2.3 Problems

Graphs and Tables

Example 1. Suppose that

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	-3	-2	4	7

Find $h'(2)$ if:

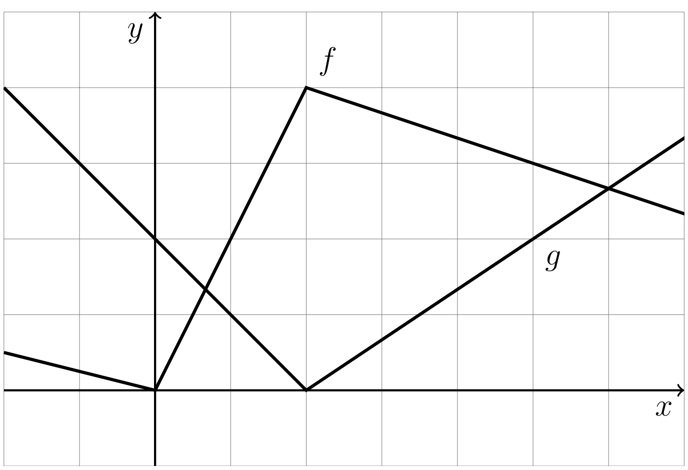
(a) $h(x) = 5f(x) - 4g(x) = -38$

(b) $h(x) = f(x)g(x) = -29$

(c) $h(x) = \dfrac{f(x)}{g(x)} = \frac{13}{16}$

(d) $h(x) = \dfrac{g(x)}{1 + f(x)} = -\frac{3}{2}$

Example 2. If f and g are the functions whose graphs are shown below, let $u(x) = f(x)g(x)$ and $v(x) = \dfrac{f(x)}{g(x)}$.



(a) Find $u'(1)$ $\quad u'(1) = f'(1)g(1) + f(1)g'(1) = 2 \cdot 1 + 1 \cdot (-2) = 0$

(b) Find $v'(5)$ $\quad v'(5) = \frac{f'(5)g(5) - f(5)g'(5)}{g^2(5)} = \frac{-\frac{1}{3} \cdot 2 - 3 \cdot \frac{2}{3}}{2^2} = -\frac{2}{3}$

Standard Problems

Example 3. Differentiate the functions:

(a) $f(x) = \pi^2$

Since $f(x)$ is a constant, we have $f'(x) = 0$

(b) $g(x) = (x-2)(2x+3)$

Using the product rule $g'(x) = (x-2)'(2x+3) + (x-2)(2x+3)' = 2x+3+2(x-2) = 4x-1$

(c) $h(x) = \dfrac{\sqrt{x}+x}{x^2}$

First simplify $h(x)$: $h(x) = \dfrac{\sqrt{x}}{x^2} + \dfrac{x}{x^2} = x^{-3/2} + x^{-1}$.

Then take the derivative using the fact that $(x^n)' = nx^{n-1}$, $h'(x) = (x^{-3/2})' + (x^{-1})' = -\frac{3}{2}x^{-5/2} - x^{-2}$

(d) $k(x) = \dfrac{x}{x+\frac{c}{x}}$

First simplify $k(x)$: $k(x) = \dfrac{x}{\frac{x^2+c}{x}} = \dfrac{x^2}{x^2+c}$

Then take the derivative using the quotient rule: $k'(x) = \dfrac{(x^2)'(x^2+c) - x^2(x^2+c)'}{(x^2+c)^2} = \dfrac{2x(x^2+c) - x^2 \cdot 2x}{(x^2+c)^2} = \dfrac{2xc}{(x^2+c)^2}$

Example 4. Find the equation of the tangent line of the curve $y = \frac{3x+1}{x^2+1}$ through the point $(1, 2)$.

Solution: Denote the given function by $f(x)$: $f(x) = \frac{3x+1}{x^2+1}$. To find the tangent line of $f(x)$ at $(1, 2)$, we need to find $f'(1)$. Using the quotient rule

$$f'(x) = \frac{(3x+1)'(x^2+1) - (3x+1)(x^2+1)'}{(x^2+1)^2} = \frac{3(x^2+1) - (3x+1)(2x)}{(x^2+1)^2} = \frac{-3x^2-2x+3}{(x^2+1)^2}$$

Plugging $x = 1$, we get $f'(1) = -1/2$. Plugging $f'(1) = -1/2$, $f(1) = 2$ into the equation of the tangent line, we get

$$y - 2 = -\frac{1}{2}(x-1)$$

Example 5. Find the points of the curve $y = 2x^3 + 3x^2 - 12x + 1$ where the tangent is horizontal.
Solution: Tangent being horizontal means the derivate equals 0. The derivative of the given function is $(2x^3 + 3x^2 - 12x + 1)' = 6x^2 + 6x - 12$, setting it to 0 and solve for x,

$$\begin{aligned} & 6x^2 + 6x - 12 = 0 \\ \iff & x^2 + x - 2 = 0 \\ \iff & (x+2)(x-1) = 0 \\ \iff & x = 1, -2 \end{aligned}$$

The curve has a horizontal tangent at $x = 1, -2$.

Example 6. Let $f(x) = \begin{cases} ax^2 & \text{if } x \le 1 \\ 3x+b & \text{if } x > 1 \end{cases}$. Find the values of a and b that make f differentiable everywhere.
Solution: No matter what a and b are, $f(x)$ is differentiable on the intervals $(-\infty, 1)$ (as when $x < 1$, $f(x) = ax^2$ is a polynomial and polynomial is differentiable), and $f(x)$ is differentiable on the intervals $(1, \infty)$. The only point left out is $x = 1$. If we can take a and b such that $f'(1)$ exists, then $f(x)$ is differentiable everywhere. What values of a and b make $f'(1)$ exist? $f'(1)$ exists means f is differentiable at 1, then it must first be continuous. Continuity means $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$. You can work out these one sided limits: $\lim_{x\to 1^-} f(x) = a$, $\lim_{x\to 1^+} f(x) = 3 + b$. Since the left and right limits equal, we have

$$a = 3 + b$$

But continuity is not enough to ensure differentiability. Recall that by definition, $f'(1) = \lim_{x\to 1} \frac{f(1+h)-f(1)}{h}$. For this limit to exist, we need $\lim_{x\to 1^-} \frac{f(1+h)-f(1)}{h} = \lim_{x\to 1^+} \frac{f(1+h)-f(1)}{h}$. You can work out these one sided limits: $\lim_{x\to 1^-} \frac{f(1+h)-f(1)}{h} = 2a$ and $\lim_{x\to 1^+} \frac{f(1+h)-f(1)}{h} = 3$. Set them equal, we get

$$2a = 3$$

Combining the above two equations, we get $a = 3/2, b = -3/2$.