3.1 Problems

Extreme Values

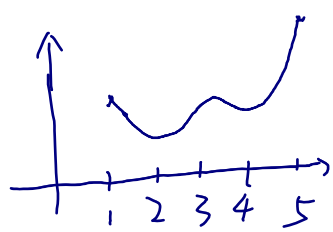
Example 1. What are the two ways in which a function $f(x)$ can have a critical value?

When $f'(x) = 0$ or $f'(x)$ is undefined

we have a critical value.

Example 2. Sketch the graph of a function f that is continuous on [1,5] and has the given properties

- Absolute maximum at 5
- Absolute minimum at 2
- Local maximum at 3
- Local minima at 2 and 4



Example 3. Find the critical numbers of the functions

(a) $g(t) = t^4 + t^3 + t^2 + 1$

$$g'(t) = 4t^3 + 3t^2 + 2t$$

$$= t\,(4t^2 + 3t + 2)$$

this polynomial is always > 0

So $g'(t) = 0$ only when $t = 0$

(b) $f(x) = x^{3/4} - 2x^{1/4}$ domain ($x \geq 0$)

$$f'(x) = \frac{3}{4}x^{-\frac{1}{4}} - \frac{2}{4}x^{-3/4}$$

$f'(x)$ is undefined at $\boxed{x = 0}$

$f'(x)$ is 0 when $\frac{3}{4}x^{-\frac{1}{4}} - \frac{2}{4}x^{-3/4} = 0$

critical points are $0, \frac{4}{9}$

$$\Rightarrow \frac{3}{4}x^{-\frac{1}{4}} = \frac{2}{4}x^{-3/4}$$

multiply both sides by $x^{3/4}$: $x^{3/4} \cdot \frac{3}{4}x^{-\frac{1}{4}} = \frac{2}{4}x^{-3/4} \cdot x^{3/4}$

$$\Rightarrow \frac{3}{4} \cdot x^{1/2} = \frac{1}{2} \quad \Rightarrow \boxed{x = \frac{4}{9}}$$

Example 4. Find the absolute maximum and absolute minimum values of f on the given interval

(a) $f(x) = 2x^3 - 3x^2 - 12x + 1$ on $[-2, 3]$

step 1: find the critical points

$$f'(x) = 6x^2 - 6x - 12$$
$$= 6(x^2 - x - 2)$$
$$= 6(x-2)(x+1)$$

$x = 2, -1$ are critical points

step 2: Evaluate $f(2), f(-1), \underbrace{f(-2)\ f(3)}_{\text{boundary points}}$

absolute min: $\boxed{f(2) = -19}$ max: $\boxed{f(-1) = 8}$ $f(-2) = -3$ $f(3) = 1$.

(b) $f(x) = x + \dfrac{1}{x}$ on $[-1, 1]$ $(x \neq 0)$

step 1: find the critical points

$$f'(x) = 1 - x^{-2}$$

$f'(x)$ is undefined at 0

$f'(x)$ is 0 when $1 - x^{-2} = 0$

$$\Leftrightarrow x^2 = 1$$
$$\Leftrightarrow x = \pm 1$$

step 2. evaluate

$f(0), f(1), f(-1)$

$f(1) = 2$, $f(-1) = -2$, $f(0) = \pm\infty$

So the absolute max/min do not exist

(c) $f(x) = x + \dfrac{1}{x}$ on $[0.2, 4]$

The critical points of f found in (b) are 0, ±1, only 1 lies in the current interval [0.2, 4], so 1 is the critical point here.

Evaluate $f(1)$, $f(0.2)$, $f(4)$

boundary points

$f(1) = \boxed{2}$ min $\quad f(0.2) = \frac{1}{5} + 5 = \boxed{5.2}$ max $\quad f(4) = 4 + \frac{1}{4} = 4.25$

(d) $f(x) = \sin x$ on $\left[\dfrac{-2\pi}{3}, \dfrac{\pi}{6}\right]$

$f'(x) = \cos x$

$f'(x)$ is never undefined

$f'(x) = 0$ when $x = \frac{\pi}{2} + k\pi$ for all integers k

among these only $-\frac{\pi}{2}$ lies in the interval $[-\frac{2\pi}{3}, \frac{\pi}{6}]$

Evaluate $f(-\frac{\pi}{2}) = \boxed{-1}$ min $\quad f(\frac{-2\pi}{3}) = -\frac{\sqrt{3}}{2}$, $\quad f(\frac{\pi}{6}) = \boxed{\frac{1}{2}}$ max

(e) $f(x) = x\sqrt{4 - x^2}$ on $[-1, 2]$

$f'(x) = \sqrt{4-x^2} + x \cdot \frac{1}{2} \frac{-2x}{\sqrt{4-x^2}} = \sqrt{4-x^2} - \frac{x^2}{\sqrt{4-x^2}}$

$f'(x)$ is undefined when $4 - x^2 = 0 \iff \underline{x = \pm 2}$

$f'(x) = 0$ when $\sqrt{4-x^2} - \frac{x^2}{\sqrt{4-x^2}} = 0$

$\iff \sqrt{4-x^2} = \frac{x^2}{\sqrt{4-x^2}}$

$\iff 4 - x^2 = x^2 \iff x^2 = 2 \iff \underline{x = \pm\sqrt{2}}$

critical points in $[-1, 2]$ are

$c = \sqrt{2}, 2$

boundary points:

$a = -1$, $b = 2$

$f(-1) = -\sqrt{3} \leftarrow$ min

$f(2) = 0$

$f(\sqrt{2}) = 2 \leftarrow$ max