Name: ____________________ **PID:** ____________________

1. (4 points) For what value of the constant c is the function

$$f(x) = \begin{cases} \sin \pi x & \text{if } x < 1 \\ x^2 - cx & \text{if } x \geq 1 \end{cases}$$

continuous on $(-\infty, \infty)$

Solution: From the definition, we immediately see that $f(x)$ has two parts, one is on $x < 1$ and the other is on $x \geq 1$. Inside each part, $f(x)$ is continuous because it is a polynomial (no matter what c is). In order for $f(x)$ to be continuous at $c = 1$ as well, it must satisfy the definition of continuity $\lim_{x\to 1} f(x) = f(1)$, or equivalently $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = f(1)$. We compute each of these one sided limit as follows

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \sin \pi x = 0$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} (x^2 - cx) = 1 - c$$

$$f(1) = 1 - c$$

So the function is continuous at c if and only if

$$0 = 1 - c$$

solving for c we get $c = 1$.

2. (2 points) Use the Intermediate Value Theorem to show that $f(x) = x^2 + x - 3$ has a root.

Solution: f is continuous everywhere. To prove $f(x)$ has a root ($f(c) = 0$ for some c), we need to find $f(a) < 0 < f(b)$, and use IVT. Observing

$$f(-3) = 3 > 0, \quad f(0) = -3 < 0$$

By IVT, there is a $c \in (-3, 0)$, such that $f(c) = 0$.

3. (2 points) Let $f(x) = x^2$, use the definition of derivatives to find $f'(1)$.

Solution 1: Directly evaluate $f'(1)$: $f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{(1+h)^2-1^2}{h} = \lim_{h\to 0} \frac{2h+h^2}{h} = \lim_{h\to 0}(2+h) = 2$

Solution 2: First find $f'(x)$, then plug in $x = 1$. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2-x^2}{h} = \lim_{h\to 0} \frac{2xh+h^2}{h} = \lim_{h\to 0}(2x + h) = 2x$.
Hence $f'(1) = 2$.