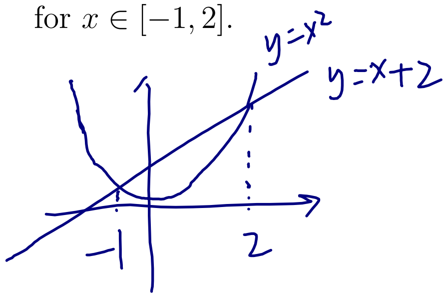
3.7 Problems

Example 1. What is the maximum vertical distance between the line $y = x + 2$ and the parabola $y = x^2$ for $x \in [-1, 2]$.



V: vertical distance

$V = x+2-x^2 \quad x \in [-1,2]$

$V' = 1-2x$

critical point $c = \frac{1}{2}$

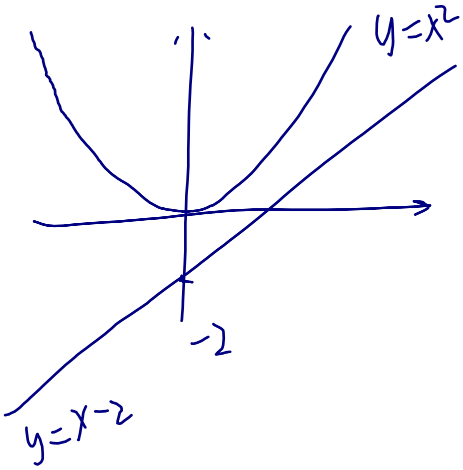
$V(\frac{1}{2}) = 2\frac{1}{4}$

$V(-1) = 0$

$V(2) = 0$

so the max vertical distance is $2\frac{1}{4}$

Example 2. What is the minimum vertical distance between the line $y = x - 2$ and the parabola $y = x^2$.



V: vertical distance

$V = x^2-(x-2)$

$V' = 2x-1$

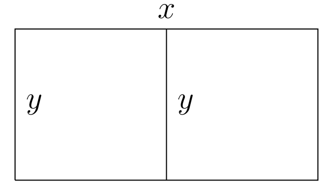
critical point $x = \frac{1}{2}$

$V(\frac{1}{2}) = \frac{1}{4} - \frac{1}{2} + 2 = \frac{7}{4}$

the min dist is $7/4$

Example 3.

A farmer with 300 ft of fencing wants to enclose a rectangular area and then divide it into two pens with fencing parallel to one side of the rectangle (see the picture to the right).



(a) Write an expression for the total area of the two pens.

A: Area

$A = x \cdot y \quad (x \geq 0, y \geq 0) \quad (1)$

(b) Use the given information to write an equation that relates the variables.

the fence is 300 ft long so $300 = 3y + 2x \quad (2)$

(c) Use part (b) to write the total area as a function of one variable.

From (2) : $x = \frac{300-3y}{2} \ldots (3)$ insert (3) into (1)

$A = \frac{300-3y}{2} \cdot y$ Also due to (3) $x \geq 0$ means $\frac{300-2y}{2} \geq 0$

$\Rightarrow y \leq 150$

(d) Find the largest possible total area of the two pens. Prove that it is the largest.

From (c) we know

$A = \frac{300-3y}{2} \cdot y \quad , \quad y \in [0, 150]$

$= 150y - \frac{3}{2}y^2$

$A' = 150 - 3y$

critical point is $y = 50$

$A(50) = 50 \cdot 75 = 3750$

$A(0) = 0$

$A(150) = 0$

The max area is 3750

Example 4. I get my sandwiches toasted at Subway®. The temperature of my sandwich is well approximated by the curve $T(t) = 20 + \frac{800t}{t^2+t+4}$ where at $t = 0$ my sandwich starts getting toasted.

(a) What is the initial temperature of my sandwich?

$T(0) = 20$

(b) How long did my sandwich get toasted for?

From 0 to $T(t)$ stops increasing, the sandwich is being toasted

To find when $T(t)$ stops increasing, we only need to find its critical point.

$$T'(t) = \frac{800}{t^2+t+4} - \frac{800t(2t+1)}{(t^2+t+4)^2}$$

$$T'(t) = 0$$

$$\Leftrightarrow \frac{800}{t^2+t+4} - \frac{800t(2t+1)}{(t^2+t+4)^2} = 0$$

$\Downarrow \leftarrow$ To simplify, multiply the common denominator $(t^2+t+4)^2$ on both sides.

$$(t^2+t+4)\cancel{800} - \cancel{800}t(2t+1) = 0$$

$$-t^2+4 = 0$$

$$\Rightarrow t = 2 \text{ or } \cancel{-2}$$

$T'(t)$ $+$ | $-$ at 2

the sandwich is toasted for 2 seconds.

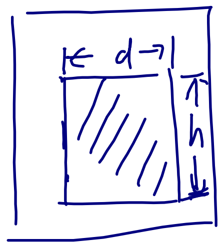
(c) What is the maximum temperature my sandwich achieved?

From (b), the maximum occur at 2

So the max temperature is $T(2) = 20 + \frac{800 \cdot 2}{10}$

$= 180$

Example 5. A poster is to have a total area of 180 in^2 with 1-inch margins at the bottom and sides and a 2-inch margin at the top. What dimensions will have the largest printed area?



A : shaded area

$A = d \cdot h \quad (d \geq 0, h \geq 0) \quad (1)$

The total area of the paper is 180 so

$(d+2)(h+3) = 180 \Rightarrow d = \frac{180}{h+3} - 2 \quad (2)$

plug (2) into (1) :

$A = \left(\frac{180}{h+3} - 2\right) \cdot h \quad (3)$, on the other hand, (2) and $d \geq 0$

implies $\frac{180}{h+3} - 2 \geq 0 \Rightarrow h \leq 87$.

From (3), the critical point of A occur at $A' = 0 \Leftrightarrow \frac{180}{h+3} - \frac{180h}{(h+3)^2} - 2 = 0$

$\Rightarrow h = \sqrt{270} - 3$. $A(\sqrt{270} - 3) = 186 - 4\sqrt{270} \leftarrow$ max area

Example 6. At which point(s) on the curve $y = 1 + 40x^3 - 3x^5$ does the tangent line have the largest slope?

S : slope of the tangent line

$S = y' = 120x^2 - 15x^4$

Find the critical points of S

$S' = 240x - 60x^3$

$S' = 0$ means $240x - 60x^3 = 0$

$\Leftrightarrow 60x(4 - x^2) = 0$

$\Leftrightarrow x = 0, \pm 2$

$S(0) = 0$

$S(2) = S(-2) = 480 - 240 = 240$.