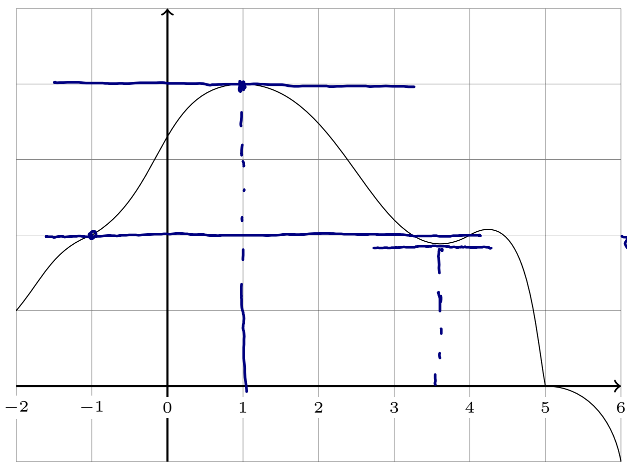
3.2 Problems

A Couple of Graph Problems

Example 1. Consider the graphs below. Estimate the value(s) c that satisfy the conclusion of the Mean Value Theorem on the given interval.

(a)

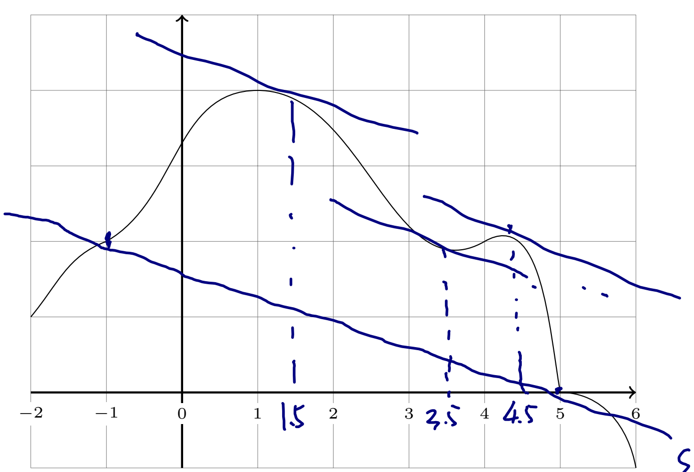


$c = 1, 3.5$

← c are the points whose tangent lines are parallel to this secant line

On the interval $[-1, 4]$.

(b)



$c = 1.5, 3.5, 4.5$

secant line

On the interval $[-1, 5]$.

Bread and Butter Problems

$a \downarrow$ $\downarrow b$

Example 2. Consider the function $f(x) = \sqrt[3]{x}$ on the interval $[0, 1]$.

(a) Why does f satisfy the hypotheses for the MVT?

$f'(x) = \frac{1}{3}x^{-2/3}$

$f(x)$ is continuous on $[0,1]$ and is differentiable on $(0,1)$

(b) Find all c values that satisfies the conclusion of the MVT.

$$\boxed{\begin{aligned} f(x) &= \sqrt[3]{x} \\ f'(x) &= \frac{1}{3}x^{-2/3} \end{aligned}}$$

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{f(1)-f(0)}{1-0} = \frac{\sqrt[3]{1}-\sqrt[3]{0}}{1-0} = 1$$

$f'(c) = 1$

$\Leftrightarrow \quad \frac{1}{3}c^{-2/3} = 1$

$\Leftrightarrow \quad c = \pm 3^{-3/2}$, $\quad -3^{-3/2}$ is not in $(0,1)$, so $3^{-3/2}$ is the c that satisfies MVT

Example 3. Consider the function $f(x) = x^3 - 3x + 2$ on the interval $[-2, 2]$.

(a) Why does f satisfy the hypotheses for the MVT?

both f and f' are polynomials, so

$f(x)$ is continuous on $[-2,2]$ and diff on $(-2,2)$

(b) Find all c values that satisfies the conclusion of the MVT.

$$\boxed{\begin{aligned} f(x) &= x^3 - 3x + 2 \\ f'(x) &= 3x^2 - 3 \end{aligned}}$$

$$\frac{f(b)-f(a)}{b-a} = \frac{f(2)-f(-2)}{2-(-2)} = \frac{4-0}{4} = 1$$

$f'(c) = 1$

$\Leftrightarrow \quad 3c^2 - 3 = 1$

$\Leftrightarrow \quad 3c^2 = 4$

$\Leftrightarrow \quad c = \pm \frac{2}{\sqrt{3}}$

Example 4. Let $f(x) = (x-3)^{-2}$. Show that there is no value $c \in (1,4)$ such that $f(4)-f(1) = f'(c)(4-1)$. Why does this not contradict the Mean Value Theorem?

$$f(x) = (x-3)^{-2}$$
$$f'(x) = -2(x-3)^{-3}$$

$$f(4) - f(1) = f'(c)(4-1)$$
$$\Leftrightarrow \quad 1 - \frac{1}{4} = f'(c) \cdot 3$$
$$\Leftrightarrow \quad \frac{1}{4} = f'(c)$$
$$\Leftrightarrow \quad \frac{1}{4} = -2(c-3)^{-3}$$
$$\Leftrightarrow \quad -\frac{1}{8} = (c-3)^{-3}$$
$$\Leftrightarrow \quad \left(-\frac{1}{8}\right)^{-\frac{1}{3}} = c-3$$
$$\Leftrightarrow \quad -2 = c-3$$
$$\Leftrightarrow \quad 3-2 = c$$
$$\Leftrightarrow \quad 1 = c$$

but this c is not in $(1,4)$

so there is no $c \in (1,4)$ that satisfies the conclusion of the MVT. This happens because the assumption of MVT that $f(x)$ is continuous on $[1,4]$ is failed here as the given $f(x) = (x-3)^{-2}$ is not continuous at $x=3$

A Beautiful Final Exam Problem

Example 5. Show that the equation $3x + 2\cos x + 5 = 0$ has exactly one real root by:

(a) Using the IVT to show that $3x + 2\cos x + 5 = 0$ has at least one root.

Let $f(x) = 3x + 2\cos x + 5$

$f(\pi) = 3\pi - 2 + 5 > 0$

$f(-\pi) = 3(-\pi) - 2 + 5 < 0$

$f(x)$ is continuous everywhere

By IVT, there exist $c \in (-\pi, \pi)$ such that $f(c) = 0$

(b) Using the MVT to show that $3x + 2\cos x + 5 = 0$ has at most one root.

prove by contradiction. Suppose there were two roots

$f(a) = f(b) = 0$, Then MVT implies that there is $c \in (a, b)$.

such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = 0$$

However, $f'(c) = 0$ cannot happen because for any x

$f'(x) = 3 - 2\sin x > 0$

This contradiction implies that there cannot be two roots.

In other words, there is at most one root.