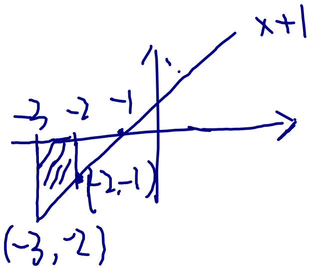
4.2 Problems

Question 1. Evaluate the integral by interpreting it in terms of area.

(a) $\displaystyle\int_{-3}^{-2} (x+1)\,dx = -3/2$

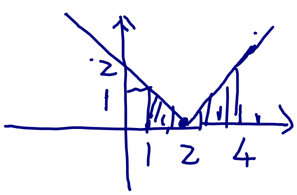


Area = −(Area of Big tiangle − Area of small triangle)

$= -\left(\frac{1}{2}\cdot 2\cdot 2 - \frac{1}{2}\cdot 1\cdot 1\right)$

$= -3/2$

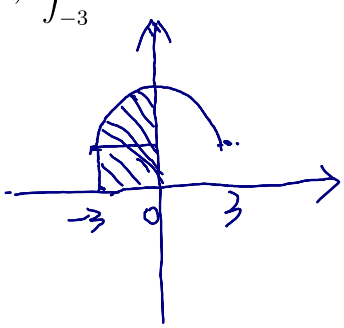
(b) $\displaystyle\int_{1}^{4} |x-2|\,dx = \frac{1}{2}\cdot 1\cdot 1 + \frac{1}{2}\cdot 2\cdot 2 = 5/2$



(c) $\displaystyle\int_{-3}^{0} 1+\sqrt{9-x^2}\,dx \rightarrow y = 1+\sqrt{9-x^2} \Rightarrow (y-1)^2 = 9-x^2 \Rightarrow x^2+(y-1)^2 = 9$

$y \geq 1$

this is a circle centered at (0,1) with radius 3



Area = a quater circle + rectangle

$= \frac{1}{4}\pi\cdot 3^2 + 1\cdot 3$

Question 2. Use the properties of definite integrals allow with:

$$\int_1^3 f(x)dx = 5 \qquad \int_1^7 f(x)dx = -1 \qquad \int_1^3 g(x)dx = 7 \qquad \int_3^7 g(x)dx = 10$$

to evaluate:

(a) $\int_3^1 f(x)dx$ $= -5$

(b) $\int_3^7 (1 + f(x))dx$ $= \int_1^7 (1+f(x))\,dx - \int_1^3 (1+f(x))\,dx$

$= \int_1^7 1\,dx + \int_1^7 f(x)\,dx - \int_1^3 1\,dx - \int_1^3 f(x)\,dx$

$\downarrow$ geometric $\quad \downarrow$ given $\quad \downarrow$ geometric $\quad \downarrow$ given

$= 7 + -1 - 3 - 7$

$= -4$

(c) $\int_7^1 (2f(x) - g(x))dx$

$= 2\int_7^1 f(x)dx - \int_7^1 g(x)\,dx \quad = -2\int_1^7 f(x) + \int_1^7 g(x)\,dx$

$= -2(-1) + \int_1^3 g(x)\,dx + \int_3^7 g(x)\,dx$

$= -2(-1) + 7 + 10 \quad = 19$

(d) $\int_3^7 (3 + f(x)g(x))dx$

A. $3 + (-1)(10)$

B. $3(4) + (-1)(10)$

C. $3 + (-6)(10)$

D. $3(4) + (-6)(10)$

E. None of the above.

Question 3. Express the following as a Riemann sum:

(a) $\int_0^{\pi} \sin x\, dx \quad = \lim_{n\to\infty} \Delta x \sum_{i=1}^{n} \sin x_i = \lim_{n\to\infty} \frac{\pi}{n} \sum_{i=1}^{n} \sin\left(\frac{i\pi}{n}\right)$

$\Delta x = \frac{\pi - 0}{n} = \frac{\pi}{n}$

$x_i = x_0 + i\cdot\Delta x$

$= 0 + \frac{i\pi}{n}$

(b) $\int_1^{5} \sqrt{x^3+1}\, dx \quad = \lim_{n\to\infty} \Delta x \sum_{i=1}^{n} \sqrt{x_i^3+1} = \lim_{n\to\infty} \frac{4}{n} \sum_{i=1}^{n} \sqrt{\left(1+i\cdot\frac{4}{n}\right)^3+1}$

$\Delta x = \frac{5-1}{n} = \frac{4}{n}$

$x_i = 1 + i\cdot\frac{4}{n}$

Question 4. Express the following as a definite integral:

(a) $\lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\frac{1-x_i^2}{4+x_i^2}}_{= f(x_i)} \Delta x$ on the interval $[\underbrace{2}_{=a}, \underbrace{6}_{=b}]$.

So $f(x) = \frac{1-x^2}{4+x^2}$

$\Rightarrow$ the given expression is $\int_2^6 \frac{1-x^2}{4+x^2}\, dx$

(b) $\lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\frac{\cos(\overbrace{\pi + i\pi/n}^{x_i})}{i\pi/n}}_{= f(x_i)} \underbrace{\frac{\pi}{n}}_{= \Delta x}$ on the interval $[\pi, 2\pi]$.

$\downarrow \quad \downarrow$

$a \quad b$

$f(x_i) = \frac{\cos x_i}{x_i - \pi}$ So $f(x) = \frac{\cos x}{x-\pi}$

So the given expression $= \int_{\pi}^{2\pi} \frac{\cos x}{x-\pi}\, dx$

Question 5. Evaluate the definite integrals using Riemann sums.

(a) $\int_1^5 (2x+1)dx$ $= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$ $= \lim_{n\to\infty} \sum_{i=1}^{n} \left(2\left(1+4\frac{i}{n}\right)+1\right)\cdot \frac{4}{n}$

$\Delta x = \frac{5-1}{n} = \frac{4}{n}$

$x_i = x_0 + i\Delta x$

$= 1 + i\frac{4}{n}$

$f(x) = 2x+1$

$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(3+\frac{8i}{n}\right)\frac{4}{n}$

$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{12}{n} + \frac{32i}{n^2}$

$= \lim_{n\to\infty} \left(\frac{12}{n}\sum_{i=1}^{n} 1 + \frac{32}{n^2}\sum_{i=1}^{n} i\right)$

$= \lim_{n\to\infty} \frac{12}{n}\cdot n + \frac{32}{n^2}\cdot\frac{n(n+1)}{2}$

$= 12 + \lim_{n\to\infty} \frac{n(n+1)}{n^2}\cdot 16$

$= 12 + 16 = 28$

(b) $\int_0^2 (2x - x^2)dx$

$\Delta x = \frac{2}{n}$

$x_i = 0 + \frac{2i}{n} = \frac{2i}{n}$

$f(x) = 2x - x^2$

$= \lim_{n\to\infty} \Delta x \sum_{i=1}^{n} f(x_i)$

$= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} (2x_i - x_i^2)$

$= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left(2\frac{2i}{n} - \left(\frac{2i}{n}\right)^2\right)$

$= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left(\frac{4i}{n} - \frac{4i^2}{n^2}\right)$

$= \lim_{n\to\infty} \left(\frac{8}{n^2}\sum_{i=1}^{n} i - \frac{8}{n^3}\sum_{i=1}^{n} i^2\right)$

ignoring low order terms

$= \lim_{n\to\infty} \left(\frac{8}{n^2}\frac{n(n+1)}{2} - \frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right) = \lim_{n\to\infty} \frac{8\cancel{n^2}}{2\cancel{n^2}} - \frac{8}{\cancel{n^3}}\frac{2\cancel{n^3}}{6} = 4 - \frac{8}{3} = 4/3$