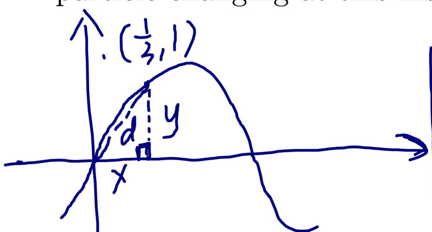
2.8 Problems

Level 1 Problems

Example 1. A particle is moving along the curve $y = 2\sin(\pi x/2)$. As the particle passes through the point $\left(\frac{1}{3}, 1\right)$ its $x-$coordinate is increasing at a rate of $\sqrt{10}$ cm/s. How fast is the distance from the origin to the particle changing at this instant?



$(x(t_0), y(t_0)) = (\frac{1}{3}, 1)$

$x'(t) = \sqrt{10}$

$d'(t_0) = ?$

$y = 2\sin(\pi x/2) \Rightarrow y' = 2\cos\left(\frac{\pi x}{2}\right)\cdot\frac{\pi}{2}x' \Rightarrow y'(t_0) = 2\cos\left(\frac{\pi}{2}\cdot\frac{1}{3}\right)\frac{\pi}{2}\sqrt{10}$

$= \frac{\sqrt{30}}{2}\pi$

$d^2 = x^2 + y^2 \Rightarrow d(t_0) = \sqrt{x(t_0)^2 + y(t_0)^2} = \frac{\sqrt{10}}{3}$

$\cancel{2}dd' = \cancel{2}xx' + \cancel{2}yy'$

$d(t_0)d'(t_0) = x(t_0)x'(t_0) + y(t_0)y'(t_0)$

$\frac{\sqrt{10}}{3}d'(t_0) = \frac{1}{3}\cdot\sqrt{10} + 1\cdot\frac{\sqrt{30}}{2}\pi$

$\Rightarrow d'(t_0) = 1 + \frac{3}{2}\sqrt{3}\pi$ cm/s

Example 2. A snowball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which the diameter decreases when the diameter is 10 cm.

Set $A(t)$: surface area

$d(t)$: diameter

$A'(t) = -1$ $\quad d(t_0) = 10$

$d'(t_0) = ?$

(1) $A = \frac{4}{3}\pi r^2 = \frac{4}{3}\pi\left(\frac{d}{2}\right)^2 = \frac{1}{3}\pi d^2$

diameter is 2·radius

(2) differentiate $A = \frac{1}{3}\pi d^2$

$A' = \frac{1}{3}\pi 2d d'$

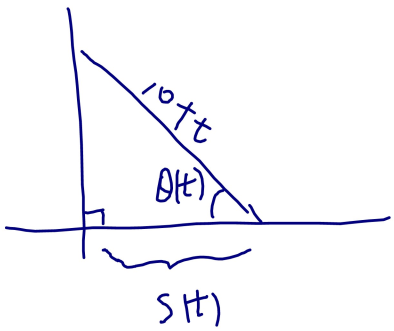
(3) $A'(t_0) = \frac{2}{3}\pi\, d(t_0)\, d'(t_0)$

$-1 = \frac{2}{3}\pi\cdot 10\cdot d'(t_0)$

$d'(t_0) = -\frac{3}{20\pi}$ cm/min

Level 2 Problems

Example 3. A 10 ft ladder rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 2 ft/s, how fast is the angle between the ladder and the ground changing when the bottom of the ladder is 6 ft from the wall?



$s'(t) = 2$

$s(t_0) = 6$

$\theta'(t_0) = ?$

(1) $\cos\theta = \dfrac{s}{10}$

(2) $-\sin(\theta)\cdot\theta' = \dfrac{s'}{10}$

(3) $-\sin(\theta(t_0))\cdot\theta'(t_0) = \dfrac{s'(t_0)}{10}$

$-\dfrac{4}{5}\cdot\theta'(t_0) = \dfrac{2}{10}$

$\Rightarrow \theta'(t_0) = -\dfrac{1}{4}$ ft/s

Get $\sin(\theta(t_0))$ from (1):

$\cos\theta(t_0) = \dfrac{s(t_0)}{10}$

$= \dfrac{6}{10} = 3/5$

use $\cos^2\theta + \sin^2\theta = 1$

we get

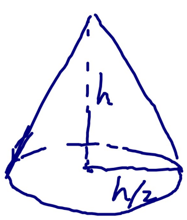
$\sin^2(\theta(t_0)) = 1 - \cos^2(\theta(t_0))$

$= 1 - 9/25 = 16/25$

$\Rightarrow \sin(\theta(t_0)) = \sqrt{\dfrac{16}{25}} = \dfrac{4}{5}$

Example 4. Gravel is being dumped from a conveyor belt at a rate of 3 ft^3/min. It forms a pile in the shape of a cone whose base diameter and height are always the same. How fast is the height of the pile increasing when the pile is 10 ft high?

Set $V(t)$: volume of Gravel



$V'(t) = 3$

$h(t_0) = 10$

$h'(t_0) = ?$

(1) $V = \dfrac{1}{3}\pi\left(\dfrac{h}{2}\right)^2\cdot h$

$= \dfrac{\pi}{12}h^3$

(2) $V' = \dfrac{\pi}{12}3h^2h'$

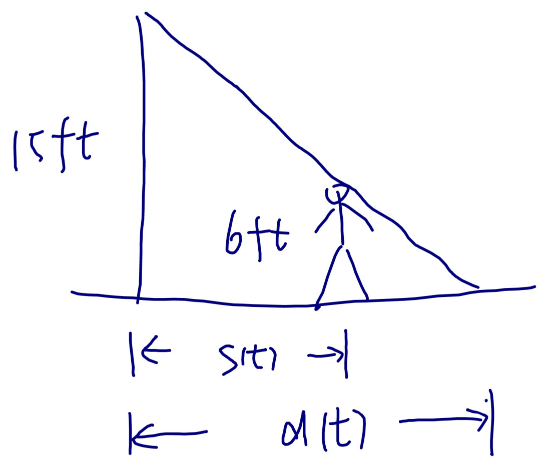
(3) $V'(t_0) = \dfrac{\pi}{4}h(t_0)^2\,h'(t_0)$

$3 = \dfrac{\pi}{4}\cdot 10^2\cdot h'(t_0)$

$\Rightarrow h'(t_0) = \dfrac{12}{100\pi}$ ft/min

Level 3 Problems

Example 5. A street light is mounted at the top of a 15 foot tall pole. A man 6 ft tall walks away from the pole with a speed of 5 ft/s along a straight path. How fast is the tip of his shadow moving when he is 40 ft from the pole?



similar triangle:

(1) $\frac{6}{15} = \frac{d-s}{d}$

simplify

$\Leftrightarrow \quad 6d = 15d - 15s$

$\Leftrightarrow \quad 15s = 9d$

(2) $15s' = 9d'$

$\Rightarrow \quad d'(t_0) = \frac{15}{9} s'(t_0) = \frac{15}{9} \cdot 5 = \frac{25}{3}$ ft/s

$s'(t) = 5$

$d'(t) = ?$