2)(E) = 0$, then the above equals zero and so $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$. The same computation shows the claimed equality by the uniqueness in the Lebesgue–Radon–Nikodym theorem. $\square$

4. (a) If $\nu(E) = 0$, then necessarily $E = \emptyset$ and so $\mu(E) = 0$. Hence $m \ll \nu$. Suppose, towards a contradiction, that $dm = f d\nu$ for some $f$. Then for each $t \in \mathbb{R}$

$$0 = m(\{t\}) = \int_{\{t\}} f \; d\nu = f(t).$$

Hence $f \equiv 0$, but then $m([0,1]) = 1 \neq 0 = \int_{[0,1]} f \; d\nu$, a contradiction. $\square$

(b) Suppose towards a contradiction that $\nu = \lambda + \rho$ for $\lambda \perp \mu$ and $\rho \ll m$. Let $[0,1] = E \cup F$ where $E$ is $\lambda$-null and $F$ is $m$-null (and hence $\rho$-null). For each $t \in [0,1]$, $\rho(\{t\}) = 0$ and so $\lambda(\{t\}) = \nu(\{t\}) = 1 > 0$. Thus we must have $\{t\} \subset F$ for each $t \in [0,1]$ and therefore $F = [0,1]$. But this set is not $m$-null. $\square$

5. (a) We have $\frac{d\nu}{d\mu} = \frac{d\nu^+}{d\mu} - \frac{d\nu^-}{d\mu}$, and $\frac{d\nu^\pm}{d\mu} \geq 0$ since $\nu^\pm$ are positive. Let $X = P \cup N$ be a Hahn decomposition for $\nu$. Then

$$\int_P \frac{d\nu^-}{d\mu} \; d\mu = \nu^-(P) = 0,$$

and so $\frac{d\nu^-}{d\mu}(x) = 0$ for $\mu$-almost every $x \in P$ by Proposition 2.16. Similarly $\frac{d\nu^+}{d\mu}(x) = 0$ for $\mu$-almost every $x \in N$. Hence $\frac{d\nu^+}{d\mu} = \frac{d\nu^+}{d\mu} 1_P$ and $\frac{d\nu^-}{d\mu} = \frac{d\nu^-}{d\mu} 1_N$ and since $P \cap N = \emptyset$ we have

$$\left|\frac{d\nu}{d\mu}\right| = \left|\frac{d\nu^+}{d\mu} 1_P - \frac{d\nu^-}{d\mu} 1_N\right| = \frac{d\nu^+}{d\mu} 1_P + \frac{d\nu^-}{d\mu} 1_N = \frac{d\nu^+}{d\mu} + \frac{d\nu^-}{d\mu} = \frac{d|\nu|}{d\mu}.$$

$\square$

(b) By part (a),

$$|\nu|(X) = \int_X \frac{d|\nu|}{d\mu} \; d\mu = \int_X \left|\frac{d\nu}{d\mu}\right| d\mu.$$

Thus $|\nu|(X) < \infty$ (i.e. $\nu$ is finite) iff $\frac{d\nu}{d\mu} \in L^1(X, \mu)$. $\square$

(c) We have $\frac{d\nu}{d\lambda} \geq 0$ since $\nu$ is positive. Observe that for $E := \{x \in X \colon \frac{d\nu}{d\lambda}(x) \geq 1\}$ we have

$$0 \leq \int_E 1 - \frac{d\nu}{d\lambda} \; d\lambda = \lambda(E) - \nu(E) = \mu(E).$$

Thus $\mu(E) = 0$ and so $\frac{d\nu}{d\mu} < 1$ $\mu$-almost everywhere. Now, Theorem 3.10 implies

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d(\lambda - \nu)}{d\lambda} = \frac{d\nu}{d\mu}\left(1 - \frac{d\nu}{d\lambda}\right).$$

Solving for $\frac{d\nu}{d\mu}$ yields the claimed equality. $\square$