1.6 Problems

Limits Properties

Question 1. Consider the limits below:

$$\lim_{x\to 1} f(x) = 2 \qquad \lim_{x\to 1} g(x) = 3 \qquad \lim_{x\to 1} h(x) = 5$$
$$\lim_{x\to 2} f(x) = -2 \qquad \lim_{x\to 2} g(x) = 7 \qquad \lim_{x\to 2} h(x) = -4$$
$$\lim_{x\to 5} f(x) = 0 \qquad \lim_{x\to 5} g(x) = -1 \qquad \lim_{x\to 5} h(x) = 1$$

Compute the following limits:

(a) $\lim_{x\to 1} f(x)g(x) = 6$

(b) $\lim_{x\to 1} \frac{g(f(x))}{h(x)} = \frac{7}{5}$

(c) $\lim_{x\to 5} g(f(h(x))) = 7$

(d) $\lim_{x\to 2}[3f(x) + g(x)] = -6 + 7 = 1$

Fractions and Cancellation

Question 2. Evaluate the following limits:

(a) $\lim_{x\to 2} \frac{x^2-4}{x^2+x-4} = \frac{0}{2} = 0$

(b) $\lim_{x\to 2} \frac{x^2-4}{x^2+x-6} = \lim_{x\to 2} \frac{(x+2)(x-2)}{(x+3)(x-2)} = \lim_{x\to 2} \frac{x+2}{x+3} = 4/5$

(c) $\lim_{x\to 9} \frac{\sqrt{x}-3}{9-x} = \lim_{x\to 9} \frac{\sqrt{x}-3}{9-x} \cdot \frac{\sqrt{x}+3}{\sqrt{x}+3} = \lim_{x\to 9} \frac{x-9}{(9-x)(\sqrt{x}+3)} = \lim_{x\to 9} \frac{-(9-x)}{(9-x)(\sqrt{x}+3)} = \lim_{x\to 9} \frac{-1}{\sqrt{x}+3} =$
$-1/6$

(d) $\lim_{h\to 0} \frac{\frac{8}{h-4}+2}{h} = \lim_{h\to 0} \frac{\frac{8}{h-4}+\frac{2(h-4)}{h-4}}{h} = \lim_{h\to 0} \frac{\frac{2h}{h-4}}{h} = \lim_{h\to 0} \frac{2}{h-4} = -1/2$

Limits with Abs Values

Question 3. Evaluate the following:

(a) $|5-1| = 4$

(b) $|1-5| = 4$

Question 4. Prove that $|a-b| = |b-a|$.

Solution: $|a-b|$ = the distance between a and $b = |b-a|$.

Question 5. Evaluate the limits

(a) $\lim_{x\to 1^-} \frac{|1-x|}{x-1} = \lim_{x\to 1^-} \frac{1-x}{x-1} = -1$

(b) $\lim_{x\to 1^-} \frac{2x(x+2)|1-x|}{x-1} = \lim_{x\to 1^-} \frac{2x(x+2)(1-x)}{x-1} = -6$

Inequality Limits

Question 6. If $9 - x^2 \leq g(x) \leq 9\cos(2x)$ for all x, then find $\lim_{x\to 0} g(x)$

Solution: since $\lim_{x\to 0}(9 - x^2) = 9$, and $\lim_{x\to 0} 9\cos(2x) = 9$, the squeeze theorem implies that $\lim_{x\to 0} g(x) = 9$

Question 7. If $12x - 53 \leq f(x) \leq x^2 + 4x - 37$ for all x, then find $\lim_{x\to 4} f(x)$

Solution: since $\lim_{x\to 4} 12x - 53 = -5$, and $\lim_{x\to 4} x^2 + 4x - 37 = -5$, the squeeze theorem implies that $\lim_{x\to 0} f(x) = -5$

Question 8. Evaluate the limit: $\lim_{x\to 0}\left[x^4 \sin\left(\frac{-3}{x}\right)\right]$

Solution: we start with the fact $-1 \leq \sin\left(\frac{-3}{x}\right) \leq 1$ for all x. Multiplying each term in these inequalities by x^4 yields $-x^4 \leq x^4 \sin\left(\frac{-3}{x}\right) \leq x^4$. This provides a lower and an upper bound on the target function. Observe that $\lim_{x\to 0} -x^4 = 0 = \lim_{x\to 0} x^4$, the squeeze theorem implies that $\lim_{x\to 0}\left[x^4 \sin\left(\frac{-3}{x}\right)\right] = 0$.