2.6 Problems

Implicit differentiation

Example 1. Find $\frac{dy}{dx}$ by implicit differentiation.

(a) $\frac{1}{x} + \frac{1}{y} = 1$

differentiate both hand sides w.r.t. x

with repect to

$$-x^{-2} + -y^{-2} \cdot \frac{dy}{dx} = 0$$

solve for $\frac{dy}{dx}$ $\Downarrow$

$$\frac{dy}{dx} = -y^2 x^{-2}$$

(b) $4 \cos x \sin y = 1$

differentiate both hand sides (take implicit diff)

$$\cancel{4}(-\sin x) \cdot \sin y + \cancel{4} \cos x \cos y \cdot \frac{dy}{dx} = 0$$

$\Downarrow$ solve for $\frac{dy}{dx}$

$$\frac{dy}{dx} = \tan x \cdot \tan y$$

Example 2. Consider the curve $y^2 = x^3 + 3x^2$ (1)

(a) Find an equation for the tangent line to this curve at the point $(1, -2)$

Implicit diff: $2y\frac{dy}{dx} = 3x^2 + 6x$

$$\frac{dy}{dx} = \frac{3x^2+6x}{2y} \qquad (2)$$

at $(1, -2)$

$$\frac{dy}{dx} = \frac{3+6}{2(-2)} = -\frac{9}{4}$$

$$y-(-2) = -\frac{9}{4}(x-1)$$

(b) At what points does this curve have horizontal tangents?

horizontal tangent means

$$\frac{dy}{dx} = 0$$

From (2) above this means

$$\frac{3x^2+6x}{2y} = 0 \qquad (3)$$

From (1), we know $y = \pm\sqrt{x^3+3x^2}$, plug this into (3)

$$\frac{3x^2+6x}{\pm 2\sqrt{x^3+3x^2}} = 0$$

simplify : $\dfrac{3x+6}{\pm 2\sqrt{x+3}} = 0 \iff 3x+6 = 0 \iff x = -2$

Example 3. Use implicit differentiation to find the equation of the tangent line to the curve at the given point.

(a) $\sin(x+y) = 2x - 2y$ at the point (π, π).

Implicit diff: $\cos(x+y)\left(1+\frac{dy}{dx}\right) = 2 - 2\frac{dy}{dx}$

at (π,π): $\underbrace{\cos(2\pi)}_{=1}\left(1+\frac{dy}{dx}\right) = 2 - 2\frac{dy}{dx}$

$\Downarrow$ solve

$3\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{3}$

tangent line $\quad y - \pi = \frac{1}{3}(x-\pi)$

(b) $x^2 + xy + y^2 = 3$ at the point where $x = 1$ (**Hint**: *there are two equations*)

$x^2 + xy + y^3 = 3 \xrightarrow{x=1} 1 + y + y^2 = 3 \Rightarrow y = -2, 1$ so when $x=1$, there're 2 pts $(1,1)$ & $(1,-2)$

Implicit diff:

$2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0 \quad (1)$

$\Downarrow$

at $(1,1)$ $\quad 2 + 1 + \frac{dy}{dx} + 2\frac{dy}{dx} = 0 \xRightarrow{\text{solve}} \frac{dy}{dx} = -1$

at $(1,-2)$: $\quad 2 - 2 + \frac{dy}{dx} - 4\frac{dy}{dx} = 0 \xRightarrow{\text{solve}} \frac{dy}{dx} = 0$

Tangent lines $\quad y - 1 = (-1)(x-1)$

$y + 2 = 0(x-1)$

Example 4. Find y'' if $x^4 + y^4 = 16$

Implicit diff: $4x^3 + 4y^3 \frac{dy}{dx} = 0 \xRightarrow{\text{solve}} \underline{\frac{dy}{dx} = -\frac{x^3}{y^3}}$

diff again: $12x^2 + 12y^2 \cdot \left(\frac{dy}{dx}\right)^2 + 4y^3 \underline{\frac{d^2y}{dx^2}} = 0$

solve for y'': $y'' = \dfrac{-3y^2\left(\frac{dy}{dx}\right)^2 - 3x^2}{y^3}$

$$= \frac{-3y^2\left(-\frac{x}{y}\right)^6 - 3x^2}{y^3}$$

$$= \frac{-3x^6y^{-4} - 3x^2}{y^3}$$