2.5b Problems

Tables and Graphs

Example 1. A table of values for f, g, f', and g' are given.

(a) Find the derivative of $f(g(x))$ at $x = 1$.

(b) Find the derivative of $g(f(x))$ at $x = 1$.

(c) Find the derivative of $f(f(x))$ at $x = 2$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	2	3	3	0
2	1	-3	-5	6
3	4	-1	11	1

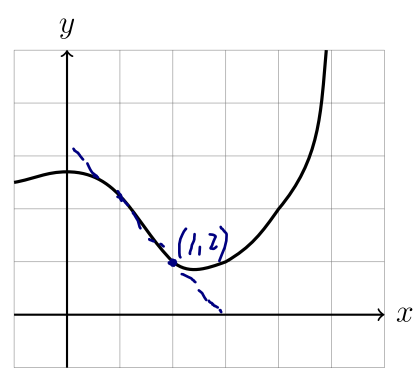
(a) $f'(g(1))\cdot g'(1) = f'(3)\,g'(1) = 11\cdot 0 = 0$

(b) $g'(f(1))\,f'(1) = g'(2)f'(1) = 6\cdot 3 = 18$

(c) $f'(f(2))\cdot f'(2) = f'(1)f'(2) = 3\cdot(-5) = -15$

Example 2. If f is the function whose graph is given to the right. Use the graph of f to estimate the value of each derivative:

1. $f(f(x))$ at $x = 2$.
2. $f(x^2)$ at $x = 2$.



$(f(f(x)))'\big|_2 = f'(f(x))f'(x)\big|_2 = f'(f(2))f'(2)$

$f(2)=1$, $f'(2) = -1$, $f'(1) = -1$ due to the graph

So $f'(f(2))f'(2) = f'(1)\cdot f'(2) = 1$

Standard Problems

Example 3. Find the derivatives of the following functions:

(a) $f(x) = \dfrac{3}{x} \cos^{-4} x$

$$f'(x) = \left(\frac{3}{x}\right)' \cos^{-4} x + \frac{3}{x} \left(\cos^{-4} x\right)' \quad \text{product rule}$$

$$= -3x^{-2} \cos^{-4} x + \frac{3}{x} \underbrace{(-4) \cos^{-5} x \cdot (-\sin x)}_{\text{chain rule}}$$

(b) $g(x) = ((4x + x^3)^{-2} + 3x)^4$ ← outer

$$g'(x) = 4\left((4x+x^3)^{-2}+3x\right)^3 \cdot \left(-2(4x+x^3)^{-3} \cdot (4+3x^2) + 3\right)$$

(c) $h(t) = \sin(\cos(\tan(2t)))$

$$h'(t) = \sin'(\cos(\tan(2t))) \cdot \cos'(\tan(2t)) \cdot \tan'(2t) \cdot (2t)'$$

$$= \cos(\cos(\tan(2t))) \cdot (-\sin(\tan(2t))) \cdot \sec^2(2t) \cdot 2$$

Example 4. Find an equations of the tangent line to the curve at the given point:

(a) $f(x) = (1 + 2x)^{10}$ at $x = 0$.

$$f'(x) = 10(1+2x)^9 \cdot 2$$

$$f'(0) = 20$$

$$f(0) = 1$$

$$y - f(0) = f'(0)(x - 0)$$

$$\Leftrightarrow \quad y - 1 = 20x$$

(b) $g(x) = \sqrt{1 + x^3}$ at $x = 2$

$$g'(x) = \frac{1}{2}(1+x^3)^{-\frac{1}{2}} \cdot 3x^2$$

$$g'(2) = \frac{1}{2} \frac{1}{3} 3 \cdot 4 = 2$$

$$g(2) = \sqrt{1+8} = 3$$

$$y - g(2) = g'(2)(x - 2)$$

$$\Leftrightarrow \quad y - 3 = 2(x - 2)$$

(c) $h(x) = \sin x + \sin^2 x$ at $(0, 0)$

$$h'(x) = \cos x + 2 \sin x \cos x$$

$$h'(0) = 1 + 0 = 1$$

$$y - 0 = 1(x - 0) \Leftrightarrow y = x$$

Non-Standard (Fun) Problems

Example 5. If $h(x) = \sqrt{4 + 3f(x)}$ where $f(1) = 7$ and $f'(1) = 4$, find $h'(1)$.

$$h'(x) = \frac{1}{2}(4+3f(x))^{-\frac{1}{2}} \cdot f'(x)$$

$$\Rightarrow h'(1) = \frac{1}{2}(4+3f(1))^{-\frac{1}{2}} \cdot f'(1)$$

$$= \frac{1}{2}(4+3\cdot 7)^{-\frac{1}{2}} \cdot 4$$

$$= \frac{1}{2} \cdot \frac{1}{5} \cdot 4 = \frac{2}{5}$$

Example 6. Write $|x| = \sqrt{x^2}$ and use the chain rule to prove that $\dfrac{d}{dx}|x| = \dfrac{x}{|x|}$

$$\frac{d\sqrt{x^2}}{dx} = \frac{1}{\cancel{2}}(x^2)^{-\frac{1}{2}} \cdot \cancel{2}x \longrightarrow \begin{cases} \text{outer function} : \sqrt{x} \\ \text{inner function} : x^2 \end{cases}$$

$$= \frac{x}{\sqrt{x^2}} = \frac{x}{|x|}$$

Example 7. If $f(x) = |\sin x|$, find $f'(x)$. Where is f not differentiable?

f is not diff when the input of $|\cdot|$ is 0, in this case,

it means $\sin x = 0 \Leftrightarrow x = k\pi$, for all $k \in \mathbb{Z}$